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Artin–Lang property for analytic manifolds of dimension two

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1 Introduction

In this paper we study the so called Artin–Lang property over a two dimensional paracompact real connected analytic manifold M. To be more precise, let $\mathcal{O}(M)$ be the ring of analytic functions on M and let K be its quotient field. Then K is a formally real field and we have

Theorem 1.1 (Artin–Lang property) *Let $f_1, \ldots, f_r \in \mathcal{O}(M)$. Then there is an ordering β in K in which they are simultaneously positive if and only if there exists a point $x \in M$ such that $f_1(x) > 0, \ldots, f_r(x) > 0$.*

This property was first proven by Artin for the case $K = \mathbb{R}[X_1, \ldots, X_n]$, [Ar], and it was the main step towards his solution of 17th Hilbert problem. Later on it was extended by Lang, [Lg], to finitely generated extensions of a real closed field k. Since then it has been in the base of Real Algebraic Geometry whose specialists have given the name of Artin–Lang property to these kind of statements. In particular, besides the already mentioned cases, the property has been also studied in local analytic geometry, [Rs], [Fe-Re-Rz], for compact real analytic manifolds, [Rz], [Jw1], for one dimensional paracompact real analytic manifolds, [An-Be], and very recently for the noncompact analytic manifolds, in the particular situation that the functions $f_1, \ldots, f_r$ have compact support, cf. [Jw2], [Ca].

Our proof is based on the attachment to each ordering $\beta \in \operatorname{Spec}_r(K)$ of an ultrafilter $\mathcal{U}_\beta$ of closed global semianalytic subsets of M, which allows to give some geometric criteria for a function f to be positive in β. This method has

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already been used in [An-Be], [Jw2] and [Ca]. We want to emphasize that as it has already been pointed out, Artin–Lang property had been the key step in the proof of 17th Hilbert problem in all the contexts mentioned above. However, for two dimensional analytic manifolds, 17th Hilbert problem was known, cf. [Bo-Rs], [Jw3], while no author seems to have paid attention to Artin–Lang property, inverting therefore the order between these two results. Here we do not use the solution of Hilbert's problem for our proof, but rather the local analytic case together with Cartan's theorem B to globalize the local data. Thus, although in the proof we use specific facts of the two dimensional hypothesis, as the decomposition described in Proposition 3.3, we have the hope of proving Artin–Lang property for higher dimensions, and from that 17th Hilbert's problem. We would like to thank Piotr Jaworski for the fruitful conversations about the problem. Some of the ideas of the proof of Theorem 1.1 are due to him.

2 Ultrafilter attached to an ordering β

We start by recalling the following

Definition 2.1 *A subset $S \subset M$ is called global semianalytic if it admits a representation of the form*

$$S = \bigcup_{i=1}^{s} \{x \in M \mid f_i(x) = 0, g_{i1}(x) > 0, \ldots, g_{ij_i} > 0\}$$

where $f_i, g_{i1}, \ldots, g_{ij_i} \in \mathcal{O}(M)$ for each $i = 1, \ldots, s$.

We denote by $\mathcal{S}$ the family of *closed* global semianalytic subsets of M. It follows from the definition that $\mathcal{S}$ is closed under finite unions and intersections, so that it makes sense to consider filters of sets of this family in the sense of [An-Be, Section 1].

Let $\mathcal{O}_b(M)$ denote the ring of bounded analytic functions on M. Let β be an ordering of K. This β can be seen either as a binary relation or as a cone of positive elements. We will use indistinctly both objects. We will attach to β an ultrafilter of elements of $\mathcal{S}$ in the following way. Let W_β be the convex hull of $\mathbb{R}$ in K, that is,

$$W_\beta = \{f \in K \mid -r <_\beta f <_\beta r \text{ for some } r \in \mathbb{R}\}\ .$$

