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Surjective-Buchsbaum modules over Cohen-Macaulay local rings

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Dedicated to Professor Takeshi Ishikawa on the occasion of his sixtieth birthday

Introduction

Let A be a Noetherian local ring and M a finitely generated A-module. We say that M is a Buchsbaum module if the difference $\ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)$ is an invariant of M, not depending on the choice of a parameter ideal $\mathfrak{q}$ for M. Goto [6] gave a structure theorem for maximal Buchsbaum modules over regular local rings, that is, if A is a regular local ring of dimension $d > 0$, then there exist exactly d isomorphism classes of indecomposable maximal Buchsbaum A-modules and any maximal Buchsbaum A-module is a direct sum of finite copies of them and the residue class field k, where an A-module M is said to be maximal if $\dim_A M = \dim A$. In this paper we are interested in improving his theorem, so that it shall work not only for regular local rings but also for Cohen-Macaulay local rings possessing dualizing complexes. For instance, Yoshino [22] explored maximal Buchsbaum modules over a Gorenstein local ring A, and gave a univalent correspondence between certain maximal Buchsbaum modules and representations of some quivers. In his theorem he assumed the modules considered to have finite projective dimension; this assumption seems reasonable, since modules over regular local rings have finite projective dimension.

Stückrad and Vogel [20] gave a sufficient condition, so called the *surjectivity criterion*, for modules to be Buchsbaum. In general, it is not a necessary condition and so we refer to those modules satisfying the condition obtained by Stückrad and Vogel as surjective-Buchsbaum modules over A; see the next section for the detail of definition. However, if A is regular, then the condition is also necessary; therefore we may particularly regard Goto's theorem as an assertion on surjective-Buchsbaum modules. In our paper we shall establish, over Cohen-

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Macaulay local rings, a structure theorem for maximal surjective-Buchsbaum modules which satisfy the finiteness of some homological invariants.

Firstly we shall prove the following theorem.

Theorem 3.1 *Let A be a Cohen-Macaulay local ring of dimension $d > 0$. Assume that A is not regular and that A has a dualizing complex. Then there exist exactly $d+1$ isomorphism classes of indecomposable maximal surjective-Buchsbaum A-modules of finite injective dimension, and any maximal surjective-Buchsbaum A-module of finite injective dimension is a direct sum of finite copies of them.*

Secondly we shall explore the relationship between the surjective-Buchsbaum modules of finite injective dimension and the surjective-Buchsbaum modules of finite projective dimension. Our results are summarized into the following.

Theorem 3.3 *Let A be a Cohen-Macaulay local ring possessing the canonical module K_A. Let M be a finitely generated A-module of finite projective dimension. Then $M \otimes_A K_A$ has finite injective dimension, and if $M \otimes_A K_A$ is a surjective-Buchsbaum A-module, so is M.*

Let us call M a *typical* surjective-Buchsbaum A-module, if it has finite projective dimension and if $M \otimes_A K_A$ is a surjective-Buchsbaum A-module. We shall give the following structure theorem, similar to Theorem 3.1, for typical maximal surjective-Buchsbaum modules over Cohen-Macaulay local rings.

Corollary 3.5 *Let A be a Cohen-Macaulay local ring of dimension $d > 0$. Assume that A is not regular and that A has a dualizing complex. Then there exist exactly $d+1$ isomorphism classes of indecomposable maximal typical surjective-Buchsbaum A-modules, and any maximal typical surjective-Buchsbaum A-module is a direct sum of finite copies of them.*

It should be noted here that there exist, over certain Cohen-Macaulay local rings, infinitely many non-isomorphic and non-typical indecomposable maximal surjective-Buchsbaum modules of finite projective dimension, by which we find the most essential assumption in Goto's theorem is the finiteness of injective dimension, *not* the finiteness of projective dimension.

Throughout this paper, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let k be the residue class field of A. We assume $d = \dim A > 0$. For each A-module M, $\ell_A(M)$ denotes the length of M.

Surjective-Buchsbaum modules

Firstly we shall give the definition and characterizations of surjective-Buchsbaum modules. Let M be an A-module and put

$$\operatorname{Soc} M = 0 \underset{M}{:} \mathfrak{m} \quad \text{and} \quad \Gamma_{\mathfrak{m}} E = \bigcup_{n=1}^{\infty} \left[0 \underset{M}{:} \mathfrak{m}^n \right] .$$

The derived functor of Soc(–) is $\mathrm{Ext}_A^i(k,-)$. The one of $\Gamma_{\mathfrak{m}}$ is denoted by $H_{\mathfrak{m}}^i(-)$; see [8]. If M is finite generated, then $\mathrm{Ext}_A^i(k,M)$ is finite-dimensional k-vector space. We put $\mu_A^i(M) = \ell_A(\mathrm{Ext}_A^i(k,M))$ and call it the i-th Bass number of M; see [2]. Similarly we put $\beta_i^A(M) = \ell_A(\mathrm{Tor}_i^A(k,M))$ and call it the i-th Betti number of M. For any finitely generated A-module M, the i-th local cohomology $H_{\mathfrak{m}}^i(M)$ of M with respect to $\mathfrak{m}$ is an Artinian module but not necessarily finitely generated. We say that M has finite local cohomologies, if $H_{\mathfrak{m}}^i(M)$ are finitely generated for all $i \neq \dim_A M$.

The inclusion map $\mathrm{Soc}\, M \hookrightarrow \Gamma_{\mathfrak{m}} M$ induces the natural map

$$\phi_M^i : \mathrm{Ext}_A^i(k,M) \to H_{\mathfrak{m}}^i(M)$$

for all $i \geq 0$.

