Math. Z. 217, 567–576 (1994)

Mathematische Zeitschrift

Blow-up for nonlinear wave equations with slowly decaying data★

Hiroyuki Takamura

Institute of Mathematics, University of Tsukuba, Tsukuba-shi Ibaraki 305, Japan
(Fax +81 298 536501)

Received 9 March 1993; in final form 12 July 1993

1 Introduction

We are concerned with classical solutions to the following initial value problem for nonlinear wave equations:

$$(1.1)\qquad \begin{aligned} &\Box u \equiv u_{tt} - \Delta u = |u_t|^p && \text{in } \mathbb{R}^n \times [0,\infty)\,, \\ &u(x,0) = f(x), u_t(x,0) = g(x), && x \in \mathbb{R}^n\,, \end{aligned}$$

where u is a scalar unknown function, $p > 1, n \geqq 2$, and f, g are given smooth functions.

In the case where f, g are large in some sense, it is known by Glassey [4] that solutions to (1.1) blow-up in finite time. Hence, we may formulate a question as follows: what is the critical value of p, say $p_0(n)$, depending on n, with the property that (1.1) admits a global solution for all "small" f, g if $p > p_0(n)$, and that there are blowing-up solutions if $1 < p \leqq p_0(n)$.

First, consider (1.1) with compactly supported initial data. In the case $n = 3$, John [6] proved that solutions to (1.1) blow-up in finite time for $p = 2$ under some positivity condition on initial data. One can see that his result is still valid for $1 < p \leqq 2$. See the result and its proof of R. Agemi [1]. On the other hand, Sideris [16] proved that a radially symmetric global solution exists for all $p > 2$. Hence, these two results imply that $p_0(3) = 2$. Schaeffer [14] verified $p_0(5) = 3/2$ by establishing global existence of a radially symmetric global solution and showing the finite-time blow-up for a fairly large class of initial data. In other space dimensions, it seems to be difficult to determine $p_0(n)$, especially the existence-part in even space dimensions. In the case $n = 2$, Agemi [1] showed that solutions to (1.1) with "positive" data blow-up in finite time for $1 < p \leqq 3$. See also Masuda [12] for $p = 2$ or John [7] and Schaeffer [15] for $p = 3$. In higher space dimensions, $n \geqq 4$, Rammaha [13] proved that

★ Dedicated to Professor Tosinobu Muramatu on his sixtieth birthday radially symmetric solutions to (1.1) with "positive" data blow-up in finite time provided $p=(n+1)/(n-1)$ for odd n and $1<p<(n+1)/(n-1)$ for even n. For the existence-part, Klainerman [8] [9] established Sobolev inequalities in Minkowski space and proved a general existence theorem which when applied to the special case (1.1) implies, one can see actually, the global in time existence of a solution provided $p>(n+1)/(n-1)$ and p is an even integer or larger number. See Introduction in Agemi [1]. It has been conjectured that

$$p_0(n)=\frac{n+1}{n-1}\,. \tag{1.2}$$

On contrast, for semilinear wave equations:

$$\square u=|u|^p \quad \text{in} \quad \mathbb{R}^n\times[0,\infty)\,, \tag{1.3}$$

one can observe the similar phenomenon to (1.1). There is a critical value of p, say $p_0^*(n)$. For $2\leqq n\leqq 4$, we have already known that $p_0^*(n)$ is a positive root of the quadratic equation: $(n-1)p^2-(n+1)p-2=0$. In the case $n\geqq 5$, only the blow-up in $1<p<p_0^*(n)$ was shown. See Introduction in Takamura [18] for this survey. It has also been conjectured that, for all $n\geqq 2$,

$$p_0^*(n)=\frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}>p_0(n)=\frac{n+1}{n-1}\,. \tag{1.4}$$

The remarkable result on (1.3) is that, if the initial data has noncompact support, we get blowing-up solutions even for the existence $p>p_0^*(n)$ in the compactly supported case because of the slow spatial decay of the data. This fact has been verified in the case $2\leqq n\leqq 5$. Moreover, the critical decay was obtained in the case $n=2,3$. Slow decay yields blowing-up solutions and rapid decay implies that we have a global solution. We note that such a critical decay never depends on dimensions n and that, just in the critical, we have a global solution. See Introduction in Takamura [18].