Then W_β is a valuation ring with residue field $\mathbb{R}$ and $\mathcal{O}_b(M) \subset W_\beta$. Let $\mathfrak{m}_\beta$ be the center of W_β in $\mathcal{O}_b(M)$, that is the intersection $\mathfrak{n}_\beta \cap \mathcal{O}_b(M)$ where $\mathfrak{n}_\beta$ is the maximal ideal of W_β. Since the residue field of W_β is $\mathbb{R}$, we get that $\mathfrak{m}_\beta$ is a maximal ideal. A straightforward computation shows that

$$\mathcal{U}_\beta = \{X \in S \mid X \cap f^{-1}([-\delta, \delta]) \neq \emptyset \text{ for all } \delta > 0 \text{ and all } f \in \mathfrak{m}_\beta\}$$

is an ultrafilter among the filters of sets of the family $\mathcal{S}$, which we will call the *ultrafilter associated to the order β.*

Let λ_β: $W_\beta \to \mathbb{R}$ be the place defined by W_β. Since W_β is convex with respect to β it follows that λ_β is compatible with it, that is, for $f \in W_\beta$ with

$f \geqq_\beta 0$ we have $\lambda_\beta(f) \geqq 0$. Using this, we will be able to interpret geometrically some facts about β in terms of λ_β and $\mathscr{U}_\beta$.

Recall that given a filter $\mathscr{F}$ of elements of $\mathscr{S}$ we say that $a \in M$ is a limit point of $\mathscr{F}$ if $a \in \bar{X}$ for all $X \in \mathscr{F}$. It follows immediately that if $\mathscr{F}$ is an ultrafilter then it has at most one limit point, and $\mathscr{F}$ is the principal ultrafilter defined by a. Also, if f: $M \to \mathbb{R}$ is an analytic map, we define the image filter of $\mathscr{F}$ by f as

$$f(\mathscr{F}) = \{Y \subset \mathbb{R} \mid f(X) \subset Y \text{ for some } X \in \mathscr{F}\}\,.$$

Note that if $f \in \mathcal{O}_b(M)$ then $f(X)$ is compact for every $X \in \mathscr{F}$ and therefore $f(\mathscr{F})$ has some limit point. If furthermore $\mathscr{F}$ is an ultrafilter then $f(\mathscr{F})$ has a unique limit point. We have:

Proposition 2.2 *For every $f \in \mathcal{O}_b(M)$, $\lambda_\beta(f)$ is the limit point of the image filter of $\mathscr{U}_\beta$ by f.*

Proof. Let $g \in \mathfrak{m}_\beta$ such that $f = \lambda_\beta(f) + g$. Then it is enough to prove that the limit point of $g(\mathscr{U}_\beta)$ is zero. Let Y be an element of $g(\mathscr{U}_\beta)$ and let δ be a positive real number. Since $g \in \mathfrak{m}_\beta$, obviously $g^{-1}([-\delta, \delta]) \in \mathscr{U}_\beta$ and on the other hand there is $X \in \mathscr{U}_\beta$ such that $g(X) \subset Y$. Then $X \cap g^{-1}([-\delta, \delta]) \neq \emptyset$ and therefore $Y \cap [-\delta, \delta] \neq \emptyset$. Thus $0 \in \bar{Y}$ as claimed.

Corollary 2.3 *A function $f \in \mathcal{O}_b(M)$ is a unit if and only if it is bounded from zero over some element $X \in \mathscr{U}_\beta$, that is, there is some $\delta > 0$ such that $|f(x)| > \delta$ for all $x \in X$.*

Another immediate consequence of this result is the following geometric characterization of the positive units:

Corollary 2.4 *Let $u \in \mathcal{O}_b(M)$, $u \notin \mathfrak{m}_\beta$. Then $u >_\beta 0$ if and only if $u|_X > 0$ for some $X \in \mathscr{U}_\beta$.*

Proof. Since u is a unit we have $\lambda_\beta(u) \neq 0$ and therefore $u >_\beta 0$ if and only if $\lambda_\beta(u) > 0$. Thus we have to show that $\lambda_\beta(u) > 0$ if and only if $u|_X > 0$ for some $X \in \mathscr{U}_\beta$.