Definition 2.1 *Let M be a finitely generated A-module. Then M is said to be a* **surjective-Buchsbaum** *A-module, if the natural map ϕ_M^i is surjective for all $i \neq \dim_A M$.*

Being surjective-Buchsbaum depends on the choice of base rings. In fact, let B be a homomorphic image of A and M a B-module. Then M is not necessary a surjective-Buchsbaum B-module, even if M is a surjective-Buchsbaum A-module; see, for example, [18, §2]. A surjective-Buchsbaum module is Buchsbaum [20] and has finite local cohomologies. Naturally, every Cohen-Macaulay module M is surjective-Buchsbaum, because $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq \dim_A M$.

For each A-module M of finite local cohomologies, we put

$$h_A^i(M) = \begin{cases} \ell_A(H_{\mathfrak{m}}^i(M)), & \text{if } i \neq s = \dim_A M; \\ \ell_A(\mathrm{Im}\, \phi_M^s), & \text{if } i = s. \end{cases}$$

The next result is basically due to [13] and [21]; inequalities (2.2.1) were given by Miyazaki [13, Corollary 1.14], when A is regular. The second part of Lemma 2.2 is called the *Bass number criterion* and due to Yamagishi [21, Theorem 1.2]. Because it plays a key role in this paper, we shall note here a brief proof for the sake of completeness.

Lemma 2.2 *Let M be a finitely generated A-module of finite local cohomologies and assume $s = \dim_A M > 0$. Then we have the inequalities*

$$\text{(2.2.1)} \qquad \mu_A^i(M) \leq \sum_{j=0}^{i} \beta_j^A(k) \cdot h_A^{i-j}(M) \quad \text{for all } i \leq s\,.$$

Furthermore the following statements are equivalent to each other:

(i) *M is a surjective-Buchsbaum A-module;*
(ii) *the equalities in (2.2.1) hold for all $i < s$.*

When this is the case, we also have the equality in (2.2.1) for $i = s$.

Proof . Take a minimal injective resolution $I^\bullet$ of M and a minimal free resolution $F_\bullet$ of k. Then the double complex $C^{\bullet\bullet} = \mathrm{Hom}_A(F_\bullet, \Gamma_\mathfrak{m} I^\bullet)$ gives rise to spectral sequences $('E_r^{pq}, 'd_r^{pq})$ and $(''E_r^{pq}, ''d_r^{pq})$:

$$'E_1^{pq} = \mathrm{Hom}_A(F_p, H_\mathfrak{m}^q(M)) \quad \text{and} \quad ''E_1^{pq} = \begin{cases} \mathrm{Soc}\, I^p, & \text{if } q = 0; \\ 0, & \text{otherwise.} \end{cases}$$

By the second spectral sequence, we have $H^n(C^{\bullet\bullet}) \cong \mathrm{Ext}_A^n(k, M)$, while the first one shows

$$\mu_A^i(M) = \sum_{j=0}^{i} \ell_A('E_\infty^{j,i-j}) \leq \sum_{j=0}^{i} \ell_A('E_1^{j,i-j}) \,. \tag{2.2.2}$$

Thus we get the inequalities (2.2.1) for all $i < s$. Because the composite map

$$H^s(C^{\bullet\bullet}) \twoheadrightarrow {'E_\infty^{0,s}} \hookrightarrow {'E_1^{0,s}}$$

coincides with the natural map ϕ_M^s, we also have the inequality (2.2.1) for $i = s$.

Suppose that M is a surjective-Buchsbaum A-module. Then for all $i < s$, any element of $H_\mathfrak{m}^i(M)$ is represented by an element of the k-vector space $\mathrm{Soc}\, I^i$. Hence since $F_\bullet$ is minimal, $'d_r^{pq}$ must be zero for all $q < s$. Thus $'E_1^{pq} = {'E_\infty^{pq}}$ for all p, q with $p+q \leq s$ and $q \neq s$, and so by (2.2.2) we have the equalities (2.2.1).

Conversely, assume the equalities in (2.2.1) for all $i < s$. Then we have $'E_1^{pq} = {'E_\infty^{pq}}$ for all $p + q < s$ by (2.2.2). Hence the composite of the maps

$$H^i(C^{\bullet\bullet}) \twoheadrightarrow {'E_\infty^{0,i}} = {'E_1^{0,i}}$$

is necessarily surjective for all $i < s$; thus so is the natural map ϕ_M^i. □

Let $X_\bullet$ be a complex of A-modules and we denote by $d_\bullet^X$ its differentiations. We say that a homomorphism $X_\bullet \to Y_\bullet$ of complexes is said to be a quasi-isomorphism, if it induces an isomorphism of homology. A quasi-isomorphism $X_\bullet \to Y_\bullet$ is said to be a free resolution of $Y_\bullet$ if $X_\bullet$ is a complex consisting of free modules. If $\mathrm{Im}\, d_i^X \subset \mathfrak{m}X_{i-1}$ for all i, then $X_\bullet$ is said to be minimal. A complex $X_\bullet$ bounded below whose homologies are finitely generated has a unique minimal free resolution up to isomorphisms; see [14].

Definition 2.3 ([9] or [14]). *A complex $D_A^\bullet$ of A-modules is said to be a dualizing complex of A, if it satisfies the following two conditions*:

(i) $D_A^i = \bigoplus_{\substack{\mathfrak{p}\in \mathrm{Spec}\, A \\ \dim A/\mathfrak{p} = -i}} E_A(A/\mathfrak{p})$ *for all integers i, where $E_A(-)$ denotes the injective envelope*;

(ii) $H^i(D_A^\bullet)$ *are finitely generated.*

A dualizing complex of A is uniquely determined up to isomorphisms if it exists. A finitely generated A-module K_A is said to be the canonical module of A if $K_A \otimes_A \hat{A}$ is isomorphic to $\mathrm{Hom}_A(H_\mathfrak{m}^d(A), E_A(A/\mathfrak{m}))$*, where $\hat{A}$ denotes the $\mathfrak{m}$-adic completion* *of A. The canonical module of A is uniquely determined up to isomorphisms if it exists.*

From now on, we assume that A *possesses a dualizing complex* $D_A^\bullet$. Let $D(-) = \mathrm{Hom}_A(-, D_A^\bullet)$. The next result is fundamental; see [8], [14], [16], or [19].