The aim of this paper is to show that, for $n=4,5$, the problem (1.1) has the similar blow-up phenomenon to the above noncompactly supported case on (1.3). For $n=2,3$, see Kubo [10]. As for the global existence for (1.1) with noncompactly supported data in the case $n\geqq 4$, one can actually remove the compactness assumption on the support of initial data in Klainerman's general existence theorem, but has to assume the rapid spatial decay on the data. For slowly decaying data, Asakura [2] proved the global existence for $n=5$ and $p=2$. If the nonlinearity is of the form u_t^p with integer p, Takamura [17] proved the global existence for $n=3$ and $p>2=p_0(3)$. In the same situation, Kubo [11] proved the global existence for odd $n\geqq 5$ and $p\geqq 2$. It is very difficult to determine the critical decay, especially to show the global existence in even space dimensions.

The main difficulty to show the blow-up in high space dimensions, $n\geqq 4$, is that the fundamental solution of $\square$ yields derivative loss. To avoid this, we consider the spherically symmetric case of (1.1). But, then, fundamental solution is no longer positive for full space. Rammaha [13] used the full space integral of solutions and reduced to the nonexistence theory for ordinary differential inequalities. Such a method is never applicable to the noncompactly supported case.

Our problem is

$$(1.5)\qquad \begin{aligned} &u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u_t) && \text{in } [0,\infty)^2\,,\\ &u(r,0) = \varepsilon\varphi(r),\quad u_t(r,0) = \varepsilon\psi(r), && r \in [0,\infty)\,,\end{aligned}$$

where ε is a positive small parameter.

Theorem 1.1 *Let n = 4, 5. Assume that $F \in C^1$ ($\mathbb{R}$) satisfies*

$$(1.6)\qquad F(s) \geqq A|s|^p \quad \textit{for} \quad A > 0, \quad p > 1$$

and that smooth data φ, ψ satisfies

$$(1.7)\qquad \begin{aligned} &\varphi(r) \equiv 0, \quad \psi(r) \geqq \frac{M}{(1+r)^\kappa}, \quad r \in [0,\infty)\\ &\text{for } M > 0, \quad 0 < \kappa < \frac{1}{p-1}\,.\end{aligned}$$

Then, there exists a positive constant C, independent of ε, such that the life-span $T(\varepsilon)$, maximal existence time, of classical solutions to (1.5) *satisfies*

$$(1.8)\qquad T(\varepsilon) \leqq C\varepsilon^{-(p-1)/(1-(p-1)\kappa)}\,.$$

Remark 1.2 In further high dimensions, $n \geqq 6$, Theorem 1.1 should be true. But our method is not directly applicable. We may get more complicated restrictions on space-time variables in gaining a sufficient positivity of fundamental solution to prove this blow-up. See the proof of Lemma 2.3 below.

Remark 1.3 By Theorem 1.1 and Kubo [10], one can conjecture that the critical decay of the initial data which has an order $O(|x|^{-\kappa})$ as $|x| \to \infty$ for the problem (1.1) is $\kappa = 1/(p-1)$ for all $n \geqq 2$. On the problem (1.3), it is known that the critical decay is $\kappa = (p+1)/(p-1)$ for $n = 2, 3$. Both two critical decays may be independent of space dimensions n.

2 Preliminaries

We shall proceed the proof via point-wise estimate as in Takamura [18]. Let u^0 be a solution to the following initial value problem:

$$(2.1)\qquad \begin{aligned} &\Box u^0 = 0 && \text{in } \mathbb{R}^n \times [0,\infty)\,,\\ &u^0|_{t=0} = 0, \quad u^0_t|_{t=0} = \psi(|x|), && x \in \mathbb{R}^n\,.\end{aligned}$$

As is well-known, the classical formulas for u^0 are given by

$$(2.2)\qquad u^0 = \frac{1}{(2\pi)^2}\frac{1}{t}\frac{\partial}{\partial t}\int_0^t \frac{\rho^3 d\rho}{\sqrt{t^2-\rho^2}} \int_{|\omega|=1} \psi(|x+\rho\omega|)dS_\omega$$

for $x \in \mathbb{R}^4$ and

$$(2.3)\qquad u^0 = \frac{1}{2(2\pi)^2}\frac{1}{t}\frac{\partial}{\partial t}t^3 \int_{|\omega|=1} \psi(|x+t\omega|)\mathrm{d}S_\omega$$

for $x \in \mathbb{R}^5$. For instance, see Courant & Hilbert [3].