Assume that $\lambda_\beta(u) > 0$. Then $X_\varepsilon = u^{-1}([\lambda_\beta(u) - \varepsilon, \lambda_\beta(u) + \varepsilon])$ belongs to $\mathscr{U}_\beta$ for all $\varepsilon > 0$, in particular for $\varepsilon = \lambda_\beta(u)/2$ we have $u|_{X_\varepsilon} > 0$. Conversely, if there is $Y \in \mathscr{U}_\beta$ such that $u|_Y > 0$, since $\lambda_\beta(u) \neq 0$, it is clear that $\lambda_\beta(u) > 0$. □

In case u is not a unit, the above corollary can be partially extended to the following result which will be of utmost importance in the sequel:

Lemma 2.5 *Let $f \in \mathcal{O}_b(M)$ and assume that $f|_X > 0$ for some $X \in \mathscr{U}_\beta$. Then $f \in \beta$.*

Proof. Let v^*: $M \to \mathbb{R}$ be a continuous function such that $v^*(x) > 0$ for all $x \in M$, and $v^*|X = f|_X$, which exists by Tietze's Theorem. Now let v: $M \to \mathbb{R}$ be an analytic approximation of v^* such that $|v(x) - v^*(x)| < \frac{1}{4}\, v^*(x)$. In particular we have that $v(x) > 0$ for all $x \in M$. We set $u_1 = f/v$ and $u = u_1/(1 + u_1^2) \in \mathcal{O}_b(M)$. Since for $x \in Y$ we have that $u_1(x) > 4f(x)/5v^*(x) = 4/5$

and $u_1(x) < 4f(x)/3v^*(x) = 4/3$, we get that u is bounded from below on Y. Hence u is a unit and, by Corollary 2.4, we have $u \in \beta$ as an element of $\mathcal{O}(M)$. Now, $u(1 + u_1^2) = f/v$, and since $1 + u_1^2$ is a sum of squares then it is positive in β. Altogether we get $f = vu(1 + u_1^2) \in \beta$. □

An immediate consequence of this lemma is the following interesting result:

Lemma 2.6 *Let $f_1, \ldots, f_r \in \beta$. Then the set $\{x \in M | f_1(x) \geqq 0, \ldots, f_r(x) \geqq 0\}$ belongs to $\mathcal{U}_\beta$ and in particular is not empty.*

Proof. Let f be one of the functions $f_1, \ldots, f_r$. Assume that $Z(f) \notin \mathcal{U}_\beta$, where $Z(f)$ is the set of zeros of f. Then, there exists $Y \in \mathcal{U}_\beta$ such that $Z(f) \cap Y = \emptyset$ and we can divide up Y into two closed global semianalytic sets Y_1 and Y_2 defined by

$$Y_1 = \{x \in Y | f(x) > 0\} \quad \text{and} \quad Y_2 = \{x \in Y | f(x) < 0\} .$$

Thus, since $\mathcal{U}_\beta$ is an ultrafilter we have that either $Y_1 \in \mathcal{U}_\beta$ or $Y_2 \in \mathcal{U}_\beta$. By the above Lemma we get that $Y_1 \in \mathcal{U}_\beta$. In particular, we have shown that for each $i = 1, \ldots, r$ there exists an element $Y \in \mathcal{U}_\beta$ such that $f_i|_Y \geqq 0$. Taking the intersection of all of them the result follows at once. □

3 Square free function associated to a given function

Before entering in the proof of Theorem 1, we need some technical results. In particular we will show how to decompose an analytic function f on M as a product of a new analytic function $\tilde{f}$ which is locally square free at all points, and a sum of squares in $\mathcal{O}(M)$.

Let f be an analytic function on M with zero set $Z(f)$. We recall that $Z(f)$ has a decomposition into a locally finite family of global analytic irreducible components. We denote by f_x the analytic germ of f at x and by $(Z(f))_x$ is the germ of $Z(f)$ at x. Finally we denote by $\mathcal{J}(Z(f))_x$ the ideal of $(Z(f))_x$ in $\mathcal{O}_x(M)$.