Proposition 2.4 *Let M be a finitely generated A-module. Then*

(i) *the canonical map $M \to DD(M)$ is a quasi-isomorphism;*
(ii) $H_\mathfrak{m}^i(M) \cong \mathrm{Hom}_A(H^{-i}(D(M)), E_A(A/\mathfrak{m}))$ *for all i.*

In particular, $K_A(d) \to D_A^\bullet$ is a quasi-isomorphism if A is Cohen-Macaulay.

Let M be a finitely generated A-module and $H_\bullet$ a minimal free resolution of $D(M)$. Then $D(H_\bullet)$ is an injective resolution of M, which gives rise to the commutative diagram

$$\begin{array}{ccc} \mathrm{Soc}\, D(H_\bullet) & \longrightarrow & \Gamma_\mathfrak{m} D(H_\bullet) \\ \| & & \| \\ \mathrm{Hom}_A(H_\bullet, \mathrm{Soc}\, E_A(A/\mathfrak{m})) & \longrightarrow & \mathrm{Hom}_A(H_\bullet, E_A(A/\mathfrak{m})) \\ \| & & \\ \mathrm{Hom}_A(H_\bullet \otimes_A k, E(A/\mathfrak{m})) & & \end{array}$$

Hence we have the natural map $H_i(H_\bullet) \to H_i(H_\bullet \otimes_A k)$ to be the Matlis dual of the natural map $\phi_M^i : \mathrm{Ext}_A^i(k, M) \to H_\mathfrak{m}^i(M)$. From this we immediately get the following.

Lemma 2.5 *Let M be a finitely generated A-module of dimension $s > 0$ and $H_\bullet$ a minimal free resolution of $D(M)$. Then the following statements are equivalent:*

(i) *M is a surjective-Buchsbaum A-module;*
(ii) *the natural map $H_i(H_\bullet) \to H_i(H_\bullet \otimes_A k)$ is injective for all $i < s$;*
(iii) $\mathrm{Ker}\, d_i^H \cap \mathfrak{m}H_i = \mathrm{Im}\, d_{i+1}^H$ *for all $i < s$.*

When M is a surjective-Buchsbaum A-module, one can get more information on $H_\bullet$. Let $F_\bullet$ be a minimal free resolution of k. For each i, let us define the subcomplex $F_\bullet^{(i)}$ of $F_\bullet$ as follows:

$$F_j^{(i)} = \begin{cases} F_j & \text{for all } j \le i; \\ 0 & \text{for all } j > i. \end{cases}$$

For each complex $X_\bullet$ and integer n, let $X_\bullet(n)$ denote the shifting of $X_\bullet$ in degree n. Then we have

Proposition 2.6 *Let M be a finitely generated A-module of dimension $s > 0$ and $H_\bullet$ a minimal free resolution of $D(M)$. Then the following statements are equivalent:*

(i) *M is a surjective-Buchsbaum A-module;*
(ii) *the subcomplex $H_\bullet^{(s)}$ of $H_\bullet$*

$$0 \to H_s \to H_{s-1} \to \cdots \to H_1 \to H_0 \to 0$$

is a direct sum of finite copies of $\{F_\bullet^{(s-i)}(-i)\}_{0\le i\le s}$.

Proof. (i) $\Rightarrow$ (ii). By induction on t, we shall prove the complex

$$H_\bullet^{(t)}: 0 \to H_t \to H_{t-1} \to \cdots \to H_1 \to H_0 \to 0$$

is isomorphic to

$$\bigoplus_{i=0}^{t} \left(F_\bullet^{(t-i)}(-i)\right)^{h_A^i(M)}$$

for all $t \le s$. We may assume that $s \ge t \ge 0$ and that our assertion is true for $t-1$; hence we have

$$\operatorname{Ker} d_{t-1}^H = \bigoplus_{i=0}^{t-1} \left(\operatorname{Ker} d_{t-1-i}^F\right)^{h_A^i(M)} .$$

Since M is a surjective-Buchsbaum A-module, we get $\operatorname{Ker} d_{i-1}^H \cap \mathfrak{m}H_{i-1} = \operatorname{Im} d_i^H$ for all $i \le s$. Therefore we have

$$\begin{aligned}\operatorname{Im} d_t^H &= \operatorname{Ker} d_{t-1}^H \cap \mathfrak{m}H_{t-1} \\ &= \left\{\bigoplus_{i=0}^{t-1} \left(\operatorname{Ker} d_{t-1-i}^F\right)^{h_A^i(M)}\right\} \cap \mathfrak{m}H_{t-1} \\ &= \bigoplus_{i=0}^{t-1} \left(\operatorname{Im} d_{t-i}^F\right)^{h_A^i(M)} .\end{aligned}$$

This decomposition of $\operatorname{Im} d_t^H$ causes a direct sum decomposition

$$H_t = \left\{\bigoplus_{i=0}^{t-1} \left(F_{t-i}\right)^{h_A^i(M)}\right\} \oplus A^\alpha \quad \text{and} \quad d_t^H = \left\{\bigoplus_{i=0}^{t-1} \left(d_{t-i}^F\right)^{h_A^i(M)}\right\} \oplus 0^\alpha,$$

with $\alpha = \operatorname{rank} H_t - \sum_{j=1}^{t} \operatorname{rank} F_j \cdot h_A^{t-j}(M)$. Since $\operatorname{rank} H_{t'} = \mu_A^t(M)$, we have $\alpha = h_A^t(M)$. Thus we have the required decomposition of $H_\bullet^{(t)}$, because $A = F_\bullet^{(0)}$.