Lemma 2.1 *Let $v \in C^0([0,\infty))$. Then, for all $t > 0$ and $x \in \mathbb{R}^n$, we have*

$$(2.4)\qquad \int_{|\omega|=1} v(|x+t\omega|)\mathrm{d}S_\omega = \frac{2\omega_{n-1}}{(2rt)^{n-2}}\int_{|r-t|}^{r+t}[(\lambda^2-(r-t)^2)((r+t)^2-\lambda^2)]^{(n-3)/2}\lambda v(\lambda)\mathrm{d}\lambda\,,$$

where $r \equiv |x| > 0$ and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is an area of the unit sphere in $\mathbb{R}^n$.

Lemma 2.1 immediately follows from a classical result of the fundamental identity for iterated spherical means. See John [5]. We omit the proof.

Lemma 2.2 (The representation formula) *Let $n = 4, 5$ and u be a solution to (1.5) with $\varphi(r) \equiv 0$ for $r \equiv |x|$, $x \in \mathbb{R}^n$. Then, we have*

$$(2.5)\qquad u = \varepsilon u^0 + L(F(u_t)) \quad \text{in} \quad [0,\infty)^2\,,$$

where, for $n = 4$,

$$(2.6)\qquad u^0(r,t) = \frac{1}{\pi r^2}\int_0^t \frac{\rho\mathrm{d}\rho}{\sqrt{t^2-\rho^2}} \times \int_{|r-\rho|}^{r+\rho}\frac{\lambda^2+r^2-\rho^2}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}}\lambda\psi(\lambda)\mathrm{d}\lambda\,,$$

$$(2.7)\qquad L(F(u_t))(r,t) = \frac{1}{\pi r^2}\int_0^t \mathrm{d}\tau\int_0^{t-\tau}\frac{\rho\mathrm{d}\rho}{\sqrt{(t-\tau)^2-\rho^2}} \times \int_{|r-\rho|}^{r+\rho}\frac{\lambda^2+r^2-\rho^2}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}}\lambda F(u_t(\lambda,\tau))\mathrm{d}\lambda\,,$$

for $n = 5$,

$$(2.8)\qquad u^0(r,t) = \frac{1}{4r^3}\int_{|r-t|}^{r+t}(\lambda^2+r^2-t^2)\lambda\psi(\lambda)\mathrm{d}\lambda\,,$$

$$(2.9)\qquad L(F(u_t))(r,t) = \frac{1}{4r^3}\int_0^t \mathrm{d}\tau\int_{|r-t+\tau|}^{r+t-\tau}(\lambda^2+r^2-(t-\tau)^2)\lambda F(u_t(\lambda,\tau))\mathrm{d}\tau\,.$$

Proof of Lemma 2.2 First, let $n = 4$. We note that $\omega_3 = 4\pi$. It follows from (2.2) and (2.4) that

$$(2.10)\quad u^0(r,t)=\frac{1}{2\pi r^2}\frac{1}{t}\frac{\partial}{\partial t}\int_0^t\frac{\rho d\rho}{\sqrt{t^2-\rho^2}}$$
$$\times\int_{|r-\rho|}^{r+\rho}[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)]^{(n-3)/2}\lambda\psi(\lambda)d\lambda\,.$$

Integration by parts in the ρ-integral yields that

$$(2.11)\quad u^0(r,t)=\frac{1}{2\pi r^2}\frac{1}{t}\frac{\partial}{\partial t}\int_0^t\sqrt{t^2-\rho^2}d\rho$$
$$\times\frac{\partial}{\partial\rho}\int_{|r-\rho|}^{r+\rho}[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)]^{(n-3)/2}\lambda\psi(\lambda)d\lambda$$
$$=\frac{1}{2\pi r^2}\int_0^t\frac{d\rho}{\sqrt{t^2-\rho^2}}$$
$$\times\int_{|r-\rho|}^{r+\rho}\frac{\partial}{\partial\rho}[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)]^{(n-3)/2}\lambda\psi(\lambda)d\lambda\,.$$

Since

$$(2.12)\qquad \frac{\partial}{\partial\rho}[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)]^{(n-3)/2}$$
$$=\frac{2(n-3)\rho(\lambda^2+r^2-\rho^2)}{[(\lambda^2-(r-\rho)^2)((r+\rho)^2-\lambda^2)]^{(5-n)/2}}\,,$$

we get (2.6). Therefore, (2.5) for $n=4$ immediately follows from Duhamel's principle.

