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Affine isoparametric hypersurfaces

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1 Introduction

An isoparametric hypersurface in Euclidean space is one whose principal curvatures are constant. In this paper, we introduce the notion of isoparametric hypersurface in affine differential geometry. The affine shape operator is symmetric with respect to the affine metric, but the affine metric is in general indefinite. Therefore, the shape operator need not be diagonalizable and the principal curvatures may not be real.

Much recent work in affine differential geometry has involved the study of hypersurfaces which are isoparametric in our sense. In particular, the affine homogeneous surfaces, studied classically, and classified by Nomizu and Sasaki, [NS], are isoparametric. Also the affine spheres, whose affine normal lines are all parallel or all pass through a common point, are included. Unlike the Euclidean case though, there are immediate examples of affine isoparametric hypersurfaces and even affine spheres which are not homogeneous. In particular there are families of non-homogeneous ruled surfaces having both principal curvatures identically zero [MR], [V]. The latter paper includes examples for which the shape operator is not diagonalizable.

In the study of Riemannian isoparametric hypersurfaces, the concept of parallel hypersurface is important, since isoparametric hypersurfaces occur in parallel families. It would be hoped that the study of parallel hypersurfaces would also be useful in discussing affine isoparametric hypersurfaces. However, it is not immediately clear that the concept of parallel hypersurface is reasonable, since there is no guarantee in the affine case that the affine normal of the parallel hypersurface will agree with the affine normal of the original hypersurface. In the present paper, we show (Theorem 3.1) that an affine hypersurface has this prop-

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erty precisely when it is isoparametric. In §4 we briefly discuss a few examples, and apply our results to the study of symmetry about the focal set.

Precise definitions are given in §2. For further information about isoparametric hypersurfaces, see [CR], and for affine hypersurfaces, see [N] and [MR]. All manifolds are assumed connected unless stated otherwise. Manifolds and maps are assumed smooth, i.e. C^∞.

2 Preliminaries

Let f be an immersion of a smooth manifold M^n, $n \geq 2$, into $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$ with a fixed parallel volume element ω defined by

$$\omega(X_1, \dots, X_{n+1}) = \det(X_1, \dots, X_{n+1}).$$

Choose any transversal vector field ξ locally on M. Then for X, Y tangent to M we have

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi,$$

$$D_X \xi = -f_*(SX) + \tau(X)\xi,$$

where ∇ is the *induced affine connection* on M, the symmetric bilinear tensor h is the *affine fundamental form* , S is the *shape operator*, and τ is the *transversal connection form*. We also have the induced volume form θ, defined by

$$\theta(X_1, \dots, X_n) = \omega(f_*X_1, \dots, f_*X_n, \xi).$$

A frame on which $\theta > 0$ will be called *positively oriented*. Whether h is degenerate or not depends only on f, and is independent of the choice of ξ. If h is nondegenerate, we say that f is a *nondegenerate* affine immersion. The possibly indefinite Riemannian metric h determines a volume element ν which we choose to have the same sign as θ. For a positively oriented frame $(X_1, \dots, X_n)$, we have

$$\nu(X_1, \dots, X_n) = \sqrt{|\det[h(X_i, X_j)]|}.$$

It is well known that we can choose ξ uniquely (up to a sign) so that $\tau = 0$ and $\nu = \theta$. This makes θ parallel with respect to ∇. This choice of ξ is called the *affine normal* and the corresponding h is the *affine metric*.

A nondegenerate affine immersion $f : M^n \to \mathbb{R}^{n+1}$ is said to be *isoparametric* if the characteristic polynomial of the shape operator S has constant coefficients. The eigenvalues of S are called the *affine principal curvatures* or simply the *principal curvatures* if the context is clear. If f is isoparametric, the principal curvatures are constant. If the principal curvatures are all real, we will represent them by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Notice that selecting the opposite sign for ξ simply reverses the sign of the λ_i's. If all principal curvatures are equal, the hypersurface is called an *affine sphere*.

3 Parallel hypersurfaces and focal sets

For a given immersion f, and any constant t, define the map $f_t : M^n \longrightarrow \mathbb{R}^{n+1}$ by $f_t = f + t\xi$. For local calculations, one can assume that M is orientable, with affine normal ξ. If x is not a critical point of f_t, then f_t is an immersion on a neighborhood of $x \in M$, and f_t is called the *parallel hypersurface to f at distance t*.

