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On the formality of equivariant classifying spaces

Mel Rothenberg[1,★], **Georgia Triantafillou**[2]

[1] Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
[2] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

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1 Statement of results

First we define $BU(\alpha)$. Let $L^2(G)$ be the usual complex (in this case finite dimensional) Hilbert space of the complex regular representation of G, and let

$$\mathscr{H}^\infty(G) \equiv \sum_{j=1}^{\infty} L_j^2(G)\,,$$

where $L_j^2(G) = L^2(G)$. Let $BU_n(G)$ be the Grassmann G-space of complex n-planes in $\mathscr{H}^\infty(G)$ and let $\pi : E_n(G) \to BU_n(G)$ be the tautological complex G-vector bundle over $BU_n(G)$. The latter is a universal $G - U(n)$-vector bundle [LR] with associated principal $G - U(n)$-bundle $P_n(G) \to BU_n(G)$, the G-bundle of orthonormal n-frames in $\mathscr{H}^\infty(G)$.

As we shall see below $BU_n(G)$ is not convenient for our purposes, especially since the fixed point sets $BU_n(G)^H$ are not connected. We correct for this by restricting to G-vector bundles modeled by a given complex representation $\alpha : G \to U(n)$ as follows:

Definition 1 *We define $BU(\alpha)$ to be the G-subspace of $\mathscr{H}^\infty(G) \times BU_n(G)$ of all pairs (v, ρ) such that the action of the isotropy group $G_{(v,\rho)}$ on $E_n(G)_\rho = \pi^{-1}(\rho)$ is equivalent to $\alpha | G_{(v,\rho)}$.*

Let $E(\alpha)$ and π_α be defined as pullbacks

$$\begin{array}{ccc} E(\alpha) & \longrightarrow & E_n(G) \\ \big\downarrow{\scriptstyle \pi_\alpha} & & \big\downarrow{\scriptstyle \pi} \\ BU(\alpha) & \longrightarrow & BU_n(G) \end{array}\,,$$

where the map $BU(\alpha) \to BU_n(G)$ is the projection to the second factor.

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As we will show later π_α is universal for complex G-vector bundles $p: E \to X$ over G-CW-complexes X such that for all $x \in X$ the action of G_x on $E_x = \pi^{-1}(x)$ is equivalent to $\alpha | G_x$. Henceforth such bundles will be called α-bundles. The analogous notion in the case when α is an orthogonal representation was introduced in our earlier paper [RT].

The study of the equivalent classifying space $BU(\alpha)$ is crucial for the understanding of equivariant characteristic classes. In the nonequivariant case it is well known that $BU(n)$ splits rationally into a product of Eilenberg–Mac Lane spaces. Therefore the classifying map of a complex n-bundle over X is determined rationally by cohomology classes namely the Chern classes. In contrast to the nonequivariant case however, $BU(\alpha)$ or $BU_n(G)$ does not split rationally into a product of Eilenberg–Mac Lane G-spaces even when the group G is as simple as $\mathbf{Z}_p$. In [RT] we gave counterexamples for real representations α but the same argument carries over for complex representation of $\mathbf{Z}_p$ as well. We recall further from [RT] that

$$BU(\alpha)^H \simeq BC(\alpha | H),$$

where $C(\alpha | H)$ is the centralizer of $\alpha(H)$ in $U(n)$, and $BC(\alpha | H)$ is a product of $BU(n_i)$'s, i.e. the fixed point sets $BU(\alpha)^H$ do split rationally into a product of Eilenberg–Mac Lane spaces. The question is whether we can still utilize the simple form of the fixed point sets in order to compute the G-rational homotopy type of $BU(\alpha)$.

In the nonequivariant case the rational homotopy type of a simply connected space X is determined by the minimal model $\mathcal{M}_X$ of X which is a free and minimal differential graded algebra (DGA) over $\mathbf{Q}$. We recall from [S] that a space X is called *formal* if there is a cohomology isomorphism

$$\rho: \mathcal{M}_X \to H^*(X;\mathbf{Q}).$$

In other words the rational homotopy type of X is determined by the rational cohomology of X.

Now let X be a G-CW complex such that X^H is nonempty and simply-connected for all $H \subseteq G$. The notion of homotopy we consider in this context is equivariant homotopy of G-maps. We recall a result of [B] which characterizes G-homotopy equivalences for G-CW complexes, namely a G-map $f: X \to Y$ is a G-homotopy equivalence if and only if it induces isomorphisms $(f^H)_*: \pi_*(X^H) \to \pi_*(Y^H)$ for all subgroups H of G. An appropriate notion of rational G-homotopy type was developed in [T] where all fixed point sets are rationalized at the same time. Also for any space X as above an equivariant minimal model was constructed which determines the equivariant rational homotopy type of X.