Definition 3.1 *We say that f* changes sign *at x if the two semianalytic germ sets at x $\{f > 0\}_x$ and $\{f < 0\}_x$ are not empty, or equivalently if there is a neighbourhood basis $\{V_i\}$ of x such that for all i both germ sets $\{y \in V_i | f(y) > 0\}$ and $\{y \in V_i | f(y) < 0\}$ are not empty. We will denote by Z' the set of points at which f changes sign, and by Z'' its complement in $Z(f)$.*

It is obvious from the definition that the Z' is closed. The next result shows that actually it is analytic.

Proposition 3.2 *Let Z' (resp. Z'') be as above. Then Z' (resp. $\overline{Z''}$) is a union of irreducible analytic components of $Z(f)$. In particular it is a global analytic subset of $Z(f)$.*

Proof. Let $Y \subset Z(f)$ be an irreducible analytic component of $Z(f)$. If $\dim Y = 0$ it is obvious that $Y \subset Z''$. Suppose now that $\dim Y = 1$. Since Y is irreducible it is of pure dimension and connected. Let $D_1 = Y \cap (\bigcup Y')$ where the union runs over the rest of irreducible components of X. D_1 is an analytic subset of Y, and therefore it is discrete. Assume first that $(Y \backslash D_1) \cap Z' \neq \emptyset$, and take $a \in (Y \backslash D_1) \cap Z'$. Since $\mathcal{O}_a(M)$ is a unique factorization domain, the ideal $\mathcal{J}((Y)_a)$, is principal, say $\mathcal{J}((Y)_a) = (h_a)\mathcal{O}_a(M)$. Moreover, by Cartan's Theorem A, there is a global analytic function $g \in \mathcal{O}(M)$ vanishing on Y, such that $\mathcal{J}((Y)_a) = (g_a)\,\mathcal{O}_a(M)$. In particular, it follows that $\mathcal{J}((Y)_b) = (g_b)\,\mathcal{O}_b(M)$ for all $b \in (Y \backslash D_2)$, where $D_2 \subset Y$ is a discrete subset.

Obviously $f_a \in (g_a)$, and since f changes sign at a, we have $f_a = g_a^{\alpha_a} u_a$, for some unit $u_a \in \mathcal{O}_a(M)$ and an odd positive integer α_a. Let $\mathcal{F}$ be the analytic coherent sheaf of modules defined by

$$\mathcal{F} = (g^{\alpha_a}\mathcal{O}_M + f\mathcal{O}_M)/f\mathcal{O}_M\,.$$

Thus the support of $\mathcal{F}$ is a closed analytic subset and we have

$$\operatorname{Supp}\mathcal{F} = \{x \in Z(f) \mid f_x\mathcal{O}_x(M) \not\supset g_x^{\alpha_a}\,\mathcal{O}_x(M)\}\,.$$

Therefore the intersection $D_3 = \operatorname{Supp}\mathcal{F} \cap Y$ is an analytic subset of Y, which is proper because $a \notin \operatorname{Supp}\mathcal{F}$. Since Y is irreducible, it follows that D_3 is discrete. Thus, setting $D = D_1 \cup D_3$, we have that f changes sign at all points of $Y \backslash D$, that is, at all Y except for a discrete subset. Since Z' is closed it follows that $Y \subset Z'$.

Assume now that $(Y \backslash D_1) \cap Z'' \neq \emptyset$. Then arguing as above we get that $Y \backslash Z''$ is contained in a discrete set, or in other words that $Y \cap Z'$ is a discrete subset of Y, and that $Y \subset \overline{Z''}$. Thus it follows at once that any irreducible component Y of $Z(f)$ is contained either in Z' or in $\overline{Z''}$ as claimed. □

For the next result, following [Bo-Rs] we say that a germ $\xi_x \in \mathcal{O}_x(M)$ is elliptic if its zero set is an isolated point, namely x.

