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Applications of symplectic homology I

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Table of contents

1 Introduction 577
1.1 Some general remarks 577
1.2 Symplectic homology of ellipsoids 578
1.3 Symplectic Homology of Polydisks 582
1.4 Symplectic capacities 583
1.5 Some Remarks 585
2 Computation of symplectic homology 585
2.1 Generic approximations 585
2.2 The boundary operator 591
2.3 The directed system 592
2.4 Computations for ellipsoids and polydisks 594
2.5 Some properties of induced homomorphisms 595
3 Applications 597
3.1 Properties of the nonlinear order structure 597
3.2 Polydisk classification 600
3.3 Non isotopic symplectic embeddings 602
3.4 Construction of a symplectic capacity 602

1 Introduction

1.1 Some general remarks

The purpose of this paper is to illustrate the symplectic homology construction in [7] via some applications concerned with ellipsoids and symplectic polydisks.Moreover we show how a symplectic capacity, see[19, 3, 4, 16, 15, 14, 22],can be constructed from the symplectic homology theory.A few results have been announced in [16]. We assume the reader to be familiar with the results and notation in [7]. For basic symplectic geometry we refer to [25].

* Andreas Floer died on May 15th, 1991

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1.2 Symplectic homology of ellipsoids

Although ellipsoids and polydisks are the most simple shapes in symplectic geometry we shall see that there are still some interesting open questions. Let us define a set $\Gamma \subset (0,+\infty)^n$ by

$$\Gamma = \{r \in (0,+\infty)^n \mid r = (r_1,\ldots,r_n)\,,\ r_1 \leq r_2 \leq \ldots \leq r_n\}\ .$$

We denote by $\Gamma_{reg} \subset \Gamma$ the subset of those r such that the numbers $r_1^2,\ldots,r_n^2$ are linearely independent over the integers. An element of $r \in \Gamma_{\text{reg}}$ will be called a regular element of Γ. Given $r \in \Gamma$ we denote by $E(r)$ the open ellipsoid defined by

$$E(r) = \{z \in \mathbb{C}^n \mid \sum \left|\tfrac{z_k}{r_k}\right|^2 < 1\}\ .$$

Generally an ellipsoid in $\mathbb{C}^n$ is a set of the form

$$E = q^{-1}((-\infty,1))\ ,$$

where q is a positive definite (reel) quadratic form. The linear symplectic group Sp acts on the collection of all ellipsoids, say $\mathscr{E}$ via

$$Sp \times \mathscr{E} \longrightarrow \mathscr{E} : (\Psi, E) \longrightarrow \Psi(E)\ .$$

The quotient space $\hat{\mathscr{E}}$ of $\mathscr{E}$ by the Sp–action can be identified with Γ via the map

$$\Gamma \longrightarrow \hat{\mathscr{E}} : r \longrightarrow [E(r)]\ .$$

This is an exercise in linear symplectic algebra, see [17]. Hence an ellipsoid $E \in \mathscr{E}$ has a natural symplectic numerical invariant $r(E) \in \Gamma$.

The map $\mathscr{E} \longrightarrow \Gamma : E \longrightarrow r(E)$ is a "fibration" with the fibre over a given $r \in \Gamma$ consisting of all linearley equivalent ellipsoids. Ellipsoids are very good objects to demonstrate the difference between linear and nonlinear symplectic geometry.

The following result is well–known, see [17] for example.

Proposition 1 *For two ellipsoids $E, F \in \mathscr{E}$ the following statements are equivalent*

(i) There exists $\Psi \in Sp$ with $\Psi(E) \subset F$
(ii) $r(E) \leq r(F)$.