Next, let $n=5$. We note that $\omega_4=2\pi^2$. It follows from (2.3) and (2.4) that

$$(2.13)\quad u^0(r,t)=\frac{1}{16r^3}\frac{1}{t}\frac{\partial}{\partial t}\int_{|r-t|}^{r+t}[(\lambda^2-(r-t)^2)((r+t)^2-\lambda^2)]^{(n-3)/2}\lambda\psi(\lambda)d\lambda$$
$$=\frac{1}{16r^3}\frac{1}{t}\int_{|r-t|}^{r+t}\frac{\partial}{\partial t}[(\lambda^2-(r-t)^2)((r+t)^2-\lambda^2)]^{(n-3)/2}\lambda\psi(\lambda)d\lambda\,.$$

Using (2.12) with $\rho=t$, we get (2.8). As in the case $n=4$, (2.5) for $n=5$ immediately follows from Duhamel's principle again.

Lemma 2.3 *Let $n=4,5$ and u be a solution to (1.5) with $\varphi(r)\equiv 0,\psi(r)\geqq 0$ for $r\equiv|x|,x\in\mathbb{R}^n$, and with $F\geqq 0$. Then, for $r\geqq t>0$, we have*

$$(2.14)\qquad u(r,t)\geqq\varepsilon u^0(r,t)+\frac{1}{8r^3}\int_0^t d\tau\int_{r-t+\tau}^{r+t-\tau}\lambda^3F(u_t(\lambda,\tau))d\lambda\,,$$

$$(2.15)\qquad u^0(r,t)\geqq\frac{1}{8r^3}\int_{r-t}^{r+t}\lambda^3\psi(\lambda)d\lambda\,.$$

Proof of Lemma 2.3 First, let $n = 4$. It follows from (2.6) that

$$u^0(r,t) \geqq \frac{1}{\pi r^2}\int_0^t \frac{\rho d\rho}{\sqrt{t^2-\rho^2}} \times \int_{r-\rho}^{r+\rho} \frac{\lambda^3\psi(\lambda)d\lambda}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}} \tag{2.16}$$

because $r-\rho \geqq r-t \geqq 0$. Then, inverting the order of (λ,ρ)-integral, we find that

$$u^0(r,t) \geqq \frac{1}{\pi r^2}\int_{r-t}^{r+t} \lambda^3\psi(\lambda)d\lambda \times \int_{|r-\lambda|}^{t} \frac{\rho d\rho}{\sqrt{t^2-\rho^2}\sqrt{\rho^2-(r-\lambda)^2}\sqrt{(r+\lambda)^2-\rho^2}}. \tag{2.17}$$

In the domain of (ρ,λ)-integral, we know that

$$\begin{aligned}(r+\lambda)^2-\rho^2 &= (r+\lambda+\rho)(r+\lambda-\rho)\\ &\leqq 2(r+t)(r+\lambda-|r-\lambda|)\\ &\leqq 8r^2.\end{aligned} \tag{2.18}$$

Since

$$\int_a^b \frac{\rho d\rho}{\sqrt{\rho^2-a^2}\sqrt{b^2-\rho^2}} = \frac{1}{2}B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\pi}{2} \quad \text{for} \quad 0 \leqq a < b, \tag{2.19}$$

it follows from (2.17) and (2.18) that

$$u^0(r,t) \geqq \frac{1}{4\sqrt{2}r^3}\int_{r-t}^{r+t} \lambda^3\psi(\lambda)d\lambda. \tag{2.20}$$

In view of (2.5) and (2.7), replacing t by $t-\tau$ in (2.16)–(2.18) and using $r-t \geqq 0$, we have

$$u(r,t) \geqq \varepsilon u^0(r,t) + \frac{1}{4\sqrt{2}r^3}\int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^3 F(u_t(\lambda,\tau))d\lambda. \tag{2.21}$$

We therefore obtain (2.14) and (2.15) for $n = 4$.