Proposition 3.3 *Let f be an analytic function on M. Then there is a unique (up to units) analytic function $\tilde{f}$ such that:*

a) $Z(\tilde{f}) = \{x \in M \mid f(x) = 0$ *and f changes sign at* $x\}$;

b) *for all $x \in Z(\tilde{f})$, $\mathcal{J}(Z(\tilde{f})_x) = \tilde{f}_x\mathcal{O}_x(M)$; in particular, $\tilde{f}$ changes sign at all points of $Z(\tilde{f})$;*

c) $f = \tilde{f}\sum_{i=1}^{z} A_i^2$, *with* $A_i \in \mathcal{O}(M)$.

Proof. Let x be a point in M. Since $\mathcal{O}_x(M)$ is a unique factorization domain, f_x has a local factorization

$$f_x = g_x^2 h_x \xi_x$$

where ξ_x is an elliptic germ, g_x is a product

$$g_x = g_1^{\alpha_1} \cdots g_s^{\alpha_s}$$

of irreducible germs with 1-dimensional zero set and h_x is a product

$$h_x = h_1 \cdots h_t$$

of distinct irreducible germs with zero set of dimension 1. Obviously we have $\mathscr{J}(Z') = h_x\mathcal{O}_x(M)$, so that we have to show that there is a global analytic function $\tilde{f}$ whose germ at each $x \in M$ coincides with h_x up to some unit.

Let D be the subset of points $x \in M$ such that f_x contains an elliptic germ ξ_x in its factorization at x. Since $\dim M = 2$ it follows that there is a neighbourhood U of x such that for every $y \in U$ the germs $(g_i)_y$ and $(h_j)_y$ are irreducible, and therefore it follows at once that D is a discrete subset of M. Thus, by [Bo-Rs, Lemma 7], [Jw3, Corollary 2i)], there is a global analytic function δ on M, which is a sum of (exactly two since M is noncompact, although this is irrelevant for us) squares and such that for each $x \in M$, $\delta_x = \xi_x v_x$ where v_x is a unit. Thus the function $f' = f/\delta \in \mathcal{O}(M)$ and it has not elliptic factors at any point of M.

Now let $\mathscr{F}$ be a coherent sheaf of ideals defined by

$$\mathscr{F}_y = g_y\mathcal{O}_y\,,$$

for $y \in M$. Since for each $y \in M$, the stalk $\mathscr{F}_y$ is generated by one element, the $\mathcal{O}(M)$-module $\Gamma(\mathscr{F}, M)$ of global sections of $\mathscr{F}$ is generated by finitely many sections, say $l_1, \ldots, l_s$, [Co] (in fact we may take $s = 3$). In particular, for each $y \in M$, we have $(l_{1,y}, \ldots, l_{s,y})\mathcal{O}_y = g_y\mathcal{O}_y$, and a straightforward computation shows that $(l_{i,y})\mathcal{O}_y = (g_y)\mathcal{O}_y$ for some $i \in 1, \ldots, s$. Then, setting $l = \sum_{i=1}^{s} l_i^2 \in \mathcal{O}(M)$ we get

$$l_y = g_y^2 v_y$$

where v_y is an unit in $\mathcal{O}_y$.

Putting all together, we define $\tilde{f} = f/\delta l \in \mathcal{O}(M)$. By construction, it follows that $\tilde{f} = h_x w_x$ at each point $x \in M$, where w_x is a unit, and the proof is complete. □

An immediate consequence of the proposition above is the following

Corollary 3.4 *Let $f_1, \ldots, f_r \in \mathcal{O}(M)$ and let $\tilde{f}_1, \ldots, \tilde{f}_r$ be the square free functions associated to them. Let β be a total ordering of the quotient field K of $\mathcal{O}(M)$. Then*

a) *$f_i \in \beta$ if and only if $\tilde{f}_i \in \beta$;*
b) *For any open set $U \subset M$, $U \cap \{x \in M \mid f_i(x) > 0, i = 1, \ldots, r\} = \emptyset$ if and only if $Y \cap \{x \in M \mid \tilde{f}_i(x) > 0, i = 1, \ldots, r\} = \emptyset$.*