The equivariant minimal model $\mathcal{M}$ is a *system* of DGA's rather than a single DGA i.e. $\mathcal{M}$ is a particular functor from the orbit category $\mathcal{O}_G$ into the category of DGA's. Here $\mathcal{O}_G$ is the category the objects of which are the quotients G/H, $H \subseteq G$, and morphisms are the equivariant maps. To simplify notation, we will designate objects of $\mathcal{O}_G$ by H rather than by G/H. Other examples of systems of DGA's associated to a G-complex X are the system

$\underline{H}^*(X;\mathbf{Q})$ of the cohomology of the fixed point sets and the system $\underline{\mathscr{E}}_X$ of de Rham algebras of the fixed point sets defined as functors from $\mathcal{O}_G$ into the category of graded algebras over $\mathbf{Q}$ by

$$\underline{H}^*(X;\mathbf{Q})(H) \equiv H^*(X^H;\mathbf{Q})$$

and

$$\underline{\mathscr{E}}_X(H) \equiv \mathscr{E}_{X^H}$$

respectively for $H \subseteq G$.

It seems that the right definition for equivariant formality would be to require that there is a cohomology isomorphism from the equivariant minimal model of a G-space X to the system of cohomology algebras of the fixed point sets $\underline{H}^*(X;\mathbf{Q})$. However, not all systems of DGA's admit an equivariant minimal model. The crucial property needed is *injectivity* in the sense of category theory. A system of DGA's can be considered as an object of a certain abelian category namely the category of functors from $\mathcal{O}_G$ into the category of vector spaces (by neglect of structure). A system of DGA's is *injective* if it is an injective object in this abelian category.

As shown in [T] the system $\underline{\mathscr{E}}_X$ of the de Rham algebras of the fixed point sets of X is always injective and equivariant minimal models for G-spaces with nonempty nilpotent fixed point sets can be constructed. However the system $\underline{H}^*(X;\mathbf{Q})$ is almost never injective. Recently [FT] solved this problem by constructing injective envelopes for arbitrary systems of DGA's as follows:

Theorem 2 [FT] *For any system of DGA's $\mathscr{A}$ over a finite group G there is an injective system of DGA's $\mathscr{I}$ and an inclusion $i: \mathscr{A} \to \mathscr{I}$ which is a cohomology isomorphism.*

This construction of injective envelopes satisfies certain functoriality and uniqueness conditions. Now the definition of equivariant formality can be given as follows.

Definition 3 [FT] *A G-space X is said to be equivariantly formal if there is a cohomology isomorphism from the equivariant minimal model $\mathscr{M}$ of X into the injective envelope $\mathscr{I}$ of $\underline{H}^*(X;\mathbf{Q})$.*

In other words the equivariant homotopy type of X is rationally determined by the rational cohomology of its fixed point sets. We remark here that the above difficulty does not appear in the definition of equivariant formality in the dual sense in terms of differential graded Lie algebras. In fact we proved in [RT] the equivariant formality of $B(\alpha)$, where α is an orthogonal representation of an *abelian* group G, in this dual sense.

The main result of this paper is:

Theorem 4 *The space $BU(\alpha)$ is equivariantly formal.*

This means that the equivariant homotopy type of $BU(\alpha)$ rationally depends only on the cohomology of its fixed point sets. Since $BU(\alpha)$ does not decompose rationally into a product of G-Eilenberg–Mac Lane spaces, the formality of $BU(\alpha)$ is the next best property that makes the question of the equivariant characteristic classes tractable.

Remark. By using more recent work of B. Fine, who extended Triantafillou's equivariant minimal model theory to the non-connected fixed point set case, we can show equally well that the general classifying space $BU_n(G)$ is equivariantly formal. However we see several advantages in focusing on $BU(\alpha)$. First of all the cohomology of the fixed point sets is simpler being a free polynomial algebra. Even more importantly fixing the representation α simplifies the surgery exact sequence in computations involving realizing cohomology characteristic classes by bundles as in the section on applications below. Thirdly the delicate question of equivariant transversality is better dealt with in the context of a single representation. Since the formality of $BU_n(G)$ is still of theoretical if not computational importance we state it here as a theorem.

Theorem 5 *The classifying space $BU_n(G)$ is equivariantly formal.*

In order to prove that the space $BU(\alpha)$ is equivariantly formal we approximate it by a system of spaces BU_α which has the advantage that is is a limit of G-Kähler manifolds. Then we employ the equivariant formality of G-Kähler manifolds from [FT] (actually a refinement of it) and a subtle inverse limit argument. Here several technical difficulties have to be dealt with. First, as is well known minimal models are not functorial. Moreover in contrast to the nonequivariant case, surjectivity of a map $f:\mathscr{A} \to \mathscr{B}$ between systems of DGA's is not sufficient for the existence of a strict (not only up to homotopy) lift $f':\mathscr{M} \to \mathscr{N}$ of f to the minimal models. Special arguments using the geometry of $BU(\alpha)$ have to be used to make this approach work.

The proof of formality of $BU_n(G)$ is simpler since it is a limit of G-Kähler manifolds and no approximation is needed.

On the way we prove the following proposition which is useful for the lifting problem above.

Proposition 6 *Let Y be a G-simplicial complex and let X be a G-subcomplex of Y. Then the map $\varepsilon:\underline{\mathscr{E}}_Y \to \underline{\mathscr{E}}_X$ has a linear splitting in each degree, where ε is the map between the systems of de Rham algebras induced by the inclusion.*

In the last section we give an application of the formality of the classifying space $BU(\alpha)$ by realizing certain cohomology data as equivariant Pontryagin classes of a G-manifold in Theorem 14 and Corollary 18.