Here $a \leq b$ precisely means $a_i \leq b_i$ for $i = 1,\ldots,n$. As a corollary we have

Corollary 1 *$r \leq r'$ iff $\Psi(E(r)) \subset E(r')$ for some $\Psi \in Sp$.*

This motivates immediately a nonlinear generalization. Let us denote by $\mathrm{Diff}(\omega)$ the group of smooth symplectic diffeomorphisms on $\mathbb{C}^n$. We put $\leq_1 := \leq$ and define a second order relation by

Definition 1 *We define an order relation $\leq_2$ on Γ by*

$$r \leq_2 r' :\Longleftrightarrow \quad \begin{array}{l} \textit{There exists } \Psi \in \mathrm{Diff}(\omega) \textit{ with} \\ \Psi(E(r)) \subset E(r') \ . \end{array}$$

It is not difficult to show that

Lemma 1 *If $r \leq_2 r'$ and $r' \leq_2 r$ then $r = r'$.*

Obviously the map $(\Gamma, \leq_1) \longrightarrow (\Gamma, \leq_2) : r \longrightarrow r$ is order preserving. However the inverse does not preserve the order. To see this we take a symplectic embedding of $\overline{B^2(2)}$ into $E(1, \ldots, 1)$, (for the flexibility of symplectic embeddings with positive codimension see [11, 12]). Then the image has a symplectic tubular neighbourhood, i. e. there exists a symplectic embedding $B^{2n-2}(\epsilon) \times B^2(2) \longrightarrow E(1, \ldots, 1)$ for small $\epsilon > 0$. In particular we have a symplectic embedding $\tilde{\Psi}$ of $E(\epsilon, \ldots, \epsilon, 2)$ into $E(1, \ldots, 1)$. Given any $\epsilon' \in (0, \epsilon)$ and $\tau \in (0, 2)$ there exists $\Psi_{\epsilon', \tau} \in \mathrm{Diff}(\omega)$ such that $\Psi_{\epsilon', \tau} \mid E(\epsilon', \ldots, \epsilon', \tau) = \Psi \mid E(\epsilon', \ldots, \epsilon', \tau)$. (That is the well–known extension after restriction principle for symplectic maps on simple shapes, see [3]). Hence we have proved $(\epsilon', \ldots, \epsilon', \tau) \leq_2 (1, \ldots, 1)$ for every $\epsilon' \in (0, \epsilon)$, $\tau \in (0, 2)$. In particular

Proposition 2 *For all sufficiently small $\epsilon > 0$ we have*

$$\left(\epsilon, \ldots, \epsilon, \frac{3}{2}\right) \leq_2 (1, \ldots, 1) \ .$$

Proposition 2 shows that a considerable amount of symplectic rigidity is lost if we pass from the linear to to the nonlinear theory. But still we know that there is a considerable amount of rigidity left (Gromov width, capacities). To compare $\leq_1$ and $\leq_2$ should give a better idea about rigidity and flexibility in symplectic geometry.

The symplectic homology theory turns out to be a useful tool in comparing $\leq_1$ and $\leq_2$. In contrast to proposition 2 we have in $\mathbb{C}^2$:

Theorem 1 *For $n = 2$, i. e. ellipsoids in $\mathbb{C}^2$, let $\sigma \in \left(\frac{1}{\sqrt{2}}, 1\right)$. The following statements are equivalent.*

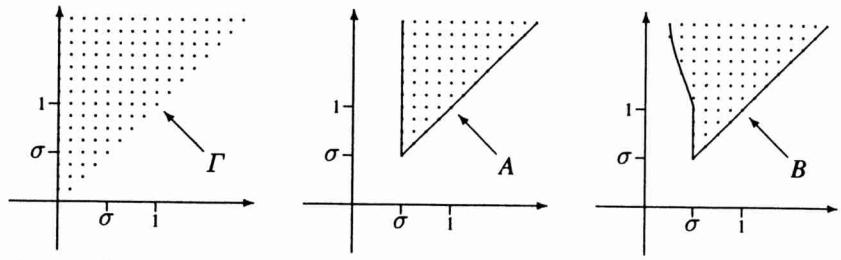
(i) $(\sigma, \sigma) \leq_1 r \leq_1 r' \leq_1 (1, 1)$
(ii) $(\sigma, \sigma) \leq_2 r \leq_2 r' \leq_2 (1, 1)$

In other words; the order intervals for $i = 1, 2$

$$[(\sigma, \sigma), (1, 1)]_i = \{r \mid (\sigma, \sigma) \leq_i r \leq_i (1, 1)\}$$

are identical and the orderings $\leq_1$ and $\leq_2$ on them coincide.