(ii) $\Rightarrow$ (i). Since $\operatorname{Im} d_i^F = \operatorname{Ker} d_{i-1}^F \cap \mathfrak{m}F_{i-1}$ for all i, the complex $F_\bullet^{(s-i)}(-i)$ satisfies the condition (iii) of Lemma 2.5. Therefore so does $H_\bullet^{(s)}$, because $H_\bullet^{(s)}$ is a direct sum of $\{F_\bullet^{(s-i)}(-i)\}$. This completes the proof. $\square$

Typical surjective-Buchsbaum modules

In this section let A be a Cohen-Macaulay local ring of dimension $d > 0$ and assume that A possesses a dualizing complex $D_A^\bullet$. Firstly we shall prove the following.

Theorem 3.1 *Suppose that A is not a regular local ring. Then there exist $d+1$ indecomposable maximal surjective-Buchsbaum A-modules $L_0, L_1, \ldots, L_d$ of finite injective dimension with $h_A^j(L_i) = \delta_{ij}$ for all i, j. Furthermore any maximal surjective-Buchsbaum A-module of finite injective dimension is isomorphic to a unique direct sum of finite copies of A-modules $L_0, L_1, \ldots, L_d$.*

Proof. Let $F_\bullet$ be a minimal free resolution of the residue class field k. For each $0 \leq i \leq d$, we put $L_i = \operatorname{Coker}(d_{d-i}^F)^* \otimes_A K_A$ where $(-)^*$ denotes the A-dual. Since $\operatorname{Ext}_A^i(k, K_A) = 0$ for all $i < d$ [10, Satz 6.1], we have the following exact sequence

$$0 \to (F_0)^* \otimes_A K_A \to \cdots \to (F_{d-i-1})^* \otimes K_A \to (F_{d-i})^* \otimes_A K_A \to L_i \to 0.$$

Since K_A has finite injective dimension [10, Bemerkung 5.4], so does L_i.

We firstly prove that L_i is an indecomposable maximal surjective-Buchsbaum A-module. Let $r_i = \sum_{j=0}^{i-1}(-1)^{i-j-1} \operatorname{rank} F_j$.

Claim . Let $i > 0$. Then $r_i > 0$.

Choose $\mathfrak{p} \in \operatorname{Spec} A \setminus \{\mathfrak{m}\}$. Then because the exact sequence

$$(F_{i-1})_\mathfrak{p} \to \cdots \to (F_1)_\mathfrak{p} \to (F_0)_\mathfrak{p} \to 0$$

is split, we have $(\operatorname{Im} d_i^F)_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module of rank r_i. Hence $r_i \geq 0$. If $r_i = 0$, then $(\operatorname{Im} d_i^F)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ and so we have $\operatorname{Im} d_i^F$ to be of finite length. Since it is an A-submodule of the free A-module F_{i-1} and A is a Cohen-Macaulay local ring of positive dimension, we get $d_i^F = 0$. Therefore A has to be regular and $i = d+1$, which contradicts our standard assumption. And so we have $r_i > 0$. This completes the proof of the claim.

Let $0 \leq i \leq d$. For each minimal prime ideal $\mathfrak{p}$ of A, we have

$$\begin{aligned}\ell_{A_\mathfrak{p}}((L_i)_\mathfrak{p}) &= \sum_{j=0}^{d-i}(-1)^{d-i-j} \operatorname{rank} F_j \cdot \ell_{A_\mathfrak{p}}((K_A)_\mathfrak{p}) \\ &= r_{d-i+1} \cdot \ell_{A_\mathfrak{p}}(K_{A_\mathfrak{p}}),\end{aligned}$$

which is positive by the claim; hence L_i is maximal.

Let us consider $D(L_i)$. Then we have the following diagram

$$\begin{array}{ccccc} & & F_\bullet^{(d-i)}(-i) & \longrightarrow & \operatorname{Hom}_A((F_\bullet^{(d-i)})^*(i), D(D_A^\bullet)) \\ & & & & \downarrow \\ D(L_i) & \longrightarrow & D((F_\bullet^{(d-i)})^*(i-d) \otimes_A K_A) & \xrightarrow{\sim} & \operatorname{Hom}_A((F_\bullet^{(d-i)})^*(i-d), D(K_A)) \end{array}$$

of quasi-isomorphisms; see the remark after Proposition 2.4, from which we find $F_\bullet^{(d-i)}(-i)$ is a minimal free resolution of $D(L_i)$; see [14, Chapter 2, Lemma 2.5]. Thus L_i is a maximal surjective-Buchsbaum A-module by Proposition 2.6. We furthermore have $h_A^i(L_j) = \delta_{ij}$ for all $i \leq d$, see (2.2.1). If L_i *were* decomposable, the complex $F_\bullet^{(d-i)}$ is also decomposable, which contradicts the fact that $F_\bullet^{(d-i)}$ is a part of a minimal free resolution of the indecomposable module k. Thus L_i is indecomposable.

Finally, let M be an arbitrary maximal surjective-Buchsbaum A-module of finite injective dimension and $H_\bullet$ a minimal free resolution of $D(M)$. Then because $\mathrm{inj.dim}_A M = \operatorname{depth} A$, we have $H_i = 0$ for all $i > d$. Hence by Proposition 2.6, we get the decomposition $H_\bullet = \bigoplus_{i=0}^{d} \left(F_\bullet^{(d-i)}(-i)\right)^{h_A^i(M)}$, whence $M = H^0(D(H_\bullet)) = \bigoplus_{i=0}^{d}(L_i)^{h_A^i(M)}$. Let $M = \bigoplus_{i=0}^{d}(L_i)^{\alpha_i}$ be another decomposition of M. Then applying the local cohomology functor $H_{\mathfrak{m}}^i(-)$ to both sides, we get an isomorphism $H_{\mathfrak{m}}^i(M) \cong \bigoplus_{i=0}^{d} H_{\mathfrak{m}}^i(L_j)^{\alpha_i}$. Hence $h_A^i(M) = \alpha_i$ for each $i < d$, because $h_A^i(L_i) = \delta_{ij}$ for all $j \geq 0$. Since $\mu_A^d(M) = \sum_{j=0}^{d} \mu_A^d(L_j) h_A^j(M) = \sum_{j=0}^{d} \mu_A^d(L_j)\alpha_j$ and $h_A^i(M) = \alpha_i$ for each $i < d$, we have $h_A^d(M)\mu_A^d(L_d) = \alpha_d \mu_A^d(L_d)$. Because $L_d = K_A$ and $\mu_A^d(K_A) = 1$, we also get $h_A^d(M) = \alpha_d$. This completes the proof of Theorem 3.1. □