2 Proofs

We begin with an easy result. Henceforth X will denote a G-space which admits a G-embedding in $\mathscr{H}^\infty(G)$ – for example, a countable G-CW-complex.

Proposition 7 *Let $E \to X$ be a numerable α bundle. (See* [LR], *Def.* 2). *Then there exists a G-map $f: X \to BU(\alpha)$ unique up to G-homotopy such that $f^*(E(\alpha)) = E$. In other words, π_α is universal for such X.*

Proof. Up to G-homotopy, E determines uniquely a map $\bar{f}: X \to BU_n(G)$. Let $j: X \mapsto \mathscr{H}^\infty$ be a G-embedding unique up to G-homotopy since $\mathscr{H}^\infty$ is G-contractible. Since $(\bar{f}\times j)^*\pi_1^*(E_n(G)) = E$, $\bar{f}\times j$ is an embedding, and E is an α-bundle $\bar{f}\times j(X)$ must lie in $BU(\alpha)$. We let $f = \bar{f}\times j$. Applying the same arguments to $X\times I$ rel $X \times \partial I$ we see that f is unique up to G-homotopy. QED

The space $BU(\alpha)$ is not so nice to work with, but it seems difficult to find simpler universal models at the level of G-spaces. However, at the level of systems of G-spaces [RT] there are more convenient models available. Recall that a G-system (of spaces) is just a contravariant functor S from the orbit category $\mathcal{O}_G$ into the category of topological spaces. A map of a system is just a transformation of functors. Given a G-space Y, the correspondence $H \mapsto Y^H$ is a G-system and this embeds G-spaces and G-maps as a full subcategory of G-systems. A system is thus a generalized G-space. The generalization consists in relaxing two properties of G-spaces: for any subgroup $K \subset H$ and N the normalizer of K in H

1. The map $S(H) \to S(K)^N$ may not be an embedding.
2. The map $S(H) \to S(K)^N$ may not be a homotopy equivalence.

The first property is an inessential difference in that we can replace a reasonable S by a homotopy equivalent system using mapping cylinders such that all mappings in the system are embeddings. The second property is the crucial difference and can lead to anomalies such as maps $j: S_1 \to S_1$ between systems such that for each H in $\mathcal{O}_G$, $j(H)$ is a homotopy equivalence between CW complexes, yet j has no homotopy inverse. Thus the usual methods of constructing maps of G-spaces step by step up the fixed point strata can fail when the domain is a system. However when the domain is a G-space and the range a G-system, those methods do apply. This is an observation that can be made precise using Elmendorf's realization transform [E] from systems to spaces, but we do not need this generality here.

In [RT] we developed the notion of bundle systems and α-bundle systems over G-systems of spaces. Specifically, for $H \in \mathcal{O}_G$, let $C(\alpha(H))$ be the centralizer of $\alpha|H$. Let $E(U(n))$ be the standard infinite join model for the universal contractible $U(n)$ space, and $BC(\alpha(H)) = E(U(n)))/C(\alpha(H))$ the corresponding model for the classifying space of $C(\alpha(H))$. Then each $BC(\alpha(H))$ has a canonical $U(n)$ bundle on it and the corresponding spaces and bundles form an α-bundle system which we denote by $BC(\alpha)$. We can then apply the criterion of Theorem 6 of [LR] and the remarks following it to conclude that $BC(\alpha)$ is universal for numerable α-bundles over G-spaces.

While we cannot expect $BC(\alpha)$, or anything else, to be universal for all numerable α-bundle systems we can refine the above to the following: Let us call S a *nice system* if each $S(H)$ is a $N(H)$-CW-complex and further that if

$K > H$ then $S(K) \to S(H)$ is a pushout of CW-embeddings. Then $BC(\alpha)$ is universal for numerable α-bundle systems over such S.

By the universality of both systems we can then conclude:

Proposition 8 *There is a unique up to homotopy map of systems $BU(\alpha) \to BC(\alpha)$ compatible with the corresponding bundle systems. For every H in $\mathcal{O}_G$ this map induces a weak homotopy equivalence $BU(\alpha)^H \to BC(\alpha)^H$.*

We wish finally to introduce a third universal model, the system BU_α, where $BU_\alpha(H)$ consists of n planes V in H^∞ invariant under the action of H and such that the action of H on V is equivalent to $\alpha|H$. BU_α is a nice system since as we show below it is a limit of systems of smooth varieties. By definition there is a map of systems $BU_\alpha \to BU_n(G)$ and the induced system of bundles over BU_α is a system of α-bundles. By the universality of $BC(\alpha)$ there is a unique up to homotopy map of systems $s: BU_\alpha \to BC(\alpha)$. To show that BU_α is universal for numerable α-bundles over G-CW-complexes it then suffices to show that the map $BU_\alpha(H) \to BC(\alpha(H))$ is a weak homotopy equivalence for every H. This follows from the following:

Proposition 9 *Given any finite complex X and f as below there exists a unique up to homotopy f' which makes the diagram commute*

$$\begin{tikzcd} & BU_\alpha(H) \arrow[d, "s(H)"] \\ X \arrow[ur, "f'"] \arrow[r, "f"] & BC(\alpha(H)) . \end{tikzcd}$$

Proof. Given f we have the pullback bundle $\alpha|H: E \to X$, where X has trivial H action. We then have the G-bundle $G \times_H E \to G/H \times X$ and this induces a unique up to G-homotopy G-map $f': G/H \times X \to BU_n(G)$. Restricting to $e \times X$ yields $f': X \to BU_n(G)^H$ and since E is a $\alpha|H$ bundle, $f'(X)$ must lie in $BU_\alpha(H)$, and since f' induces E, $s(f') = f$. The uniqueness of f' follows by making the same argument on $X \times I$ rel $X \times \partial I$. QED.