The following figure shows the sets $A = \{r \mid (\sigma, \sigma) \leq_1 r\}$ and $B = \{r \mid (\sigma, \sigma) \leq_2 r\}$.



The proof of Theorem 1 is quite difficult and involves results by McDuff, [20], and the symplectic homology theory from [7].

The obvious question is why do we need the condition $\sigma \in \left(\frac{1}{\sqrt{2}}, 1\right)$? To a certain extend it is possible to understand this constraint geometrically. In passing from the linear symplectic theory to the nonlinear theory the new flexibility is the possibility of "folding" a set. Symplectic capacities measure to some extend the area of two dimensional symplectic projections of a set. If one folds a set some projection might double in size. If for example $r = (\sigma, \sigma)$ then the smallest projection is of area $\pi\sigma^2 > \frac{\pi}{2}$. If one "folds" some projection could have area around $2\pi\sigma^2 > \pi$. The folded set would not fit into the ball $B^4(1)$. So to some extend the pinching condition in Theorem 1 excludes folding. If folding is excluded the linear and nonlinear theory coincide. This is of course extremely heuristic, but we believe it points in the right direction. Here is a "test problem".

Problem 1 Given any $\sigma \in \left(0, \frac{1}{\sqrt{2}}\right)$ does there exist a r_σ such that

$$\begin{aligned} (\sigma, \sigma) &\leq_1 r_\sigma \ , \ r_\sigma \not\leq_1 (1,1) \\ r_\sigma &\leq_2 (1,1) \ ? \end{aligned}$$

In other words: is it true that Theorem 1 is sharp?

Next we turn to the main topic of this section, namely the computation of the symplectic homology of an ellipsoid. Since symplectic homology is an invariant under symplectic diffeomorphism in $\mathscr{D}$, see [7], it suffices to compute it for the normal form $E(r)$. We shall work for simplicity only with $\mathbb{Z}_2$–coefficients. Given $r \in \Gamma$ we denote by $\sigma(r)$ the subset of $(0, +\infty)$ defined by

$$\sigma(r) = \left\{ k\pi r_j^2 \mid k \in \mathbb{N}^* \ , \ j = 1, \ldots, n \right\} \ .$$

Here $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. We have a map

$$\phi : \mathbb{N}^* \times \{1, \ldots, n\} \longrightarrow \sigma(r) : (k, j) \longrightarrow \pi k r_j^2 \ .$$

Given $d \in \sigma(r)$ we define $m(d) = \#\phi^{-1}(r)$ and call it the multiplicity of d. Consider the sequence (d_l) , $d_l = d_l(r)$, $l \in \mathbb{N}^*$

$$(d_0 =) 0 < d_1 \leq d_2 \leq d_3 \ldots$$

consisting of all elements in $\sigma(r)$, written in increasing order and each element repeated according to its multiplicity.

We need the notation of a $\mathbb{R}$–filtered chain complex. For every number $\mu \in (-\infty, +\infty]$ suppose a subcomplex (C^μ, ∂^μ) of (C, ∂) is given with $C^{+\infty} = C$. We define

$$C^{[a,b)} = C^b / C^a$$

with the induced boundary operator $\partial^{[a,b)}$. For given $r \in \Gamma$ let $(d_j)_{j \in \mathbb{N}}, \mathbb{N} = \{0, 1, \ldots\}$ be the associated sequence. We define

$$\begin{aligned} C^\mu(r) &= 0 \quad \text{for} \quad \mu \leq 0 \\ C^\mu(r) &= (\mathbb{Z}_2, n) \quad \text{for} \quad 0 < \mu \leq d_1 \\ C^\mu(r) &= (\mathbb{Z}_2, n) \oplus \ldots \oplus (\mathbb{Z}_2, n + 2m(\mu; r)) \quad \text{for} \quad d_1 \leq \mu < +\infty \\ C^{+\infty}(r) &= \oplus_{l=0}^{\infty} (\mathbb{Z}_2, n + l) \ , \end{aligned}$$

where $m(\mu, r) = \sup\{l \mid d_l < \mu\}$. Next we define the boundary map as follows. If $\mu \leq d_1$ it is the zero map, if $d_1 < \mu < +\infty$ it is given by the following diagram

