Math. Z. 217, 179–197 (1994)

Mathematische Zeitschrift

Harnack inequalities for evolving hypersurfaces★

Ben Andrews

Centre for Mathematics and its Applications, Australian National University, A.C.T. 0200, Australia

Received 19 November 1992; in final form 25 May 1993

1 Introduction

Harnack inequalities for parabolic equations originate with the work of Moser [M] who treated the case of linear divergence-form equations. In this context, the inequality estimates a solution from below, in terms of the values it attains on an earlier region of the parabolic domain. Inequalities of this type have recently appeared for many geometric evolution equations, including several quasilinear and fully nonlinear examples. These new developments began with Li and Yau [LY], who showed how to obtain a Harnack inequality for the heat equation by clever use of the parabolic maximum principle. Similar techniques were employed by Hamilton, who proved Harnack inequalities for various nonlinear evolution equations – the flow of Riemannian metrics by their Ricci curvature in two dimensions [Ha1], the mean curvature flow of hypersurfaces in Euclidean space, and several scalar equations [Ha2]. Chow has treated flows of hypersurfaces in Euclidean space by powers of the Gauss curvature [Ch3], and also the flow of Riemannian metrics by the gradient of the Yamabe functional [Ch4]. Most recently, Hamilton has proved a Harnack inequality for the higher-dimensional Ricci curvature flow [Ha3].

Throughout this paper, M^n will be a compact, smooth n-dimensional Riemannian manifold. We consider a smoothly evolving one-parameter family of immersions described by a map φ: $[0, T) \times M^n \to \mathbb{R}^{n+1}$, where the evolution is governed by an equation of the following form:

$$\frac{\partial}{\partial t}\varphi(x, t) = -F(\mathscr{W}(x, t), \nu(x, t))\nu(x, t) \tag{1.1}$$

★ This work was carried out while the author was supported by an Australian Postgraduate Research Award and an ANUTECH scholarship

where $\nu: M^n \times [0, T) \to S^n$ is a unit normal to the hypersurface $\varphi_t(M^n)$ at the point $\varphi_t(x)$, and $\mathscr{W}$ is the Weingarten map (see Sect. 2 for details). The function F gives the speed of motion of the hypersurfaces through $\mathbb{R}^{n+1}$. We require that Eq. (1.1) is a parabolic, second order system of partial differential equations, which is invariant under diffeomorphisms of M^n, translations of φ in $\mathbb{R}^{n+1}$, and translations in time. The precise form implied by these conditions will be discussed in Sect. 3. In many cases we may wish to consider the stronger condition that Eq. (1.1) must be invariant under all isometries of $\mathbb{R}^{n+1}$; equations which satisfy this are called **isotropic**, while those that only satisfy the weaker condition are called **anisotropic**.

The purpose of this paper is to prove Harnack inequalities for a wide class of such evolution equations for hypersurfaces. It is shown that natural Harnack inequalities hold for convex solutions to equations of the form (1.1), provided that the speed function S satisfies some natural concavity properties when considered as a function of the inverse of the Weingarten map $\mathscr{W}$. The first step of the proof is to show that a certain natural quantity (to leading order, the time derivative of the speed of the hypersurface) satisfies a simple and useful evolution equation (Lemma 5.1). In the cases which have been proved previously by Hamilton and Chow, this step is accomplished by performing a long and cumbersome calculation with an astonishingly simple result. Here the calculation is made transparent by a natural reparametrization of the flow equations using the Gauss map. This is developed in Sects. 2 and 3. The parabolic maximum principle can be applied to this evolution equation to deduce a differential inequality for the speed (Theorems 5.6, 5.11). In many cases this in turn can be integrated to give a Harnack inequality for the speed. This integration can be performed in various ways, two of which are described here: Theorem 5.17 gives an estimate which imitates the methods of previous work, and Theorem 5.21 describes an alternative estimate which seems more useful in many cases. An integral estimate (an *entropy inequality*) can also be obtained for certain flows by integrating the differential equation of Lemma 5.1 over the whole manifold instead of using the parabolic maximum principle (Theorem 5.26).