We have not quite constructed a map from $BU(\alpha)$ to BU_α, since $BU(\alpha)$ is not a priori a-G CW-complex. However, applying Elmendorf's realization functor replaces $BU(\alpha)$ by a G-CW-complex $r(BU(\alpha))$ which maps into it. Since in this case the realization is also universal being G-homotopy equivalent to the realization of $BC(\alpha)$, we get a map in the other direction and this gets us a weak equivalence from $BU(\alpha)$ to BU_α. In any case we have a diagram

$$\begin{tikzcd} & BU(\alpha) \\ r(BU(\alpha)) \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & BU_\alpha , \end{tikzcd}$$

where $i(H)$, $j(H)$ are weak homotopy equivalences. Thus we proved the following.

Proposition 10 *There is a weak equivalence between $BU(\alpha)$ and BU_α.*

Before we prove the main theorem we recall the result about the formality of Kähler manifolds which we will use in the proof.

Theorem 11 [FT] *Let G be a finite group and let X be a Kähler manifold which admits a holomorphic G-action with nonempty connected and simply connected fixed point sets. Then X is equivariantly formal. Moreover equivariant holomorphic maps between G-Kähler manifolds are formal, i.e. they are determined rationally by the maps induced on cohomology.*

Proof of Theorem 4 By Proposition 10 it suffices to prove that BU_α is formal. We consider the system of spaces BU_α as the limit of inclusions

$$\cdots \subset BU_{\alpha,N} \subset BU_{\alpha,N+1} \subset \cdots$$

where $BU_{\alpha,N}$ is defined in the same way as BU_α except that we use $\mathscr{H}^N(G) = \bigoplus_{j=1}^N L_j(G)$ instead of $\mathscr{H}^\infty(G)$. By construction each $BU_{\alpha,N}(H)$ is a smooth algebraic variety and NH/H acts on $BU_{\alpha,N}(H)$ preserving this structure. By a well known result $BU_{\alpha,N}(H)$ is a Kähler manifold and G preserves the structure. In fact $BU_{\alpha,N}(H)$ are complex Grassmann manifolds and they approximate the cohomology of $BU_\alpha(H)$ up to a range $k(N)$ such that $k(N) \to \infty$ if $N \to \infty$, i.e.

$$H^i(BU_\alpha(H)) \cong H^i(BU_{\alpha,N}(H)) \tag{1}$$

for $i \leqq k(N)$. Consider the sequence of the systems of de Rham algebras of $BU_{\alpha,N}$'s and their equivariant minimal models $\mathscr{M}_N \equiv \mathscr{M}_{BU_{\alpha,N}}$ namely:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \underline{\mathscr{E}}_{BU_{\alpha,N+1}} & \xrightarrow{\varepsilon_N} & \underline{\mathscr{E}}_{BU_{\alpha,N}} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \rho_{N+1}} & & \uparrow{\scriptstyle \rho_N} & & \\
\cdots & \longrightarrow & \mathscr{M}_{N+1} & \xrightarrow{\bar{\varepsilon}_N} & \mathscr{M}_N & \longrightarrow & \cdots .
\end{array} \tag{2}$$

This diagram commutes only up to homotopy in general. Moreover the inverse limit $\lim_{\overleftarrow{N}} \mathscr{M}_N$ need not be a minimal system of DGA's. It need not be even an injective system of DGA's.

In order to get a well defined inverse limit we proceed as follows. First we observe that ε_N is surjective meaning that $\varepsilon_N(H)$ is surjective for every $H \subseteq G$. Moreover by (1) ε_N induces an isomorphism on cohomology in degrees less or equal to $k(N)$. In the nonequivariant case these two properties are enough to guarantee the existence of a strict lifting $\bar{\varepsilon}_N$ in (2) up to degree $\leqq k(N)$. In the equivariant case we need in addition a linear splitting of the map ε_N in the same degrees. This is provided by Proposition 6. Then the construction of lifts of [T] can be modified by a coboundary to give a strict lifting of ε_N in the range of degrees $\leqq k(N)$. Therefore the diagram (2) commutes strictly in degrees $\leqq k(N)$ and commutes only up to homotopy in higher degrees. In fact by construction it commutes strictly when restricted to the subsystems $\mathscr{M}_{N+1}(k(N)) = \mathscr{M}_N(k(N))$ which correspond to the $k(N)$-stage of the Postnikov tower.