Although we excluded in Theorem 3.1 the case where A is regular, the above proof still works for that case and we may recover Goto's theorem [6]. In fact,when A is regular, we have $r_i = \binom{d-1}{i-1}$ for all i, so that the proof shows L_i is a maximal Buchsbaum A-module if $0 < i \leq d$, while $L_0 = k$. Clearly, they are the syzygies of k and we find, by the latter part of the proof, that any maximal Buchsbaum A-module is a direct sum of finite copies of them.

Theorem 3.1 involves the result of Sharp [17, Corollary 2.7], that is to say any maximal Cohen-Macaulay A-module of finite injective dimension is a direct sum of finite copies of K_A.

Here we state the acyclicity lemma due to Buchsbaum and Eisenbud.

Lemma 3.2 ([3]). *Let M be a finitely generated module over a Noetherian local ring R and*

$$F_\bullet : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$$

a complex of finitely generated free R-modules. Let $r_i = \sum_{j=i}^{n}(-1)^{j-i} \operatorname{rank} F_j$ and $I_{r_i}(d_i^F)$ denote the ideal of R generated by the r_i-th minors of d_i^F. Then the following statements are equivalent:

(i) *$F_\bullet \otimes_A M$ is acyclic*;
(ii) $\operatorname{depth}\left(I_{r_i}(d_i^F), M\right) \geq i$ *for all* $i \geq 1$.

Hence the complex $F_\bullet \otimes_A M$ is acyclic if and only if so is $F_\bullet$, when R is a Cohen-Macaulay local ring and M is a maximal Cohen-Macaulay R-module.

Let M be a finitely generated A-module of finite projective dimension and $H_\bullet$ a minimal free resolution of M. Then by Lemma 3.2 the complex

$$0 \to H_t \otimes_A K_A \to \cdots \to H_0 \otimes_A K_A \to M \otimes_A K_A \to 0$$

is exact. Hence the A-module $N = M \otimes_A K_A$ has finite injective dimension and $(H_\bullet)^*(-d)$ is a minimal free resolution of $D(N)$; see the proof of Theorem 3.1.

Conversely, let N be a finitely generated A-module of finite injective dimension $H_\bullet$ a minimal free resolution of $D(N)$ and $t = \operatorname{depth} N$. We note that $H_i = 0$ for all $i > d$. We put $M = \operatorname{Coker}(d_d^H)^*$ and $r_i = \sum_{j=t}^{i-1}(-1)^{j-i+1}\operatorname{rank} H_j$ for $t+1 \leq i \leq d$. Then we get $\dim A/I_{r_i}(d_i^H) \leq i-1$. In fact for all prime ideal $\mathfrak{p}$, we find that $(D_A^j)_\mathfrak{p} = 0$ for all $-j < \dim A/\mathfrak{p}$ by the definition. For all prime ideal $\mathfrak{p}$ such that $\dim A/\mathfrak{p} \geq i$, since $(H_\bullet)_\mathfrak{p} \to D(N)_\mathfrak{p}$ is also quasi-isomorphism, the sequence

$$(H_i)_\mathfrak{p} \to (H_{i-1})_\mathfrak{p} \to \cdots \to (H_t)_\mathfrak{p} \to 0$$

is exact and split, whence $I_{r_i}(d_i^H) \not\subset \mathfrak{p}$. Therefore we have the following sequence

$$0 \to (H_t)^* \xrightarrow{(d_{t+1}^H)^*} (H_{t+1})^* \to \cdots \to (H_{d-1})^* \xrightarrow{(d_d^H)^*} (H_d)^* \to M \to 0$$

to be exact. Hence M has finite projective dimension and $\operatorname{depth} M = t$; see [1, Theorem 3.7]. Because $H_\bullet = (H_\bullet)^{**}$ is a free resolution also for $D(M \otimes_A K_A)$, we have an isomorphism $M \otimes_A K_A \cong N$. Thus there is a depth-preserving one-to-one correspondence between the modules M of finite projective dimension and the modules N of finite injective dimension. Hence we get the first assertion in the next theorem so that, passing to the localizations at $\mathfrak{p}$ of $\operatorname{Supp} M \otimes_A K_A = \operatorname{Supp} M$, we have by [15, Satz 2.5] the second one of the next theorem.

Theorem 3.3 *Let M be a finitely generated A-module of finite projective dimension. Then*

(i) *M is a Cohen-Macaulay A-module if and only if so is $M \otimes_A K_A$.*
(ii) *M has finite local cohomologies if and only if so does $M \otimes_A K_A$.*
(iii) *If $M \otimes_A K_A$ is a surjective-Buchsbaum A-module, then so is M.*

Proof. We have only to show (iii). Suppose that $M \otimes_A K_A$ is a surjective-Buchsbaum A-module and let $G_\bullet$ be a minimal free resolution of K_A. Then M has finite local cohomologies by (ii) and we have the following quasi-isomorphisms

$$\begin{array}{ccccc} (H_\bullet)^*(-d) \otimes_A G_\bullet & \longrightarrow & (H_\bullet)^*(-d) \otimes_A K_A & \longrightarrow & \operatorname{Hom}_A(H_\bullet, K_A(d)) \\ & & & & \big\downarrow \\ & & D(M) & \longrightarrow & D(H_\bullet), \end{array}$$

so that we find $(H_\bullet)^*(-d) \otimes_A G_\bullet$ is a minimal free resolution of $D(M)$, see [13, Chapter 2, Lemma 2.5]. We shall use $(H_\bullet)^*(-d) \otimes_A G_\bullet$ to compute several invariants of M. First we have

$$\mu_A^i(M) = \operatorname{rank}[(H_\bullet)^*(-d) \otimes_A G_\bullet]_i = \sum_{j=0}^{i} \operatorname{rank} G_j \cdot \operatorname{rank} H_{d+j-i}\,. \tag{3.3.1}$$