$$\begin{aligned} \ldots \leftarrow 0 \leftarrow (\mathbb{Z}_2, n) \overset{\mathrm{Id}}{\leftarrow} (\mathbb{Z}_2, n+1) \quad & \overset{0}{\leftarrow} (\mathbb{Z}_2, n+2) \overset{\mathrm{Id}}{\leftarrow} \ldots \overset{\mathrm{Id}}{\leftarrow} (\mathbb{Z}_2, n+2m-1) \overset{0}{\leftarrow} (\mathbb{Z}_2, n+2m) \\ & \leftarrow 0 \leftarrow \ldots \end{aligned} \tag{1}$$

with $m = m(\mu, r)$ and $C^{+\infty}$ is the direct limit complex. We have the following result

Theorem 2 *Let $r \in \Gamma$ and $-\infty < a \leq b \leq +\infty$. Then*

$$S^{[a,b)}(E(r)) = H_* \left(C^{[a,b)}(r) \ , \ \partial^{[a,b)}(r) \right) \ .$$

As an example let $r = (1, \ldots, 1)$ so that $E(r) = B^{2n}(1)$ is the open unit ball around zero in $\mathbb{C}^n$. The sequence (d_l) is given by

$$0, \quad \underbrace{\pi, \ldots, \pi}_{n\text{–times}} , \quad \underbrace{2\pi, \ldots, 2\pi}_{n\text{–times}} , \quad \underbrace{3\pi, \ldots, 3\pi}_{n\text{–times}} , \quad 4\pi, \ldots$$

To compute the symplectic homology of $B^{2n}(1)$ consider the following diagram

$$\begin{aligned} & \ldots \leftarrow 0 \leftarrow (\mathbb{Z}_2, n; 0) \overset{\mathrm{Id}}{\leftarrow} (\mathbb{Z}_2, n+1; \pi) \leftarrow \ldots \overset{0}{\leftarrow} (\mathbb{Z}_2, 3n, \pi) \\ & \overset{\mathrm{Id}}{\leftarrow} (\mathbb{Z}_2, 3n+1, 2\pi) \leftarrow \ldots \end{aligned}$$

Here $(\mathbb{Z}_2, k; d)$ has the following meaning: $k \in \mathbb{Z}$ is the grading dimension, and with $C = (\mathbb{Z}_2, k; d)$ we have $C^\mu = 0$ for $\mu \leq d$, $C^\mu = (\mathbb{Z}_2, k)$ for $\mu > d$. For example let $\mu = 2\pi$. Then the $C^{2\pi}$–part of the above filtered complex is

$$\begin{aligned} & \ldots \leftarrow 0 \leftarrow (\mathbb{Z}_2, n) \overset{\mathrm{Id}}{\leftarrow} (\mathbb{Z}_2, n+1) \overset{0}{\leftarrow} (\mathbb{Z}_2, n+2) \\ & \ldots \overset{0}{\leftarrow} (\mathbb{Z}_2, 3n) \leftarrow 0 \leftarrow \ldots \end{aligned} \tag{2}$$

since only the groups $(\mathbb{Z}_2, n, 0), \ldots, (\mathbb{Z}_2, 3n, \pi)$ live strictly below level 2π. The groups $(\mathbb{Z}_2, 3n+1, 2\pi)$ and up are replaced by zero in order to obtain $C^{2\pi}$. In

order to continue our example we compute $C^{[0,2\pi)}(B^{2n}(1))$. The relevant complex is (2) since $C^0 = 0$. Hence

$$C^{[0,2\pi)}(B^{2n}(1)) = (\mathbb{Z}_2, 3n) \ .$$

More generally we compute from Theorem 2

Corollary 2 *Let $-\infty < a \leq b \leq +\infty$. For $a \leq 0$ and $k\pi < b \leq (k+1)\pi$, $k \in \mathbb{N}$, we have*

$$S^{[a,b)}(B^{2n}(1)) = (\mathbb{Z}_2, n + 2kn) \ .$$

For $0 \leq j\pi < a \leq (j+1)\pi \leq k\pi < b \leq (k+1)\pi$ with $j, k \in \mathbb{N}$ we have

$$S^{[a,b)}(B^{2n}(1)) = (\mathbb{Z}_2, n + 2jn + 1) \oplus (\mathbb{Z}_2, n + 2kn) \ .$$

Further for $0 \leq j\pi < a \leq (j+1)\pi$ and $b = +\infty$

$$S^{[a,+\infty)}(B^{2n}(1)) = (\mathbb{Z}_2, n + 2jn + 1) \ .$$

In all other cases it is zero.