In the same way we continue the diagram to the left for larger N and hence larger $k(N)$. Now we observe that $\mathcal{M}_N(k(N)) = \mathcal{M}_{N+i}(k(N))$ for all $i \geqq 0$ and therefore

$$\mathcal{M}(k(N)) = \mathcal{M}_N(k(N)) .$$

This shows that $\mathcal{M}$ is a minimal system of DGA's. We note that $\mathcal{M}$ does not depend on the homotopy lifting of (2) in degrees $\geqq k(N)$, in fact such a lifting beyond the $k(N)$-stage of the Postnikov tower is not necessary for the definition of $\mathcal{M}$. We will use this fact later in the proof. Because the minimal systems $\mathcal{M}_N(k(N))$ are uniquely defined up to isomorphism $\mathcal{M}$ is unique up to isomorphism as well. Since obviously

$$\underline{\mathscr{E}}_{BU_{\alpha,N}} = \varprojlim_N \underline{\mathscr{E}}_{BU_{\alpha,N}}$$

we get a map $\rho : \mathcal{M} \to \underline{\mathscr{E}}_{BU_\alpha}$ which is a cohomology isomorphism by construction of $\mathcal{M}$ and property (1).

In order to show that BU_α is equivariantly formal we have to construct a cohomology isomorphism

$$\mu : \mathcal{M} \to \mathcal{I} ,$$

where $\mathcal{I}$ is an injective envelope of $\underline{H}^*(BU_\alpha;\mathbf{Q})$. First we note that

$$\underline{H}^*(BU_\alpha;\mathbf{Q}) = \varprojlim_N \underline{H}^*(BU_{\alpha,N};\mathbf{Q})$$

by property (1). By the equivariant formality of each $BU_{\alpha,N}$ we have maps

$$\mu_N : \mathcal{M}_N \to \mathcal{I}_N ,$$

where $\mathcal{I}_N$ is an injective envelope of $\underline{H}^*(BU_{\alpha,N};\mathbf{Q})$. The injective envelope $\mathcal{I}_{N+1}$ can be chosen to be isomorphic to the injective envelope $\mathcal{I}_N$ in the range of degrees $\leqq k(N)$. Therefore we get again an isomorphism $\varepsilon'_N : \mathcal{M}_{N+1}(k(N)) \to \mathcal{M}_N(k(N))$ such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{I}_{N+1} & \longrightarrow & \mathcal{I}_N & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathcal{M}_{N+1}(k(N)) & \longrightarrow & \mathcal{M}_N(k(N)) & \longrightarrow & \cdots \end{array} \tag{3}$$

commutes. As we remarked earlier the inverse limit $\varprojlim_N \mathcal{M}_N$ does not depend on the existence of a map from $\mathcal{M}_{N+1}$ to $\mathcal{M}_N$ beyond the $k(N)$-stage. Hence $\varprojlim_N \mathcal{M}_N$ is defined by the sequence (3), and it coincides with the inverse limit defined by (2). On the other hand it is easy to see that the inverse limit of the $\mathcal{I}_N$'s is an injective envelope $\mathcal{I}$ of $\underline{H}^*(BU_\alpha;\mathbf{Q})$. Therefore the required map μ between the two inverse limits exists and is a cohomology isomorphism by a degree by degree argument. This completes the proof of the theorem.

In effect we proved a more general result. We state it as a separate proposition below since it may be useful in other situations as well.

Proposition 12 *Let S be a G-space or a system of spaces over G which is the direct limit of a sequence of inclusions*

$$\cdots \to S_n \to S_{n+1} \to \cdots,$$

where (i) each S_n is a formal G-space or a formal nice system of spaces over G, and (ii) there is a function $k(n)$ of n such that $k(n) \to \infty$ as $n \to \infty$ and such that

$$H^i(S_n(H)) = H^i(S(H))$$

for $i \leqq k(n)$ *and $H \subseteq G$. Then S is equivariantly formal.*

Proof of Proposition 6 We adapt to the equivariant case the extension map E on forms given in [BG] p. 3. This map is linear and provides for each p-form ω defined on the boundary of a simplex σ an extension ω' of ω of the same degree on σ. We will make E functorial with respect to the category $\mathcal{O}_G$, i.e. we will construct a natural transformation

$$\underline{E} : \underline{\mathscr{E}}_Y \to \underline{\mathscr{E}}_X$$

by induction on the fixed point sets and on dimension of simplices. First we define an extension map $\underline{E}(G) : \mathscr{E}_{X_n^G} \to \mathscr{E}_{Y_n^G}$ as in [BG] which is a splitting of $\varepsilon(G) : \mathscr{E}_{Y_n^G} \to \mathscr{E}_{X_n^G}$. Let H be a fixed subgroup of G. Assume inductively that $\underline{E}(H') : \mathscr{E}_{X_n^{H'}} \to \mathscr{E}_{Y_n^{H'}}$ is defined for all $H' \supsetneqq H$. Consider a form $\omega \in \underline{\mathscr{E}}_{X_n^H}$. This form has already been extended to all n-simplices of Y the isotropy groups of which are larger than H. Extend the form by E of [BG] on the simplices of Y^H the isotropy group of which is H. We have a linear map from $\mathscr{E}_{X^H}$ into $\mathscr{E}_{Y^H}$ which we average over the action of $N(H)/H$ to get an equivariant map $\underline{E}(X)$: $\mathscr{E}_{X_n^H} \to \mathscr{E}_{Y_n^H}$; here $N(H)$ is the normalizer of H in G. By translating this map by $g \in G$ we get the desired map $\underline{E}(K)$ for all subgroups K which are conjugate to H. This completes the inductive step and the proof of the proposition.