The double complex $(H_\bullet)^*(-d) \otimes_A G_\bullet$ gives rise to the spectral sequence

$$E_{pq}^1 = H_{p-d}(H_\bullet)^* \otimes_A G_q \Longrightarrow H_{p+q}((H_\bullet)^*(-d) \otimes_A G_\bullet)\,.$$

Recall that $H^i_{\mathfrak{m}}(M \otimes_A K_A)$ and $H^i_{\mathfrak{m}}(M)$ has finite length for all $i < s$, and we find $\ell_A(H_{i-d}((H_\bullet)^*)) = h^i_A(M \otimes_A K_A)$ and $\ell_A(H_i((H_\bullet)^*(-d) \otimes_A G_\bullet)) = h^i_A(M)$ [12]. Therefore we get

$$h^i_A(M) \leq \sum_{j=0}^{i} \operatorname{rank} G_j \cdot h^{i-j}_A(M \otimes_A K_A) \quad \text{for all } i < s. \tag{3.3.2}$$

On the other hand, because $M \otimes_A K_A$ is a surjective-Buchsbaum A-module, we have by Lemma 2.2 the equalities
(3.3.3)

$$\mu^i_A(M \otimes_A K_A) = \operatorname{rank} H_{d-i} = \sum_{j=0}^{i} \operatorname{rank} F_j \cdot h^{i-j}_A(M \otimes_A K_A) \qquad \text{for all } i < s\,.$$

Hence by (3.3.1), (3.3.2), and (3.3.3) we find

$$\begin{aligned}
\mu^i_A(M) &= \sum_{j=0}^{i} \operatorname{rank} G_j \cdot \operatorname{rank} H_{d-i+j} \\
&= \sum_{\substack{0 \leq j,k \\ j+k \leq i}} \operatorname{rank} F_k \cdot \operatorname{rank} G_j \cdot h^{i-j-k}_A(M \otimes_A K_A) \\
&\geq \sum_{k=0}^{i} \operatorname{rank} F_k \cdot h^{i-k}_A(M) \qquad \text{for all } i < s.
\end{aligned}$$

Thus Lemma 2.2 shows M is a surjective-Buchsbaum A-module. □

Definition 3.4 *A finitely generated A-module M is said to be a* **typical** *surjective-Buchsbaum A-module if M has finite projective dimension and if $M \otimes_A K_A$ is a surjective-Buchsbaum A-module.*

The next result is a structure theorem for typical maximal surjective-Buchsbaum A-modules.

Corollary 3.5 *If A is not a regular local ring, then there exist exactly $d + 1$ non-isomorphic indecomposable typical maximal surjective-Buchsbaum A-modules. Furthermore any typical maximal surjective-Buchsbaum A-module is a unique direct sum of finite copies of them.*

Proof. Let $F_\bullet$ be a minimal free resolution of k and we put $L'_i = \operatorname{Coker}(d^F_{d-i})^*$, for each $0 \leq i \leq d$. Then because $\operatorname{Ext}^j_A(k, A) = 0$ for all $j < d$, the A-module L'_i has finite projective dimension. Therefore, since $L'_i \otimes_A K_A = L_i$ and since L_i is by Theorem 3.1 an indecomposable maximal surjective-Buchsbaum A-module of finite injective dimension, by Theorem 3.3 we get L'_i is an indecomposable typical surjective-Buchsbaum A-module. Let M be an arbitrary typical maximal surjective-Buchsbaum A-module and $H_\bullet$ a minimal free resolution of M. Then $(H_\bullet)^*(-d)$ is a minimal free resolution of $D(M \otimes_A K_A)$ and so, by Proposition 2.8, the complex $(H_\bullet)^*(-d)$ is a direct sum of finite copies of $\{F_\bullet^{(d-i)}(-i)\}_{0\le i\le d}$. Hence $H_\bullet$ is a direct sum of finite copies of $\{(F_\bullet^{(d-i)})^*(i-d)\}_{0\le i\le d}$. Therefore we have M to be a direct sum of finite copies of $L'_0, L'_1, \dots, L'_d$. Considering the local cohomology modules, we find, similarly as is in the proof of Theorem 3.1, that L'_i appears exactly $h^i_A(M \otimes_A K_A)$ times in the decomposition. Thus the uniqueness of the decomposition follows. $\square$

Non typical surjective-Buchsbaum modules.

The converse of (iii) of Theorem 3.3 is not true in general, that is there exists, over a certain Cohen-Macaulay local ring A, a non-typical surjective-Buchsbaum A-module of finite projective dimension. More explicitly, we have the following.