1.3 Symplectic Homology of Polydisks

For $r \in \Gamma$ denote by $D^{2n}(r)$ the open polydisk

$$D^{2n}(r) = B^2(r_1) \times \ldots \times B^2(r_n) \ .$$

Our aim is to compute its symplectic homology. From the construction in [7] it is clear that the symplectic homology has certain product properties. This combined with the knowledge of the symplectic homology of a disk suffices to compute it for polydisks. Before we give a formula we give two applications. The first is Gromov's polydisk conjecture, [11].

Theorem 3 *Let $r, r' \in \Gamma$ and assume the open polydisks are symplectomorphic. Then $r = r'$.*

Another application is concerned with the topology of the space of symplectic embeddings of $D^4(r)$ into $D^4(1,1)$.

The study of such symplectic embedding spaces is in general very difficult. In particular McDuff studied in [20] symplectic embeddings of $B^4(\delta)$ into $B^4(1)$ for $\delta \in (0,1)$ and showed that they are all symplectically isotopic. This result is not known if $n \geq 3$.

However we will be able to show that polydisks behave quite differently.

Theorem 4 *Let $n = 2$ and $r \in \Gamma$ and assume $r_1^2 + r_2^2 > 1$ and $r_1, r_2 < 1$. Then there are at least two different symplectic isotopy classes of symplectic embeddings into $D^4(1,1)$.*

In order to derive the statement for the symplectic homology of the polydisk recall that the diagram for a 2–disk of radius r is

$$\ldots \leftarrow 0 \leftarrow 0 \leftarrow (\mathbb{Z}_2, 1; 0) \overset{\mathrm{Id}}{\leftarrow} (\mathbb{Z}_2, 2; \pi r^2) \overset{0}{\leftarrow} (\mathbb{Z}_2, 3; \pi r^2) \leftarrow \quad . \tag{3}$$

Again the numbers $0, \pi r^2$ etc. mean that the groups $(\mathbb{Z}_2, 1), (\mathbb{Z}_2, 2)$ etc. live above a certain level.

The relevant diagram for a polydisk is then the "tensor product" of the diagram as in (3) with different radii. The rule for the tensor product is

$$(G, k, b) \otimes (H, k', b') = (G \otimes H, k + k', b + b') \ .$$

Let us denote for $r \in (0, +\infty)$ by $(C^\mu(r), \partial^\mu)$ the complex for the r–disk in $\mathbb{C}$. Define

$$\tilde{C}^\infty(r_1, \ldots, r_n) = C^\infty(r_1) \otimes \ldots \otimes C^\infty(r_n)$$

with obvious boundary operator, which is the usual tensor product of chain complexes. For $l \in (-\infty, +\infty)$ we define

$$\tilde{C}^l(r_1, \ldots, r_n) = \bigcup_{l = \sum l_j} C^{l_1}(r_1) \otimes \ldots \otimes C^{l_n}(r_n) \ ,$$

which is a subcomplex of $\tilde{C}^\infty(r_1, \ldots, r_n)$.

Finally we put $\tilde{C}^{[\mu,\nu)}(r) = \tilde{C}^\nu(r)/\tilde{C}^\mu(r)$ for $r \in \Gamma$. Then

Theorem 5 *For $-\infty < a \leq b \leq +\infty$ we have*

$$S^{[a,b)}(D^{2n}(r)) = H_* \left(\tilde{C}^{[a,b)}(r), \ \partial^{[a,b)}\right) \ .$$

This is already quite complicated to calculate and we restrict ourselves to the computations of particular cases in order to prove Theorems 3 and 4.