Applications

We recall that in the nonequivariant case, by using surgery methods one can realize (under certain signature conditions) rational cohomology classes as Pontryagin classes of a manifold of a given rational homotopy type. As an application of our formality result we prove an equivariant analogue, Theorem 14 and Corollary 18. The formality assumption of the manifold is necessary since the nonstable equivariant classifying space is not a product of G-Eilenberg Mac Lane spaces rationally.

As before we fix a representation $\alpha : G \to U(n)$ which we also call V, where V is a complex vector space on which G acts by unitary automorphisms. Let $S(V)$ denote the unit sphere of V and $T(V)$ the one-point-compactification, where both are G-spaces. In the equivariant context we have the notion of a G-bundle $E \to Y$ with fiber X, where all spaces are G-spaces and all maps are equivariant. These come with structure groups $K(E) \subset \mathrm{Homeo}(X)$, where the inclusion is an equivariant map and the action of G on Homeo(X) is by conjugation. We will particularly consider bundles with fiber V or $S(V)$ and structure group $U(V)$.

More generally we can consider fiber spaces $E \to Y$ with fibers G-homotopy equivalent to X, the structure H-space of which is the monoid of base point preserving self homotopy equivalences of X denoted aut(X). Because of the G-fixed base point assumption, the fiber space E has a canonical crossection which we regard as part of its structure. Equivalence of two such fibrations means G-fiber homotopy equivalence preserving the crossection. Both bundles and fiber spaces have classifying spaces denoted by $BK(E)$ and $BH(E)$ respectively.

Definition 13 *A (V, W)-structure on a G-manifold M is a smooth embedding $c: M \to V \oplus W$ such that the G-normal bundle $n(c)$ of c is given a V-structure, i.e. $n(c)$ is classified by a G-map, defined up to G-homotopy, $n(c): M \to BU(V)$. Here W can be a real or complex representation whereas V is a complex representation as before.*

We will routinely identify fibrations and bundles with their classifying maps and with their total spaces. Such a structure on M yields a based G-map $T(V \oplus W) \to T(n(c))$ called the *characteristic map* of the structure. Its transverse inverse image of M is just $c(M)$.

Example. If M is G-connected, and $|G|$ is odd and if W is the tangent representation of G at a fixed point of M, and V is any unitary representation, then any G-embedding of M in $V \oplus W$ yields a canonical (V, W)-structure.

Now we wish to recall a few facts from surgery theory. We will work in the smooth category and for brevity with the homotopy rather than the simple homotopy version, although what we say with some obvious modifications applies to both.

Let P be an integral Poincaré duality space of formal dimension $n > 4$. Again for brevity's sake we assume that P is orientable. Then we have the surgery exact sequence

$$L^1 \to \mathscr{S}(P) \to \mathscr{N}(P) \to L^0 .$$

Here L^i are Abelian groups which depend only on the dimension of P and on $\pi_1(P)$. The set $\mathscr{S}(P)$ consists of homotopy equivalences of smooth manifolds to P identified up to h-cobordism over P, whereas $\mathscr{N}(P)$ is the set of normal maps of degree 1 from smooth manifolds to P identified up to cobordism over P.

The sequence can be localized at 0 (rationalized) in the following sense. We replace P by its localization at 0. More precisely we localize at 0 the universal cover of P in the category of free $\pi_1(P)$-spaces. Let P' be the orbit space. Now $\mathscr{S}(P')$ is the set of rational homotopy equivalences of smooth manifolds to P' up to rational h-cobordism over P' and $\mathscr{N}(P')$ becomes the set of smooth manifold maps of positive degree such that the rationalized stable normal bundle pulls back from P'. The corresponding L-groups depend only on the rational group ring $\mathbf{Q}(\pi_1(P))$, [Ba].

Instead of looking at fundamental groups acting freely on universal covers we consider finite groups G acting smoothly (and orientation preserving) on compact manifolds M. Then under the *gap hypothesis*, i.e. the assumption that for subgroups $K \subset H$, $\dim M^H < \dim M^K$ implies $\dim M^H < 2\dim M^K > 2$, a good version of the surgery exact sequence holds [DR]. The $\mathcal{N}$ and $\mathcal{S}$ sets are the G-category versions of the above $\mathcal{N}$ and $\mathcal{S}$. The L-groups depend on G, the dimensions of the fixed point submanifolds, the fundamental groups of the fixed point sets, and some local data. For simplicity we consider only the G-simply connected case, where the fixed point sets are nonempty and simply connected. Then the only local data we need to specify is the tangent representation of G at a fixed point which we fix once and for all.

Under the hypothesis of nonempty and simply connected fixed point sets we can rationalize G-spaces M yielding M_0 and as above the localized surgery exact sequence. The corresponding L-groups depend only on G, the fixed point dimensions and the local tangent representation. We will write them as $L^i(M_0)$ for convenience, and make some remarks about calculating later. Our main theorem can now be stated.