Proposition 4.1 *Let B be a Noetherian local ring with maximal ideal $\mathfrak{n}$ and $\mathfrak{a} \subset \mathfrak{n}^2$ an ideal of B. Assume that $A = B/\mathfrak{a}$ is not a Gorenstein ring but it is a Cohen-Macaulay ring of dimension $d \ge 2$. We furthermore assume that the field $B/\mathfrak{n}$ is infinite, $\beta_0^B(\mathfrak{a}) > 1$, and $\beta_1^A(K_A) = \beta_1^B(K_A)$. Then there exist infinitely many non-isomorphic and non-typical indecomposable maximal surjective-Buchsbaum A-modules of finite projective dimension.*

Proof . We put $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$ and $k = A/\mathfrak{m} = B/\mathfrak{n}$. Let $F_\bullet$ be a minimal free resolution of k as a B-module. First, we shall consider $M = \operatorname{Coker}(d_2^F \otimes_B A)^*$. Since $\operatorname{Ext}^i_B(k,A) = 0$ for all $i < d$, there is an exact sequence

$$0 \to (F_0 \otimes_B A)^* \to (F_1 \otimes_B A)^* \to (F_2 \otimes_B A)^* \to M \to 0$$

and M has finite projective dimension as an A-module. We can obtain that

$$\mu_A^i(M) = \begin{cases} \beta_1^A(K_A) + \beta_0^A(K_A)\cdot\beta_1^B(k) & i = d-1; \\ \beta_0^A(K_A) & i = d-2; \\ 0 & \text{for all } i < d-2, \end{cases}$$

and

$$h_A^i(M) = \begin{cases} \beta_1^B(K_A) & i = d-1; \\ \beta_0^B(K_A) & i = d-2; \\ 0 & \text{for all } i < d-2 \end{cases}$$

in the same way as the proof of Theorem 3.3. If $i > s = \dim_A M$, then $H^i_\mathfrak{m}(M) = 0$, and if $s > 0$, then $H^s_\mathfrak{m}(M)$ is not finitely generated [8]. Hence M is maximal. Moreover by the assumption, we have $\beta_1^A(K_A) = \beta_1^B(K_A)$, $\beta_0^A(K_A) = \beta_0^B(K_A)$ and $\beta_1^B(k) = \beta_1^A(k)$, therefore M is a surjective-Buchsbaum A-module. On the other hand, we also obtain that

$$\mu_A^i(M \otimes_A K_A) = \begin{cases} \beta_1^B(k) & i = d-1; \\ \beta_0^B(k) & i = d-2; \\ 0 & \text{for all } i < d-2, \end{cases}$$

and

$$h_A^i(M \otimes_A K_A) = \begin{cases} \beta_1^B(A) = \beta_0^B(\mathfrak{a}) & i = d-1; \\ \beta_0^B(A) = 1 & i = d-2; \\ 0 & \text{for all } i < d-2. \end{cases}$$

Hence $M \otimes_A K_A$ is not a surjective-Buchsbaum A-module by Lemma 2.2.

Next we shall construct more non-typical surjective-Buchsbaum modules of finite projective dimension. Let $K = \operatorname{Ker}(d_1^F \otimes_B A)$ and $L = \operatorname{Im}(d_2^F \otimes_B A)$. Then since $K/L = \operatorname{Tor}_1^B(k, A)$ is a k-vector space of dimension $\beta_0^B(\mathfrak{a}) \geq 2$, there exist infinitely many distinct submodules of K which contain L. For such a submodule L', we choose a free A-module $F_{L'}$ and an epimorphism $\phi_{L'}: F_{L'} \to L'$ such that $\operatorname{rank} F_{L'} = \beta_0^A(L')$. Then since the localization of the complex

$$\mathbb{F}_{L'}: 0 \to F_{L'} \xrightarrow{\phi_{L'}} F_1 \otimes_B A \to F_0 \otimes_B A \to 0$$

is exact and split at any $\mathfrak{p} \in \operatorname{Spec} A \setminus \{\mathfrak{m}\}$, the ideal $I_r(\phi_{L'})$ of A is $\mathfrak{m}$-primary for $r = \operatorname{rank} F_1 - \operatorname{rank} F_0$. We put $M_{L'} = \operatorname{Coker}(\phi_{L'})^*$. Then, by Lemma 3.2, we have the following exact sequence

$$0 \to (F_0 \otimes_B A)^* \to (F_1 \otimes_B A)^* \xrightarrow{(\phi_{L'})^*} (F_{L'})^* \to M_{L'} \to 0\,.$$

Hence $M_{L'}$ has finite projective dimension. We shall show that $M_{L'}$ is a maximal surjective-Buchsbaum A-module.

Since $L \subset L'$, futhermore $\mathbb{F}_{L'} \otimes_A K_A$ and $D(M_{L'})$ have the same homologies, we get

$$\operatorname{Im}(\phi_{L'} \otimes_A K_A) \supset \operatorname{Im}(d_2^F \otimes_A K_A)$$

and $h_A^{d-1}(M_{L'}) \leq h_A^{d-1}(M)$. Moreover we have

$$\begin{aligned} \beta_1^A(k) \cdot h_A^{d-2}(M_{L'}) + h_A^{d-1}(M_{L'}) &\geq \mu_A^{d-1}(M_{L'}) \\ &= \beta_1^B(k) \cdot \beta_0^A(K_A) + \beta_1^A(K_A) \\ &= \mu_A^{d-1}(M) \\ &= \beta_1^A(k) \cdot h_A^{d-2}(M) + h_A^{d-1}(M). \end{aligned}$$

Since $h_A^{d-2}(M) = h_A^{d-2}(M_{L'}) = 1$, we get $h_A^{d-1}(M) = h_A^{d-1}(M_{L'})$ and so $M_{L'}$ is a maximal surjective-Buchsbaum A-module by Lemma 2.2. Furthermore, since $\mathbb{F}_{L'}$ is a minimal free resolution of $D(M_{L'} \otimes_A K_A)$, $M_{L'}$ cannot be a typical surjective-Buchsbaum A-module unless $L' = K$. We shall show that $M_{L'}$ is indecomposable. Let $M_{L'} = M_1 \oplus M_2$ be a direct sum decomposition of $M_{L'}$. Since $\operatorname{proj.dim}_A M_{L'} = 2$, we may assume $\operatorname{proj.dim}_A M_1 = 2$. Then $\operatorname{proj.dim} M_2 \leq 1$, because $\beta_2^A(M_{L'}) = 1$. If $\operatorname{proj.dim}_A M_1 = 0$, the homomorphism $\phi_{L'}$ must be a non-trivial direct sum of some map with a zero map, which contradicts the fact that $\beta_0^A(L') = \operatorname{rank} F_{L'}$. Similarly we can prove that $\operatorname{proj.dim}_A M_2 \neq 1$. Therefore M_2 is a zero module and hence we conclude $M_{L'}$ is indecomposable.