More interesting is the following

Proposition 3.5 *Let $f_1, \ldots, f_r \in \mathcal{O}(M)$, $f_i \neq 0$, $i = 1, \ldots, r$, and set $S = \{x \in M \mid f_1(x) > 0, \ldots, f_r(x) > 0\}$. Let $g_{ij} = f_i f_j$, for $1 \leqq i < j \leqq r$ and set $T = \{x \in M \mid \tilde{f}_1(x) \geqq 0, \ldots, \tilde{f}_r(x) \geqq 0, \tilde{g}_{12}(x) \geqq 0, \ldots, \tilde{g}_{r-1,r}(x) \geqq 0\}$. Then $T \backslash \bar{S}$ is a discrete subset of M.*

Proof. Obviously it is enough to show that if $U \subset M$ is an open subset with $S \cap U = \emptyset$, then $T \cap U$ is discrete. Since we are assuming that $S \cap U = \emptyset$, by Corollary 3.4 b), it follows that $T \cap U$ is contained in the union Y of the zero sets of the functions $\tilde{f}_1, \ldots, \tilde{f}_r$. Thus it is enough to show that $\dim Y_x = 0$ for all $x \in U$.

Suppose that $\dim Y_y = 1$ for some $y \in U$. Moving to another point y, we may assume then that Y_y is irreducible. Thus Y_y is contained in a single analytic germ say $Z(\tilde{f}_1)_y$. Now by the proposition above, $\tilde{f}_1$ changes sign at all the points of Y in a small neighbourhood of x, which we still denote by U. Then, since $S \cap U = \emptyset$ and $Y_y \subset T$, it follows that there is another function, say $\tilde{f}_2$, which vanishes on Y_y, that is

$$Y_y \subseteq Z(f_1)_y \cap Z(f_2)_y .$$

Since $(\tilde{g}_{12})_y$ is square free and $\dim Y_y = 1$, then $Y_y \nsubseteq Z(\tilde{g}_{12})$. Pick any point $y' \in Y \cap U$ such that $\dim Y_y' = 1$ and with $\tilde{g}_{12}(y') \neq 0$. Then $\tilde{g}_{12}(y') > 0$ and therefore $\tilde{f}_1 \tilde{f}_2(y') > 0$, that is $\tilde{f}_1$ and $\tilde{f}_2$ have the same sign distribution on a neighbourhood of y'. In particular $\{\tilde{f}_1 > 0, \tilde{f}_2 > 0\}_{y'} \neq \emptyset$. Taking into account again that $S \cap U = \emptyset$ we obtain that there is a third function, say $\tilde{f}_3$, such that

$$Y'_y \subseteq Z(f_1)'_y \cap Z(f_2)'_y \cap Z(f_3)'_y .$$

Arguing as above with the function g_{13} we get that $\{\tilde{f}_1 > 0, \tilde{f}_2 > 0, \tilde{f}_3 > 0\} \neq \emptyset$. Thus in a finite number of steps we get a point $z \in U$ such that $Y_z \subseteq Z(f_i)_z$ and $g_{ij}(z) > 0$ for all i, j and with $\dim Y_z = 1$. But this implies that we have $U \cap \{x \in M \mid \tilde{f}_1(x) > 0, \dots, \tilde{f}_r(x) > 0\} \neq \emptyset$, contradiction. In conclusion, $Y \cap U$ is discrete, and the proof is complete. □

Corollary 3.6 *Let S be a global semianalytic subset of M. Then the closure $\bar{S}$ is a global semianalytic set.*