Theorem 14 *Let M be a G-simply connected manifold of dimension > 4 with a (V, W)-structure, which is G-formal, and satisfies the gap hypothesis. Let*

$$H(M, V) \equiv \mathrm{Hom}(\underline{H}^*(BU(V);\mathbf{Q}), \underline{H}^*(M;\mathbf{Q}))$$

be the set of natural transformations of rational cohomology thought of in the usual way as algebra functors on the orbit category of G. Then there is a family of maps $j: H(M, V) \to L^0(M_0)$ such that if $j(b) = 0$ then there exists a G-manifold M' rationally equivalent to M such that b stabilized is realized by a $(V \oplus ke, W)$-structure on M'. Here e is the trivial representation. If V itself is sufficiently large (depending on the dimension of M) we can replace $V \oplus ke$ by V.

To prove this we will construct a map from the set of homotopy classes of maps $M_0 \to BU(V)_0$ to $\mathcal{N}(M_0)$. This construction does not use the gap hypothesis or formality. This construction, the formality of M and $BU(V)$ and the surgery exact sequence will then provide Theorem 14.

For any unitary G-representation V we have $S(V \oplus e) = T(V \oplus \mathbf{R})$, where T means one-point-compactification. Thus any V-bundle E over Y gives an associated $S(V \oplus e)$-bundle $S(E + e)$ with a canonical crossection corresponding to the base point of $T(V \oplus \mathbf{R})$. Passing to fiber spaces yields $i: BU(V) \to B\mathrm{aut}S(V \oplus e)$. Let i_0 be the rationalization of this map.

Remark. One may worry about localizing $B\mathrm{aut}S(V \oplus e)$ since it is not G-simply connected, although $BU(V)$ is. The standard way to deal with this is to replace the classifying space $B\mathrm{aut}S(V \oplus e)$ by its G-simply connected covering space, say $\tilde{B}$, and lift i. Then we simply define $B\mathrm{aut}S(V \oplus e)_0 \equiv \tilde{B}_0$.

A standard argument equates $\mathrm{aut}S(V \oplus e)_0 = \mathrm{aut}(S(V \oplus e)_0)$ and thus

$$B\mathrm{aut}S(V \oplus e)_0 = B\mathrm{aut}(S(V \oplus e)_0)\,.$$

We thus have $i_0: BU(V)_0 \to B\mathrm{aut}(S(V \oplus e)_0)$.

Lemma 15 *For any subgroup K of G, $\pi_i(\mathrm{Baut}S(V \oplus e)^K)$ is finite.*

Proof. We have $\pi_i(\mathrm{Baut}S(V \oplus e)^K) = \pi_{i-1}(\mathrm{aut}_K S(V \oplus e))$, where $\mathrm{aut}_K(X)$ is the monoid of K-equivariant self homotopy equivalences of X. Note that $\mathrm{aut}_K S(V \oplus e)$ is a proper subset of $\mathrm{aut}S(V \oplus e)^K$. For $K = 1$, $\mathrm{aut}_K S(V \oplus e)$ is just the union of two components of $\Omega^m S(V \oplus e)$, where $m = \dim S(V \oplus e)$. For larger K the result follows from the equivariant Postnikov decomposition of $S(V \oplus e)$ and the fact that the fixed point sets are all spheres of odd dimension.

Corollary 16 *The space $\mathrm{Baut}S(V \oplus e)_0$ is equivariantly contractible as a G-space.*

Corollary 17 *Any two fiber spaces over X with fibers $S(V \oplus e)_0$ and with crossections are G-fiber homotopy equivalent by an equivalence preserving the crossection. The equivalence is itself determined up to G-fiber homotopy.*

Proof of Theorem 14 We now return to our (V, W)-manifold M. We have the composition.

$$i_0 \circ n(c)_0 : M_0 \to BU(V)_0 \to \mathrm{Baut}S(V \oplus e)_0 \,.$$

Let E_1 be the spherical fibration induced by this map. Let $b: M_0 \to BU(V)_0$ be any map, and let E_2 be the spherical fibration induced by $i_0 \circ b$. By Corollary 17 we have a zero section preserving G-fiber homotopy equivalence $f: E_1 \to E_2$, uniquely determined up to G-homotopy. Let $\bar{E}_i = E_i -$ (0-*section*). Then $\bar{f}: \bar{E}_1 \to \bar{E}_2$ is well defined.

Since $S(V \oplus e)_0 = T(V \oplus \mathbf{R})_0$, and M_0 and thus E_i and $\bar{E}_i$ are rational spaces, we have that $(S^1 \wedge Tn(c))_0 = T(\bar{E}_1)$. We thus have the following composite map, uniquely defined up to G-homotopy, $T(V \oplus W \oplus \mathbf{R}) \to (S^1 \wedge Tn(c))_0 = T(\bar{E}_1) \to T(\bar{E}_2)$, where the first map is the suspension of the characteristic map. We will denote this composition by h.