Next, let $n = 5$. It follows from (2.8) and $r-t \geqq 0$ that

$$u^0(r,t) \geqq \frac{1}{4r^3}\int_{r-t}^{r+t} \lambda^3\psi(\lambda)d\lambda. \tag{2.22}$$

Similarly to the case $n = 4$, using $r-t \geqq 0$ again, (2.5) and (2.9) yield that

$$u(r,t) \geqq \varepsilon u^0(r,t) + \frac{1}{4r^3}\int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^3 F(u_t(\lambda,\tau))d\lambda, \tag{2.23}$$

which completes the proof of Lemma 2.2.

3 Proof of Theorem 1.1

Let u be a solution of (1.5) with (1.6) and (1.7). If t is larger than some constant, we will have a contradiction. This implies that $T(\varepsilon)$ has an upper bound.

By (1.6) and (1.7), we find from Lemma 2.3 that

$$u(r,t) \geqq \varepsilon u^0(r,t) \geqq \frac{\varepsilon}{8r^3} \int_{r-t}^{r+t} \lambda^3 \psi(\lambda) d\lambda \geqq \frac{M\varepsilon}{8r^3} \int_{r-t}^{r+t} \lambda^3 (1+\lambda)^{-\kappa} d\lambda \quad \text{for} \quad r-t \geqq 0 . \tag{3.1}$$

First, we assume that $r - t \geqq \delta$ for fixed $\delta > 0$. Then, it follows from (3.1) that

$$u(r,t) \geqq \frac{M\varepsilon}{8r^3} \left(\frac{1+\delta}{\delta} \right)^{-\kappa} \int_{r-t}^{r+t} \lambda^{3-\kappa} d\lambda \geqq \frac{M\varepsilon}{8r^3} \left(\frac{1+\delta}{\delta} \right)^{-\kappa} (r+t)^{-\kappa} \int_{r-t}^{r+t} \lambda^3 d\lambda . \tag{3.2}$$

Hence, we have

$$u(r,t) \geqq \frac{C_0 t^3}{r^2 (r+t)^\kappa} \quad \text{for} \quad r - t \geqq \delta , \tag{3.3}$$

where we set

$$C_0 = \varepsilon \frac{M}{2} \left(\frac{\delta}{1+\delta} \right)^\kappa > 0 . \tag{3.4}$$

Now, we assume an estimate of the form

$$u(r,t) \geqq \frac{C t^a}{r^2 (r+t)^b} \quad \text{for} \quad r - t \geqq \delta . \tag{3.5}$$

All constants a, b, C are positive. We note that (3.5) is true with $a = 3, b = \kappa, C = C_0$. Then, it follows from (1.6), (1.7) and Lemma 2.3 that, for $r - t \geqq \delta$,

$$u(r,t) \geqq \frac{A}{8r^3} \int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^3 |u_t(\lambda,\tau)|^p d\lambda . \tag{3.6}$$

Inverting the order of (λ, τ)-integral, we find that, for $r - t \geqq \delta$,

$$u(r,t) \geqq \frac{A}{8r^3} \left(\int_{r-t}^{r} d\lambda \int_0^{\lambda-(r-t)} d\tau + \int_r^{r+t} d\lambda \int_0^{r+t-\lambda} d\tau \right) \lambda^3 |u_t(\lambda,\tau)|^p \geqq \frac{A}{8r^3} \int_r^{r+t} \lambda^3 d\lambda \int_0^{r+t-\lambda} |u_t(\lambda,\tau)|^p d\tau . \tag{3.7}$$

Hölder's inequality yields that

$$\left|\int_0^{r+t-\lambda} u_t(\lambda,\tau)d\tau\right|^p \leqq (r+t-\lambda)^{p-1}\int_0^{r+t-\lambda}|u_t(\lambda,\tau)|^p d\tau\,. \tag{3.8}$$

Since $u(\lambda,0)=\varepsilon\varphi(\lambda)\equiv 0$ by (1.7), we obtain from (3.7) and (3.8) that, for $r-t\geqq\delta$,

$$u(r,t)\geqq\frac{A}{8r^3}\int_r^{r+t}\lambda^3(r+t-\lambda)^{1-p}|u(\lambda,r+t-\lambda)|^p d\lambda\,. \tag{3.9}$$