Several evolution equations of the form (1.1) have been studied before: A well-known example is the mean curvature flow (see for example [Hu1]) for which $F_t(x)$ is given by the mean curvature H of the hypersurface $\varphi_t(M^n)$ at the point $\varphi_t(x)$. Tso [Ts] has considered Eq. (1.1) with speed F given by the Gauss curvature K; Chow [Ch1, Ch2] extended this work to other Gauss curvature flows, with speed $F = K^\alpha$ for $\alpha > 0$, and also to the case $F = R^{\frac{1}{2}}$, where R is the scalar curvature. Urbas [U1, U2], Huisken [Hu2], and Gerhardt [G] have considered flows where F is homogeneous of degree 1 in the principal curvatures. These are all isotropic examples; Cahn et al. [CHT] have considered some anisotropic flows, including examples of the form $F = \mu(\nu)H$. Related anisotropic equations have appeared in other papers (see [AG, GG1]). In a paper by the author [A1], a wide class of other isotropic examples is treated – all homogeneous of degree 1 in the principal curvatures. The applicability of these flows is demonstrated in [A2], where they are used to give an elegant new proof of the 1/4-pinching sphere theorem of Klingenberg, Rauch, and Berger. Further applications are given in [A3], where anisotropic flows are central to a new proof of the Aleksandrov–Fenchel inequalities for convex bodies. Section 4 gives some more examples. The results of this paper apply to all of these different flow equations.

2 Notation and conventions

The standard metric on Euclidean space $\mathbb{R}^{n+1}$ is denoted $\langle \ldots, \ldots \rangle$, and the standard connection is denoted D. Each immersion φ_t of M^n induces a metric g and a connection ∇ on TM^n, the tangent bundle of M (the dependence of these on time will not be made explicit):

$$g(u, v) = \langle T\varphi(u), T\varphi(v) \rangle \tag{2.1}$$

$$\nabla_u v = T_x^{-1}\varphi(\pi_x(D_{T\varphi(u)} T\varphi(v)))$$

for all u and v in $T_x M^n$. Here π_x is the projection of $\mathbb{R}^{n+1}$ onto the image of $T_x\varphi$.

The tangent bundle TM and its adjoint T^*M give rise to higher tensor bundles by taking tensor products. Of particular interest is the space of maps of TM, which may be identified with the bundle $T^*M \otimes TM$. The metric g allows us to relate this to the bundle of bilinear forms $T^*M \otimes T^*M$: For a bilinear form $\mathcal{T}$, the corresponding map is denoted $g^*\mathcal{T}$, and defined by the equation

$$g(u, g^*\mathcal{T}(v)) = \mathcal{T}(u, v) \tag{2.2}$$

for all vectors u and v in TM. In indicial notation, g^* is the operator which raises an index; some care is required here because we consider several different metrics, and the effect of 'raising an index' is different for each of these.

The connection can be used to define tensorial derivatives for any tensor. Suppose $\mathscr{C}$ is a covariant k-tensor (a multilinear function of k vectors). Then the derivative is a covariant $(k+1)$-tensor defined as follows:

$$(\nabla\mathscr{C})(u, v_1, \ldots, v_k) = d_u(\mathscr{C}(v_1, \ldots, v_k)) - \mathscr{C}(\nabla_u v_1, \ldots, v_k) - \cdots - \mathscr{C}(v_1, \ldots, \nabla_u v_k)\,, \tag{2.3}$$

where d_u is simply the directional derivative in direction u, which we consider to act on the function $\mathscr{C}(v_1, \ldots, v_k)$, taking the vector fields $v_1, \ldots, v_k$ fixed. Repeating this procedure gives higher tensorial derivatives. In particular, the second tensorial derivative is called the **Hessian**. In the case of functions, it is defined as follows:

$$\mathrm{Hess}_\nabla f(u, v) = d_u d_v f - d_{\nabla_u v} f\,. \tag{2.4}$$

The second fundamental form II is the symmetric tensor given by the normal component of the connection on $\mathbb{R}^{n+1}$:

$$II(u, v) = -\langle D_{T\varphi(u)} T\varphi(v), \nu\rangle \tag{2.5}$$

for all u and v in $T_x M^n$, where $T\varphi\colon TM \to \mathbb{R}^{n+1}$ is the derivative of φ.