In the Euclidean case, the unit normal ξ_x to a given hypersurface f at x is also normal to f_t at x. In the affine case though, it is not generally true that the affine normal direction ξ_x at x is also the affine normal direction to f_t at x. The following Theorem shows in fact that this is true only when f is isoparametric.

Theorem 3.1 *Let $f : M^n \longrightarrow \mathbb{R}^{n+1}$ be a nondegenerate affine immersion with shape operator S. If f is isoparametric, then for any parallel hypersurface f_t, the affine normal associated with f_t is proportional to ξ. Conversely, if for some open interval U, f_t is a hypersurface with affine normal proportional to ξ for all $t \in U$, then f is isoparametric.*

Proof. We will write $\bar{f}$ for $f_t = f + t\xi$, use bars to identify objects associated with $\bar{f}$, and omit the subscript t. If we assume that f is simply the inclusion map of a submanifold M of $\mathbb{R}^{n+1}$, then we can write $\bar{f}_* = I - tS$, where I is the identity transformation.

Let M_0 be a connected open subset of M on which $\bar{f}$ is an immersion. If $\bar{\xi}$ is any transverse vector field to $\bar{f}$ on M_0, there is a smooth function $r = r(t) : M_0 \longrightarrow \mathbb{R}$ and a vector field Z tangent to M_0 such that $\bar{\xi} = r\xi + Z$. Then for X and Y tangent to M_0,

$$\begin{aligned} D_X\bar{f}_*(Y) &= \bar{f}_*(\overline{\nabla}_X Y) + \bar{h}(X,Y)\bar{\xi} \\ &= \overline{\nabla}_X Y - tS(\overline{\nabla}_X Y) + \bar{h}(X,Y)r\xi + \bar{h}(X,Y)Z. \end{aligned} \tag{3.1}$$

But also

$$\begin{aligned} D_X\bar{f}_*(Y) &= D_X(Y - tSY) \\ &= \nabla_X Y + h(X,Y)\xi - t(\nabla_X SY + h(X,SY)\xi). \end{aligned} \tag{3.2}$$

Then matching up coefficients of ξ we see that

$$\bar{h}(X,Y) = \frac{1}{r}h(X,(I - tS)Y).$$

Thus $\bar{h}$ is automatically nondegenerate on M_0. In particular, if $\bar{\xi}$ is an affine normal for $\bar{f}$, we can write

$$\begin{aligned} D_X\bar{\xi} &= -\bar{f}_*(\bar{S}X) \\ &= -(I - tS)\bar{S}X, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} D_X\bar{\xi} &= D_X(r\xi + Z) \\ &= (Xr)\xi - rSX + \nabla_X Z + h(X,Z)\xi. \end{aligned} \tag{3.4}$$

Again matching coefficients of ξ gives $0 = Xr + h(X,Z)$, and so for a given value of t we know that r is constant if and only if $Z \equiv 0$, that is, ξ and $\bar{\xi}$ are proportional.

Now let us explicitly calculate $r(t)$. Let $X_1,\dots,X_n$ be a local frame on M_0 such that $\theta(X_1,\dots,X_n) = \omega(X_1,\dots,X_n,\xi) = 1$. Then since ξ is the affine normal, we have $\nu = \theta$, so that

$$\nu(X_1,\dots,X_n) = \sqrt{|\det[h(X_i,X_j)]|} = 1.$$

Now, let $\varphi = \varphi(t) = \det(I - tS)$. Let $\bar{\xi}$ be the affine normal of $\bar{f}$ which induces the same orientation as ξ. Then

$$\begin{aligned} \bar{\nu}(X_1,\dots,X_n) &= \sqrt{|\det[\bar{h}(X_i,X_j)]|} \\ &= \sqrt{|\det[\tfrac{1}{r}h(X_i,(I - tS)X_j)]|} \\ &= \frac{1}{|r|^{\frac{n}{2}}}|\det(I - tS)\det[h(X_i,X_j)]|^{\frac{1}{2}} \\ &= \frac{1}{|r|^{\frac{n}{2}}}|\det(I - tS)|^{\frac{1}{2}}\nu(X_1,\dots,X_n) \\ &= \frac{1}{|r|^{\frac{n}{2}}}|\varphi|^{\frac{1}{2}}. \end{aligned}$$

But we can also calculate that

$$\begin{aligned} \bar{\theta}(X_1,\dots,X_n) &= \omega(\bar{f}_*X_1,\dots,\bar{f}_*X_n,\bar{\xi}) \\ &= \omega((I - tS)X_1,\dots,(I - tS)X_n, r\xi) \\ &= r\det(I - tS)\,\omega(X_1,\dots,X_n,\xi) \\ &= r\varphi\,\theta(X_1,\dots,X_n). \end{aligned}$$