Noticing that $\lambda-(r+t-\lambda)\geqq r-t\geqq\delta$ in the domain of the integral, we get by (3.5) and (3.9) that, for $r-t\geqq\delta$,

$$\begin{aligned} u(r,t) &\geqq \frac{AC^p}{8r^3(r+t)^{pb}}\int_r^{r+t}\lambda^{3-2p}(r+t-\lambda)^{1-p+pa}d\lambda \\ &\geqq \frac{AC^p}{8r^3(r+t)^{pb+3(p-1)}}\int_r^{r+t}\lambda(r+t-\lambda)^{pa}d\lambda \\ &\geqq \frac{AC^p}{8r^2(r+t)^{pb+3(p-1)}}\int_r^{r+t}(r+t-\lambda)^{pa}d\lambda\,. \end{aligned} \tag{3.10}$$

Hence, we have

$$u(r,t)\geqq\frac{C^*t^{a^*}}{r^2(r+t)^{b^*}}\quad\text{for}\quad r-t\geqq\delta \tag{3.11}$$

with

$$a^*=pa+1, b^*=pb+3(p-1), C^*=\frac{AC^p}{8(pa+1)}\,. \tag{3.12}$$

In view of (3.3), (3.4), (3.11), (3.12), we define sequences $\{a_m\},\{b_m\},\{C_m\}$ for $m\in\mathbb{N}$ by

$$\begin{aligned} &a_{m+1}=pa_m+1, \quad b_{m+1}=pb_m+3(p-1), \quad C_{m+1}=\frac{AC_m^p}{8(pa_m+1)}, \\ &a_1=3, \qquad\qquad b_1=\kappa, \qquad\qquad . \quad C_1=C_0\,. \end{aligned} \tag{3.13}$$

Solving these sequences, we have

$$\begin{aligned} &a_{m+1}=p^m\left(3+\frac{1}{p-1}\right)-\frac{1}{p-1}, \ b_{m+1}=p^m(\kappa+3)-3 \\ &C_{m+1}\geqq N\frac{C_m^p}{p^m}, \quad\text{where}\quad N=\frac{A(p-1)}{24p}\,. \end{aligned} \tag{3.14}$$

The last line of (3.14) implies that

$$C_{m+1}\geqq\exp[p^m(\log C_0-S_p(m))]\,, \tag{3.15}$$

where we set

$$S_p(m) = \sum_{j=1}^{m} d_j, \quad d_j = \frac{j \log p - \log N}{p^j}\,. \tag{3.16}$$

We note that $d_j > 0$ for sufficiently large j. Since $\lim_{j\to\infty} d_{j+1}/d_j = 1/p, S_p(m)$ converges for $p > 1$ as $m \to \infty$ by d'Alembert's criterion on series with positive terms. Hence, there is a positive constant $S_{p,A}$ such that

$$C_{m+1} \geqq \exp(\,p^m(\log C_0 - S_{p,A}))\,. \tag{3.17}$$

Therefore, we conclude by (3.5), (3.14), (3.17) that

$$u(r,t) \geqq \frac{(r+t)^3}{r^2 t^{1/(p-1)}} \exp(\,p^m J(r,t)) \quad \text{for} \quad r - t \geqq \delta\,, \tag{3.18}$$

where we set

$$J(r,t) = \log C_0 - S_{p,A} + \left(3 + \frac{1}{p-1}\right) \log t - (\kappa + 3) \log(r+t)\,. \tag{3.19}$$

If there is a point (r_0, t_0) such that

$$J(r_0, t_0) > 0, r_0 - t_0 \geqq \delta\,, \tag{3.20}$$

we get a desired contradiction:

$$u(r_0, t_0) \to \infty \quad \text{as} \quad m \to \infty \tag{3.21}$$

by (3.18), which shows that u cannot be a solution of (1.5). So we shall find such a point.