The Weingarten map $\mathscr{W}\colon TM^n \to TM^n$ gives the rate of change in the direction of the normal along the surface:

$$\mathscr{W} = T\varphi^{-1} \circ T\nu \tag{2.6}$$

where $T\nu\colon TM \to TS^n \subset \mathbb{R}^{n+1}$ is the derivative of the Gauss map. The second fundamental form and the Weingarten map are related by the Weingarten relation:

$$II(u, v) = g(\mathscr{W}(u), v)\,. \tag{2.7}$$

The eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathscr{W}$ are called the **principal curvatures**.

This paper will be concerned particularly with the case where the immersions are strictly locally convex (the second fundamental form is positive definite everywhere). For such a hypersurface, the inverse of the Gauss map is an immersion, which we can consider as a parametrization of the hypersurface. Thus we can assume that we have an immersion $\varphi\colon S^n \to \mathbb{R}^{n+1}$ for which the Gauss map $\nu\colon S^n \to S^n$ is the identity map on S^n (In the case $n=1$, we may have to consider some covering of S^1). The standard metric and connection on S^n will be denoted $\bar{g}$ and $\bar{\nabla}$.

The support function $s\colon S^n \to \mathbb{R}$ of φ gives the perpendicular distance from the origin of the tangent plane at $\varphi(z)$. It follows that φ has this form:

$$\varphi(z) = s(z) + a(z) \tag{2.8}$$

where $a(z)$ is a vector tangent to S^n at z, for each z in S^n. Differentiating this expression in a tangential direction u gives the following result:

$$\begin{aligned} T\varphi(u) &= (D_u s)z + su + D_u a \\ &= (D_u s)z + su + \bar{\nabla}_u a - \bar{g}(u, a)z\,. \end{aligned} \tag{2.9}$$

The vector a can be deduced from the fact that the tangent space $T_z\varphi(T_z S^n)$ is parallel to the tangent space $T_z S^n$ – this implies that the component of the expression (2.9) in direction z must be zero. Hence we have $\bar{g}(u, a) = D_u s$ for every tangential vector u, and so $a(z) = \bar{\nabla} s$, the gradient of s with respect to the metric $\bar{g}$ on the sphere. The immersion is therefore given by the following expression

$$\varphi(z) = s(z)z + \bar{\nabla} s\,. \tag{2.10}$$

An expression for the curvature also follows from the calculation above: Recall that the Weingarten curvature $\mathscr{W}$ is given by the expression $\mathscr{W}(u) = T\varphi^{-1} \circ T\nu(u)$, from Eq. (2.6). In the present situation we have $T\nu = \mathrm{Id}$, and so $\mathscr{W}^{-1} = T\varphi$. The calculation (2.9) and the expression for the vector

a therefore give:

$$\mathcal{W}^{-1}(u)=\bar{\nabla}_u(\bar{\nabla}s)+s\mathrm{Id}(u) \tag{2.11}$$
$$=(\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}s+s\mathrm{Id})(u)\ .$$

For convenience we will denote the map $\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}s+\mathrm{Id}s$ by A, or by $A[s]$.

It is useful to have expressions for the metric and connection of the hypersurface in terms of s and A:

$$g(u,v)=\bar{g}(A(u),A(v)) \tag{2.12}$$

$$\nabla_u v=\bar{\nabla}_u v+A^{-1}(\bar{\nabla}A(u,v))\ , \tag{2.13}$$

Another useful equation, which can be deduced directly from the Eq. (2.11) for the map A, is a form of the Codazzi equations:

$$\bar{\nabla}A(u,v)=\bar{\nabla}A(v,u)\ . \tag{2.14}$$

The great advantage of the support function is that it allows us to consider a family of convex hypersurfaces simply as an evolving scalar function defined on the sphere. This makes things much simpler than the more abstract framework allowing arbitrary parametrizations, since we no longer have different descriptions of the same hypersurface. Furthermore, the identification with the sphere provides a time-independent metric and connection, which vastly simplifies many calculations – including especially those presented here for the proof of the Harnack inequalities.