Because $\bar{\xi}$ is the affine normal of $\bar{f}$, we know that $\bar{\nu} = \bar{\theta}$. This allows us to calculate r, and we see that

$$r(t) = \pm|\det(I - tS)|^{-\frac{1}{n+2}} = \pm|\varphi|^{-\frac{1}{n+2}}.$$

Note that the characteristic polynomial of S is just $t^n\varphi(1/t)$. Thus, if f is isoparametric, $\det(I - tS)$ is constant on M, $\bar{f}$ is an immersion on all of M, and $r(t)$ is constant. Conversely, suppose that $\bar{f} : M_0 \to \mathbb{R}^{n+1}$ is a parallel hypersurface at distance t whose affine normal $\bar{\xi}$ satisfies $\bar{\xi} = r(t)\xi$. Then $r(t)$ is constant by (3.4) and hence $\varphi(t) = \det(I - tS)$ is also constant. If this condition holds for all t in some interval U, then the coefficients of the characteristic polynomial of S can be expressed in terms of the values of φ and its derivatives at some $t_0 \in U$. Thus they must be constants and we can conclude that f is isoparametric.

□

We also have an interest in what happens to the shape operator when taking a parallel hypersurface.

Proposition 3.2 *If $f : M^n \longrightarrow \mathbb{R}^{n+1}$ is an isoparametric hypersurface, then so is $\bar{f} = f + t\xi$ provided that $I - tS$ is invertible. Its shape operator is given by*

$$\overline{S} = r(I - tS)^{-1}S.$$

In particular, if S is diagonalizable, and $X_1, \ldots, X_n$ is a basis of eigenvectors of the shape operator S, such that $SX_i = \lambda_i X_i$, then the shape operator $\overline{S}$ of $\bar{f}$ is given by

$$\overline{S}X_i = \pm\left(\prod_{j=1}^{j=n} |1 - t\lambda_j|\right)^{-\frac{1}{n+2}} \frac{\lambda_i}{1 - t\lambda_i} X_i,$$

or more briefly,

$$\overline{S}X_i = \frac{r(t)\lambda_i}{1 - t\lambda_i} X_i.$$

Proof. Match coefficients of X in equations (3.3) and (3.4).

□

Define a point $p \in \mathbb{R}^{n+1}$ to be a *focal point of multiplicity m of f* if there is some real number λ, and $x \in M$ such that $p = f(x) + (1/\lambda)\xi_x$, and the Jacobian of the function $F(x) = f(x) + (1/\lambda)\xi_x$ has nullity $m > 0$ at x. The set of all focal points is called the *focal set of f*. In the discussion of isoparametric immersions in Euclidean space, the focal set plays an important role, and the same will be true in the affine case.

Theorem 3.3 *If $f : M^n \longrightarrow \mathbb{R}^{n+1}$ is an isoparametric immersion and $1/t$ is not an eigenvalue of the shape operator S, then f_t has the same focal set as f.*

Proof. Again write $\bar{f}$ for f_t. If λ is a nonzero eigenvalue of S, then

$$\bar{\lambda} = \frac{r(t)\lambda}{1 - t\lambda}$$

is an eigenvalue of $\overline{S}$, the shape operator for $\bar{f} = f + t\xi$, with the same eigenspace as λ. For any $x \in M$,

$$\begin{aligned} \bar{F}(x) = \bar{f}(x) + (1/\bar{\lambda})\bar{\xi}_x &= f(x) + t\xi_x + \frac{1 - t\lambda}{r(t)\lambda} r(t)\xi_x \\ &= f(x) + (1/\lambda)\xi_x. \end{aligned}$$

Also, $\bar{F}_*$ and F_* have the same null space. Thus the focal points of f and $\bar{f}$ match.