We now consider the diagram:

$$\begin{array}{ccc} M' & \xrightarrow{b'} & BU(V) \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{b} & BU(V)_0 \,. \end{array}$$

Here M' and b' are pullbacks, and the right vertical map is localization. Since M_0 is rational the left vertical map is rationalization as well, and M' is rationally equivalent to M. Let E' be the bundle induced by b'. Our construction yields a fiberwise localization $T(E' \oplus \mathbf{R}) \to T(\bar{E}_2)$. We then have the diagram:

$$\begin{array}{ccc} & & T(E' \oplus \mathbf{R}) \\ & \nearrow & \downarrow \\ T(V \oplus W \oplus \mathbf{R}) & \xrightarrow{h} & T(\bar{E}_2) \,. \end{array}$$

The set of base point preserving G-homotopy classes of maps $[T(V \oplus W \oplus \mathbf{R}), T(\bar{E}_2)]_G$ form an Abelian group, since by our assumptions

W always contains at least one trivial summand. Since the vertical map is a rationalization elementary obstruction theory says there is a minimal positive integer m such that mh lifts. Suspending once yields $\overline{mh}: T(V \oplus W \oplus e) \to T(E' \oplus e)$. Since in what follows we can also consider dmh for any integer d, we are really describing an integer indexed family of liftings. That is why we speak of a family of maps in Theorem 14.

The final step is to G-homotopy $\overline{dmh}$ to a map transversal to the zero section. In general deforming maps equivariantly to transversal ones is a delicate business which cannot always be done. However we have good control. Specifically the tangent bundle of $T(V \oplus W \oplus e)$ away from the base point has a direct summand which at each point is the fiber of $E' \oplus e$. We can then apply the technique of strict transversality to deform $\overline{dmh}$ to a map transversal to M' which is unique up to G-transversal homotopy. (This makes sense even though M' is a "bad" space. Transversality is a local notion, and locally the bundle is trivial. Hence locally we can project to the fiber of the bundle and make the map transversal to zero.)

Thus without loss of generality we can assume $\overline{dmh}$ is transversal to M'. Composing $\overline{dmh}|_{(\overline{dmh})^{-1}(M')}$ with the given localization of $M' \to M_0$ yields a well defined element of $\mathcal{N}(M_0)$ which by construction is provided with a $(V \oplus e, W)$-structure. This completes the construction of the family of functions from the homotopy classes of rational maps to the normal groups.

So far we have used neither formality nor the gap hypothesis. The surgery exact sequence says that this element of $\mathcal{N}(M_0)$ is G-cobordant to a rational G-equivalence iff its image in the surgery group vanishes. We may have to add k trivial representations to make this cobordism in $V \oplus W \oplus ke$. Here k depends on $\dim M$. If V has enough trivial summands we do not have to do this. In fact we can lift our original (V, W)-structures back to $(V - ke, W)$ and work there.

Now we bring formality into play. By definition it provides us with a map from $H(M, V)$ into homotopy classes of maps $M_0 \to BU(V)_0$, which realize the given map on cohomology. Let $b \in H(M, V)$ and let $\bar{b}: M_0 \to BU(V)_0$ represent the corresponding homotopy class of maps. We say that a rational equivalence $f: M'' \to M_0$, where M'' is given a (V, W)-structure, realizes b if the following diagram commutes up to homotopy.

$$\begin{array}{ccc} M'' & \longrightarrow & BU(V) \\ \downarrow{\scriptstyle f} & & \downarrow \\ M_0 & \xrightarrow{\bar{b}} & BU(V)_0 . \end{array}$$

Here the top horizontal map is the one associated with the (V, W)-structure. Now the conclusion of Theorem 14 holds by the construction.

The fact that the correspondence j in Theorem 14 is a relation rather than a function is due entirely to the choice one has in lifting the rationalized characteristic map. The vanishing of $j(b)$ presumably depends on this choice. One can describe the effect of different choices in terms of operations on the surgery group $L(M_0)$. Since we do not know these groups in general there is not too much point in doing so. When $L(M_0)$ is nonzero it tends to have rational vector space summands and thus is quite large. Since there are real obstructions to realizing b, given by the G-signature theorem, which involve Pontryagin numbers of the various fixed point sets, as well as some G-orientation constraints, $j(b)$ can often be nonzero, and a more refined analysis shows that $j(b)$ can take on quite a wide range of values. In fact $L(M_0)$ should be thought of as precising exactly all the G-signature and G-orientation constraints, and our theorem says that these are the only constraints.

Our result says that $j(b)$ is determined by b but we are far from an explicit calculation which would be very interesting. It is possible and likely that one could realize b by an M'' which does not come with a lift of the characteristic map. Finally it is also true that it may be possible to realize some map $M_0 \to BU(V)_0$ whose map on cohomology is b, without being able to realize $\tilde{b}$. That is $\tilde{b}$ is not the unique map which on cohomology gives b.

Despite all this the existence of j does yield specific results. Most cleanly, when $|G|$ is odd, and the dimension of M is odd, then it is not difficult to show by explicit calculation over fixed point sets that $L(M_0)$ is 0. Thus we have:

Corollary 18 *If along with the hypothesis of Theorem* 14, $|G|$ *is odd and* M *is odd dimensional then any* $b \in H(M, V)$ *has a realization* $M'' \to M_0$.

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