The expression (2.11) allows us to use the support function to calculate functions of the curvature of a hypersurface. For example, a function $F(\mathcal{W})$ (such as the speed functions of Eq. (1.1)) gives rise to a 'dual' function $\Phi(A[s])$, defined according to the following equation:

$$\Phi(X)=-F(X^{-1}) \tag{2.15}$$

for every positive definite map X. The application of these ideas to the evolution equations will be developed fully in the next section.

Examples. The mean curvature H is given by the trace of the Weingarten map, which is the trace of the inverse of the map A. The harmonic mean curvature is the inverse of the trace of the inverse of the Weingarten map, or the reciprocal of the trace of A. The Gauss curvature is the determinant of $\mathcal{W}$, or the inverse of the determinant of A. More examples are given in Sect. 4.

3 The evolution equations

In this paper we will work with the support function, rather than working explicitly with the hypersurfaces. This contrasts with previous work such as [Hu1], and [A1], where the speed was considered as a function of the principal curvatures, and calculations were performed with respect to the metric and measure on the hypersurfaces. Here we write everything in terms of the support function and the map A defined by (2.4). A similar approach has been used before by Tso [Ts], and by Urbas [U1]. An important step in this approach is to rewrite Eq. (1.1) as an evolution equation for the support function.

Theorem 3.1 *Suppose* $\varphi: M^n \times [0, T) \to \mathbb{R}^{n+1}$ *is a family of strictly convex immersions satisfying* (1.1). *Then the support functions* s: $S^n \times [0, T) \to \mathbb{R}$ *satisfy the following equation*:

$$\frac{\partial}{\partial t} s(z, t) = \Phi(A[s(z, t)], z) . \tag{3.2}$$

I will not be concerned with questions of existence of solutions or preserving convexity in this paper. Some very general results on short-time existence and regularity for hypersurface evolution equations can be found in [GG2].

We require the function Φ to have the following form:

$$\Phi(A, z) = -F(\mathscr{W}, z) \tag{3.3}$$

for all z in S^n. As in Eq. (2.15), Φ can be defined in terms of F as follows:

$$\Phi(X, z) = -F(X^{-1}, z) \tag{3.4}$$

for all positive definite maps X and all points z in S^n. Φ is a real function defined on a domain Ω contained in $T^*S^n \otimes TS^n$, the space of linear maps of TS^n. Since we require Eq. (1.1) to be parabolic, Φ must satisfy a strict monotonicity condition: The derivative $\dot{\Phi}$ of Φ is in $TS^n \otimes T^*S^n$, and is defined for each point z in S^n and each map X in Ω by its action on elements $\mathscr{B}$ of the tensor bundle $T^*S^n \otimes TS^n$:

$$\dot{\Phi}(\mathscr{B}) = \frac{d}{dr} \Phi(X + r\mathscr{B}, z)\bigg|_{r=0} . \tag{3.5}$$

We require that this map be positive definite at each z in S^n and each X in Ω. The domain of definition of F is the set Ω' of maps which are inverse to maps in Ω. Note that the derivative $\dot{F}$ of F is positive definite whenever $\dot{\Phi}$ is.

Since we usually consider convex hypersurfaces, we will often take Ω to be the set of symmetric positive definite maps of TM. In some circumstances, however, other choices of domain are interesting – see for example the flows used in [A2].

If the equation is isotropic, then F and Φ are restricted further: F is given by a symmetric function of the principal curvatures $\lambda_1, \ldots, \lambda_n$. This means also that Φ is given by a symmetric function of the eigenvalues of the map $A = \bar{g}^* \mathrm{Hess}_{\bar{\nabla}} s + \mathrm{Id} s$, which are called the **principal radii of curvature**.