Proof. Let

$$S = \bigcup_{i=1}^{s} \{x \in M \mid f_i(x) = 0, g_{i1}(x) > 0, \dots, g_{ij_i} > 0\}$$

for some $f_i, g_{i1}, \dots, g_{ij_i} \in \mathcal{O}(M)$. Obviously it is enough to see that the closure of each basic piece $S_i = \{x \in M \mid f_i(x) = 0, g_{i1}(x) > 0, \dots, g_{ij_i} > 0\}$ is a global semianalytic set. If f_i is not the zero the function, then S_i is a semianalytic subset of the curve $\{f_i = 0\}$ and the result is well known. On the other hand, if $S_i = \{x \in M \mid g_{i1}(x) > 0, \dots, g_{ij_i} > 0\}$ then by the proposition above $\overline{S_i} = \{x \in M \mid \tilde{g}_{i1}(x) \geqq 0, \dots, \tilde{g}_{ij_i} \geqq 0, h(x) \neq 0\}$ where h is a global analytic function vanishing in a discrete set. This shows that S_i is global semianalytic. □

4 Proof of the theorem

In this section we will prove Artin–Lang property for orderings of the field K of meromorphic functions of M. We may assume from the very beginning that M is immersed as a submanifold of some $\mathbb{R}^n$. As already announced in the introduction, the essential ingredients that we will use are Cartan's Theorem B and the Positivstellensatz on the local rings $\mathcal{O}_p(M)$ of germs of analytic functions at points $p \in M$. For the sake of completeness we recall this last result. Recall that we have $\mathcal{O}_p(M) \approx \mathbb{R}\{X_1, X_2\}$, the ring of convergent power series. Thus, we have

Theorem 4.1 (Positivstellensatz) *Let $(f_j)_{j=1,\ldots,s}$, $(g_k)_{k=1,\ldots,t}$, $(h_l)_{l=1,\ldots,u}$ be analytic functions in $\mathbb{R}\{X_1, \ldots, X_n\}$. Let P be the cone generated by $(f_j)_{j=1,\ldots,s}$, let N be the multiplicative system generated by $(g_k)_{k=1,\ldots,t}$, and let I be the ideal generated by $(h_l)_{l=1,\ldots,u}$. Then the following conditions are equivalent:*

a) *The semianalytic germ*

$$S = \{f_j \geqq 0;\ g_k \neq 0;\ h_l = 0;\ j = 1, \ldots, s;\ k = 1, \ldots, t;\ l = 1, \ldots, u\}$$

is empty.

b) *There are $f \in P$, $g \in N$, $h \in I$ such that $f + g^2 + h = 0$.*

Proof. [Fe-Re-Rz], [An-Br-Rz]. □

So, let us go for the

Proof of Theorem 1.1. Let $f_1, \ldots, f_r \in \mathcal{O}(M)$ be positive in some ordering β of the field K. We want to show that $\{x \in M \mid f_1(x) > 0, \ldots, f_r(x) > 0\} \neq \emptyset$.

As above, we set $g_{ij} = f_i f_j$ for $i < j$. Obviously $g_{ij} \in \beta$ and

$$\{x \in M \mid f_1(x) > 0, \ldots, f_r(x) > 0\}$$
$$= \{x \in M \mid f_1(x) > 0, \ldots, f_r(x) > 0, g_{12}(x) > 0, \ldots, g_{(r-1)r}(x) > 0\}\ .$$

Now let $\tilde{f}_1, \ldots, \tilde{f}_r, \tilde{g}_{ij}$ be the square free functions associated to them. By Corollary 3.4 we are reduced to show that

$$\{x \in M \mid \tilde{f}_i(x) > 0, \tilde{g}_{ij}(x) > 0, i < j; \tilde{f}\, i, j = 1, \ldots, r\} \neq \emptyset$$

Assume the contrary. By Proposition 3.5 the semianalytic set

$$Y = \left[\bigcap_i \tilde{f}_i^{-1}([0, \infty))\right] \cap \left[\bigcap_{j<m} \tilde{g}_{jm}^{-1}([0, \infty))\right]$$

is discrete.