1.4 Symplectic capacities

One of the very fruitful notions in symplectic geometry is that of a symplectic capacity. This is a map which associates to a symplectic $2n$–dimensional manifold a number $c(M, \omega) \in (0, +\infty]$ (assuming $M \neq \emptyset$) satisfying the following axioms

- $c(M, \alpha\omega) = |\alpha| \ c(M, \omega)$ for $\alpha \neq 0$
- If there exists a symplectic embedding of (M, ω) into (N, τ) then $c(M) \leq c(N)$ (4)
- $c(B^2(1) \times \mathbb{C}^{n-1}, \text{standard}) < +\infty$

The success of the variational theory of Hamiltonian systems was very important for the development of the symplectic capacity theory, see [21, 26, 23, 18, 2]. Many constructions of symplectic capacities are known [19, 3, 4, 22, 24, 11, 17]. It is also well–known that the existence of a single capacity implies for example the C^0–rigidity phenomenon detected by Eliashberg, see [5, 11, 12].

Here we show that one can construct a symplectic capacity for open sets in $\mathbb{C}^n$ from the symplectic homology theory. For $0 < \epsilon_1 \leq \epsilon_2 < +\infty$ and $b \geq \epsilon_2$ we have the natural maps

$$S_{n+1}^{[\epsilon_1,b)}(U) \longrightarrow S_{n+1}^{[\epsilon_2,b)}(U) \ .$$

We obtain an inverse system and define

$$S_{n+1}^{(0,b)}(U) := \lim_{\leftarrow} \left(S_{n+1}^{[\epsilon,b)}(U)\right)_{\epsilon\in(o,b]} \ .$$

We shall prove

Proposition 3 *If $\pi r^2 \in (0,b)$ then*

$$S_{n+1}^{(0,b)}\left(B^{2n}(r)\right) \cong \mathbb{Z}_2 \ .$$

Next we consider for $\epsilon \leq \epsilon'$ the natural map

$$S_{n+1}^{(0,b)}\left(B^{2n}(\epsilon')\right) \longrightarrow S_{n+1}^{(0,b)}\left(B^{2n}(\epsilon)\right)$$

since $B^{2n}(\epsilon) \subset B^{2n}(\epsilon')$. We pass to the direct limit $\epsilon \longrightarrow 0$ obtaining the group Θ.

Proposition 4 *For $b > 0$ we have $\Theta \cong \mathbb{Z}_2$.*

Given any $z \in U$ and $\Psi \in \mathscr{D}$ with $\Psi(0) = z$ we have for $\epsilon > 0$ sufficiently small $\Psi(B^{2n}(\epsilon)) \subset U$. Hence we obtain an induced map

$$\Psi^* : S_{n+1}^{(0,b)}(U) \longrightarrow S_{n+1}^{(0,b)}(B^{2n}(\epsilon))$$

and passing to the direct limit we obtain a map

$$\Psi^\natural : S_{n+1}^{(0,b)}(U) \longrightarrow \Theta \ .$$

Using the isotopy invariance the map $\Psi^\natural$ only depends on U and the connected component of $z \in U$.

Hence for every connected bounded open set U there is a characteristic map denoted by σ_U

$$\sigma_U : S_{n+1}^{(0,b)}(U) \longrightarrow \Theta \ .$$

The capacity of a connected set U is defined by

$$c(U) = \inf\{b \mid \sigma_U : S_{n+1}^{(0,b)}(U) \longrightarrow \Theta \quad \text{is onto}\} \ .$$

If U has components $(U_\lambda)_{\lambda\in\wedge}$ we define $c(U) := \sup_{\lambda\in\wedge} c(U_\lambda)$. For unbounded U we define $c(U)$ by exhaustion through bounded open sets. We have

Theorem 6 *i) If $\Psi \in \mathscr{D}$ and $\Psi(U) \subset V$ the $c(U) \leq c(V)$.*
ii) $c(\alpha U) = |\alpha|^2 c(U)$ for $\alpha \in \mathbb{R} \setminus \{0\}$.
iii) $c(B^{2n}(1)) = c(B^2(1) \times \mathbb{C}^{n-1}) = \pi$.