Finally we shall check that $M_{L_1} \not\cong M_{L_2}$ if $L_1 \neq L_2$. Suppose that $M_{L_1} \cong M_{L_2}$. Then there are isomorphisms ψ_0, ψ_1, ψ_2 that make the following diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & (F_0 \otimes_B A)^* & \longrightarrow & (F_1 \otimes_B A)^* & \xrightarrow{(\phi_{L_1})^*} & (F_{L_1})^* & \longrightarrow & M_{L_1} & \longrightarrow 0 \\
& \downarrow \psi_2 & & \downarrow \psi_1 & & \downarrow \psi_0 & & \downarrow \wr & \\
0 \longrightarrow & (F_0 \otimes_B A)^* & \longrightarrow & (F_1 \otimes_B A)^* & \xrightarrow{(\phi_{L_2})^*} & (F_{L_2})^* & \longrightarrow & M_{L_2} & \longrightarrow 0
\end{array}$$

is commutative. Without loss of generality we may assume that ψ_2 is the identity map. Then taking the A-duals $(-)^*$, we have that $(\psi_1)^*(L_2) = L_1$ and $(\psi_1)^*(e) - e \in K$ for any $e \in F_1 \otimes_B A$. We would like to show that $(\psi_1)^*(L_2) = L_2$. Let $X_\bullet$ be the Koszul complex generated over A by a minimal base of $\mathfrak{m}$. We may identify $F_1 \otimes_B A$ and $F_0 \otimes_B A$ with X_1 and X_0, respectively. Hence we may lift the homomorphism $(\psi_1)^*$ to an automorphism of X_2. Then, letting $C = \operatorname{Im}(d_2^X \otimes_B A)$, we have that $(\psi_1)^*(C) = C$. Let $e \in \mathfrak{m}(F_1 \otimes_B A)$, that is, $e = \sum a_i e_i$ for some $a_i \in \mathfrak{m}$ and $e_i \in F_1 \otimes_B A$. Then because K/C is a k-vector space,

$$(\psi_1)^*(e) - e = \sum a_i \{(\psi_1)^*(e_i) - e_i\} \in C,$$

that is, $(\psi_1)^*(e) \equiv e$ modulo C. Thus we get $(\psi_1)^*(L_2) = L_2$, because $L \supset C$. Therefore we have $L_1 = L_2$ as required. This completes the proof of Proposition 4.1. □

Corollary 4.2 *Let A be a Cohen-Macaulay complete local ring with infinite residue class field. Suppose that A has dimension $d \geq 2$ and embedding codimension 2. If A is not a Gorenstein ring, then there exist infinitely many non-isomorphic and non-typical indecomposable maximal surjective-Buchsbaum A-modules of finite projective dimension.*

Proof. We choose a regular local ring B with maximal ideal $\mathfrak{n}$ and an ideal $\mathfrak{a} \subset \mathfrak{n}^2$ such that $A = B/\mathfrak{a}$. Then $\mathfrak{a}$ is a perfect ideal of height 2. By the theorem of Hilbert and Burch (see [4] or [11, p. 148]), we have a minimal free resolution

$$0 \to B^n \xrightarrow{f} B^{n+1} \xrightarrow{g} B \to B/\mathfrak{a} \to 0$$

of $B/\mathfrak{a}$ over B, where $n > 1$ is the Cohen-Macaulay type of A, $f = (f_{ij})$ is an $(n+1) \times n$ matrix and $g = (g_j)$ is a $1 \times (n+1)$ matrix such that g_j is the determinant of the matrix obtained from f by omitting the j-th row. Notice that $\beta_0^B(\mathfrak{a}) = n + 1 \geq 2$ and we have $\beta_1^B(K_A) = n + 1$, since $K_A = \operatorname{Coker} f^*$, where $(-)^*$ denotes B-dual. We want to know $\beta_1^A(K_A)$. A direct computation shows that $\mathfrak{a}B^n \subset \mathfrak{n} \operatorname{Im} f^*$; for example,

$$\begin{pmatrix} g_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = f^* \begin{pmatrix} 0 \\ d_2 \\ \vdots \\ d_{n+1} \end{pmatrix}$$

where for all $j > 1$, d_j is the determinant of the matrix obtained from f omitting the first and j-th rows and the first column. Hence we have $\beta_1^A(K_A) = n + 1$, because $\beta_1^A(K_A) = \beta_0^A(\mathrm{Im}(f^* \otimes_B A))$. Thus applying Proposition 4.1, we get the assertion. $\square$

By the well-known theorem of Auslander and Buchsbaum [1, Theorem 3.7] we find that a maximal Cohen-Macaulay module of finite projective dimension must be free. Proposition 4.1 contrasts strikingly with this assertion. Here we would like to cite the following theorem due to Goto and Nishida.

Theorem 4.3 ([7]). *Let $R = k[\![X_1, \ldots, X_n]\!]$ be a formal power series ring over an algebraically closed field k of $\mathrm{ch}\, k \neq 2$. Let $A = R/I$, where I is an ideal of R and suppose that A is a Cohen-Macaulay ring of $\dim A = d \geq 2$. Then the following statements are equivalent:*

(i) *A is a regular local ring;*
(ii) *there exist only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules.*

These results lead us to the following conjecture:

Conjecture 4.4 Let A be a Cohen-Macaulay complete local ring of dimension $d \geq 2$ with infinite residue class field. Then the following statements are equivalent:

(i) A is a Gorenstein local ring;
(ii) there exist only finitely many isomorphism classes of indecomposable maximal surjective-Buchsbaum A-modules of finite projective dimension.

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