Let $p \in Y$ and let U_p be a neighbourhood of p such that $\dot{U}_p \cap Y = \{p\}$ and define $r_p(x) = \|x - p\|^2$ where $\|\cdot\|$ represents the euclidean norm in $\mathbb{R}^n$. This is a global analytic function with an elliptic germ at p, and we have that for every $x \in U_p$ with $\tilde{f}_i(x) \geqq 0$ and $\tilde{g}_{ij}(x) \geqq 0$, then $r_p(x) = 0$. In other words, the semianalytic germ at p

$$\{\tilde{f}_{i,p}(x) \geqq 0, \tilde{g}_{jm,p}(x) \geqq 0, \|p - x\| \neq 0\}$$

is empty. Hence we can apply the Positivstellensatz in $\mathcal{O}_p(M)$ to conclude that for every $p \in Y$ there exists $h_p \in P_p$ and a natural number s_p such that

$$h_p = -r_p^{2s_p} \qquad (*)$$

where P_p is the cone generated by $\tilde{f}_{1,p}, \ldots, \tilde{f}_{r,p}, \tilde{g}_{12,p}, \ldots, \tilde{g}_{(r-1)r,p}$, that is, h_p is a finite sum of products of the form

$$a_{p,ijm} \prod_i \tilde{f}_{i,p}^{\delta_i} \prod_{j,m} \tilde{g}_{jm,p}^{\delta_{jm}}$$

where $a_{p,ijm}$ is a sum of squares in $\mathcal{O}_p(M)$. Moreover, since in $\mathcal{O}_p(M)$ any sum of squares is the sum of two squares, see [Bo-Rs], [Jw3], we may write $a_{p,ijm} = b_{p,ijm}^2 + c_{p,ijm}^2$, with $b_{p,ijm}, c_{p,ijm} \in \mathcal{O}_p(M)$. Moreover, it follows from the equation (∗) above, that h_p is a homogeneous polynomial (viewed as a series at p) of degree $4s_p$. Let $t_p \in \mathcal{O}_p(M)$ be such that $t_p \equiv h_p \pmod{m_p^{4s_p}}$. Then $t_p = h_p + f$, where $f \in \mathbb{R}\{X_1 - p_1, X_2 - p_2\}$ is a series with initial form of degree bigger than $4s_p$. In particular h_p is the initial form of t_p. Since h_p is an elliptic germ at p, it follows that t_p is elliptic too, and, consequently, non positive on a neighbourhood of p.

Thus, it follows that for any $b'_{p,ijm}, c'_{p,ijm} \in \mathcal{O}_p(M)$ verifying

$$b'_{p,ijm} - b_{p,ijm} \in \mathfrak{m}_p^{v_p}, \quad c'_{p,ijm} - c_{p,ijm} \in \mathfrak{m}_p^{v_p}$$

with $v_p \geqq 4s_p$, then

$$t_p = \sum \prod_{i=1}^{r} \tilde{f}_{i,p}^{\delta_i} \prod_{j,m} \tilde{g}_{jm,p}^{\delta_{jm}} (b'_{p,ijm} + c'_{p,ijm})$$

has an isolated zero at p and that $t_p(x) < 0$ for all $x \neq p$ in a neighbourhood of p.

Now we use once more Cartan's Theorem B to produce global functions B_{ijm} and C_{ijm} such that at each $p \in Y$ their germs coincide with $b_{p,ijm}$ and $c_{p,ijm}$, respectively, till order v_p. Finally consider the global analytic function

$$H = \sum_{i,j,m} \prod_{i=1}^{r} \tilde{f}_i^{\delta_i} \prod_{i,j} \tilde{g}_{jm}^{\delta_{jm}} (B_{ijm}^2 + C_{ijm}^2)\,.$$

Obviously $H \in \beta$, and by construction its germ H_p is elliptic and non positive at any point $p \in Y$. Hence it follows from Proposition 3.3 that the function $\tilde{H} \in \mathcal{O}(M)$ is also in β while $\tilde{H}|_Y < 0$. But then

$$\{x \in M \mid \tilde{f}_i(x) \geqq 0, \tilde{g}_{ij}(x) \geqq 0, \tilde{H}(x) \geqq 0\} = Y \cap \{x \in M \mid \mathbb{R}(x) \geqq 0\} = \emptyset$$

in contradiction with Lemma 2.6, and therefore the proof of the theorem is complete. □

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