□

4 Isoparametric hypersurfaces that are symmetric about the focal set

Let $f : M^n \longrightarrow \mathbb{R}^{n+1}$ be an isoparametric immersion into Euclidean space. It is known that f extends uniquely to an embedding of a complete manifold (sphere, cylinder, or hyperplane). So we might as well assume that the original M is complete. If ξ is a unit normal field to f, then one can verify that the map $\bar{f} = f + (2/\lambda)\xi$ has an image identical to that of f, when λ is a nonzero constant principal curvature of the immersion f. In the affine case, the situation is somewhat more complicated. For example, the set $V = \{x \in \mathbb{R}^3 | (x_1^2 - x_2^2)x_3 = \pm 1\}$ has 8 components, each of which is a homogeneous affine sphere with principal curvature $3^{-\frac{3}{4}}\sqrt{2}$ having the origin as the only focal point. With appropriate choice of ξ, we see that $\bar{f}(V) = V$. Although V does not qualify as an affine hypersurface (since it is not connected), it does exhibit symmetry about the focal set. Specifically, $\bar{f}(x) = -x$ for $x \in V$. Another standard example, the set $(x_1^2 + x_2^2)x_3 = \pm 1$ behaves similarly. It has just two components and they are interchanged by $\bar{f}$.

It should be noted that all homogeneous affine surfaces are isoparametric and are also symmetric about the focal set in this sense. These are described in [NS] and include surfaces that are not affine spheres. For instance, $(x_3 - x_2^2)^3 x_1^2 = \pm 1$, in other words, $x_3 = \pm x_1^{-\frac{2}{3}} + x_2^2$, has four connected components, and the focal set is the parabola $x_3 = x_2^2$, $x_1 = 0$. The components are separated by the plane $x_1 = 0$ and the parabolic cylinder $x_3 = x_2^2$. The principal curvatures are $\lambda_1 = 0$ and $\lambda_2 = -5^{-\frac{3}{4}}\sqrt{6}$ and the affine normal is $\xi = (-\lambda_2)(x_1, 0, x_1^{-\frac{2}{3}})$. If, for example, $f(x) = (x_1, x_2, x_1^{-\frac{2}{3}} + x_2^2)$, describes the sheet in the region $x_1 > 0$, $x_3 > x_2^2$, then $f + (2/\lambda_2)\xi$ is the sheet in the region $x_1 < 0$, $x_3 < x_2^2$.

This example is analogous to the Euclidean cylinder. It has two constant distinct principal curvatures, one of which is zero. In fact, every affine hypersurface that is symmetric about the focal set possesses this property or is an affine sphere. Specifically, we have the following Theorem:

Theorem 4.1 *Let $f : M^n \longrightarrow \mathbb{R}^{n+1}$ be an affine isoparametric hypersurface. Suppose that the principal curvatures are all real and not all zero. If for all nonzero principal curvatures λ, $f + (2/\lambda)\xi$ is an isoparametric immersion with the same principal curvatures as f, then f is an affine sphere or has exactly two distinct principal curvatures. One of these principal curvatures is 0.*

Proof. We will first show that all the principal curvatures have the same sign. Suppose that the principal curvatures are arranged such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Without loss of generality, we can assume that at least one principal curvature is positive. Let $\lambda = \lambda_k$ be the smallest positive principal curvature. Then under the map $\bar{f} = f + (2/\lambda)\xi$, the principal curvatures change as follows

$$\bar{\lambda}_i = \frac{r(2/\lambda)\lambda_i}{1 - (2/\lambda)\lambda_i}.$$

Notice that if λ_i is positive, then the sign of $\bar{\lambda}_i$ is $-\mathrm{sgn}(r(2/\lambda))$. But also notice that if λ_i is negative, then the sign of $\bar{\lambda}_i$ is also $-\mathrm{sgn}(r(2/\lambda))$. Thus all the nonzero $\bar{\lambda}_i$ have the same sign. Since f and $\bar{f}$ have the same principal curvatures, all the nonzero λ_i must be positive.

Suppose that $r(2/\lambda) < 0$. If $\lambda_i < \lambda_j$ are two nonzero principal curvatures, it is straightforward to verify that

$$\bar{\lambda}_i = \frac{r(2/\lambda)\lambda_i}{1-(2/\lambda)\lambda_i} > \bar{\lambda}_j = \frac{r(2/\lambda)\lambda_j}{1-(2/\lambda)\lambda_j}.$$

Thus $\bar{f}$ reverses the order of the principal curvatures, and hence $\bar{\lambda}_k = \lambda_n$. We can then use this to get $\lambda_n = -r(2/\lambda)\lambda$. We also know that $\bar{\lambda}_n = \lambda_k$, and from this equation we can then show that $r(2/\lambda) = -1$. It follows that all the nonzero principal curvatures are equal. On the other hand, if $r(2/\lambda) > 0$, then all the nonzero $\bar{\lambda}_i$ are negative. However, the argument just given applies to the $-\bar{\lambda}_i$ and again, all the nonzero principal curvatures are equal.

□

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