By (3.19), $J(r,t) > 0$ follows from

$$\left(\frac{1}{p-1} - \kappa\right) \log t \geqq \log \frac{3^{\kappa+3} e^{S_{p,A}}}{C_0} \quad \text{for} \quad t \geqq r - t \geqq \delta\,. \tag{3.22}$$

In view of (1.7) and (3.4), (3.22) follows from

$$t \geqq C\varepsilon^{-(p-1)/(1-(p-1)\kappa)} \quad \text{for} \quad t \geqq r - t \geqq \delta\,, \tag{3.23}$$

where

$$C = \left(\frac{2 \cdot 3^{\kappa+3} e^{S_{p,A}}}{M} \left(\frac{1+\delta}{\delta}\right)\kappa\right)^{(p-1)/(1-(p-1)\kappa)} > 0\,. \tag{3.24}$$

Therefore, we can conclude that

$$T(\varepsilon) \leqq C\varepsilon^{-(p-1)/(1-(p-1)\kappa)} \tag{3.25}$$

with C defined by (3.24). This inequality completes the proof of Theorem 1.1.

References

1. Agemi, R.: Blow-up of solutions to nonlinear wave equations in two space dimensions. Manuscripta Math. **73**, 153–162 (1991)
2. Asakura, F.: Existence of spherically symmetric global solution to the semi-linear wave equation $u_{tt} - \Delta u = au_t^2 + b(\nabla u)^2$ in five space dimensions. J. Math. Kyoto Univ. **24**, 361–380 (1984)
3. Courant, R., Hilbert, D.: Methods of Mathematical Physics II. New York: Interscience 1962
4. Glassey, R.T.: Blow-up theorems for nonlinear wave equations. Math. Z. **132**, 183–203 (1973)
5. John, F.: Plane waves and spherical means applied to partial differential equations. New York: Interscience 1955
6. John, F.: Blow-up for quasi-linear wave equations in three space dimensions. Comm. Pure Appl. Math. **34**, 29–51 (1981)
7. John, F.: Non-existence of global solutions of $\Box u = \frac{\partial}{\partial t}F(u_t)$ in two or three space dimensions. Rend. Circ. Mat. Palermo (2) Suppl. **8**, 229–249 (1985)
8. Klainerman, S.: Uniform decay estimates and the Lorentz invariance of the classical wave equation. Comm. Pure Appl. Math. **38**, 321–332 (1985)
9. Klainerman, S.: Remarks on the global Sobolev inequalities in Minkowski space $\mathbb{R}^n$. Comm. Pure Appl. Math. **40**, 111–117 (1987)
10. Kubo, H.: Blow-up for semilinear wave equations with initial data of slow decay in low space dimensions. Differential Integral Equations **7**, 315–321 (1994)
11. Kubo, H.: Asymptotic behaviors of solutions to semilinear wave equations with initial data of slow decay. Hokkaido Univ. Preprint Ser. Math. #196 (1993) (to appear) Math. Meth. Appl. Sci.
12. Masuda, K.: Blow-up of solutions for quasi-linear wave equations in two space dimensions. Lecture Notes in Num. Appl. Anal. **6**, 87–91 (1983)
13. Rammaha, M.A.: Finite-time blow-up for nonlinear wave equations in high dimensions. Comm. Partial Differential Equations. **12(6)**, 677–700 (1987)
14. Schaeffer, J.: Wave equation with positive nonlinearities. Ph. D. Thesis, Indiana University (1983)
15. Schaeffer, J.: Finite-time blow-up for $u_{tt} - \Delta u = H(u_r, u_t)$ in two space dimensions. Comm. Partial Differential Equations. **11(5)**, 513–543 (1986)
16. Sideris, T.C.: Global behavior of solutions to nonlinear wave equations in three space dimensions. Comm. Partial Differential Equations. **8(12)**, 1291–1323 (1983)
17. Takamura, H.: Global existence for nonlinear wave equations with small data of noncompact support in three space dimensions. Comm. Partial Differential Equations. **17(1&2)**, 189–204 (1992)
18. Takamura, H.: Blow-up for semilinear wave equations in four or five space dimensions. Mathematical Reseach Note 93-001, University of Tsukuba (to appear in Nonlinear Anal.)

Added in proof

Using Rammaha's representation formulas in [13], one can see that the assertion of Theorem 1.1 is true for all $n \geqq 2$. The estimate $\lambda^2 + r^2 - t^2 \geqq \lambda^2$ in the proof of Lemma 2.3 should be replaced by $\lambda^2 + r^2 - t^2 \geqq r^2 - t^2$. Details of this proof will be published elsewhere. Anyway, this paper essentially proves the blow-up for slowly decaying data in high space dimensions.