Examples. In the case of the flow by mean curvature, the elliptic operator $\mathscr{L} = \dot{F} g^* \mathrm{Hess}_{\nabla}$, naturally associated with the flow in the standard parametrization, is just the Laplacian on the hypersurface:

$$\mathscr{L} = \mathrm{tr}_g \mathrm{Hess}_{\bar{\nabla}} = \Delta .$$

In local coordinates this may be written $g^{ij}\nabla_i\nabla_j$, employing the summation convention. Similarly, the flow by the Gauss curvature K has $\mathscr{L}=K(II^{-1})^{ij}\nabla_i\nabla_j$, where II^{-1} is the inverse of II with respect to g. More generally, for a curvature function F the associated elliptic operator is given by $\mathscr{L}=\dot{F}^{ij}\nabla_i\nabla_j$, where $\dot{F}^i_j$ gives the components of $\dot{F}$, and $\dot{F}^{ij}=g^{jk}\dot{F}^i_k$.

The mean curvature flow is not as simple when written in terms of Φ and the Gauss map parametrization – then we have an elliptic operator $\bar{\mathscr{L}}=\dot{\Phi}\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}$, which becomes in this case $H^2(A^{-2})^{ij}\bar{\nabla}_i\bar{\nabla}_j$.

The flow by harmonic mean curvature has $F=-H_{-1}$ where $H_{-1}=\left(\frac{1}{n}\sum\lambda_i^{-1}\right)^{-1}$. The flow Eq. (1.1), written in terms of the principal curvatures, is rather complicated: The coefficients of the elliptic operator $\mathscr{L}=\dot{F}g^*\mathrm{Hess}_\nabla$ associated with this flow are given by $H_{-1}^2\mathscr{W}^{-2}$. In terms of the support function, however, this flow is much simpler: The Eq. (3.2) becomes in this case:

$$\frac{\partial}{\partial t}s=-(\bar{\Delta}s+ns)^{-1}$$

where $\bar{\Delta}$ is the Laplacian on S^n. Even simpler is the outward flow by the inverse of the harmonic mean curvature, which has the form:

$$\frac{\partial}{\partial t}s=\bar{\Delta}s+ns\ .$$

The Eq. (1.1) extends the parametrization φ_0 of the initial hypersurface to later hypersurfaces by identifying points on trajectories normal to the hypersurfaces. This will be referred to as the **standard parametrization** of the flow. The approach adopted here is somewhat different – we identify points which have the same normal direction. This will be referred to as the **Gauss map parametrization** of the flow.

The definitions of Sect. 1 allow us to find induced evolution equations for various interesting geometric quantities. In the Gauss map parametrization the details are slightly different from the standard parametrization – compare the calculations in [A1].

Theorem 3.6 *The following evolution equations hold under the Gauss map parametrization of the flow* (1.1):

(3.7) $$\frac{\partial}{\partial t}(\mathrm{Hess}_{\bar{\nabla}}s+\bar{g}s)=\mathrm{Hess}_{\bar{\nabla}}\Phi+\Phi\bar{g}$$

(3.8) $$\frac{\partial}{\partial t}A=\bar{g}^*(\mathrm{Hess}_{\bar{\nabla}}\Phi)+\mathrm{Id}\,\Phi$$

(3.9) $$\frac{\partial}{\partial t}\Phi=\bar{\mathscr{L}}\Phi+\dot{\Phi}(\mathrm{Id})\Phi\ ,$$

where $\bar{\mathscr{L}}$ *is the elliptic operator* $\dot{\Phi}\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}$.

Proof. The first equation follows simply by differentiating the Eq. (3.2), since the metric $\bar{g}$ and connection $\bar{\nabla}$ are independent of time. The second follows immediately from this. Since Φ depends only on A and z, where z is independent of time, we have:

$$\frac{\partial}{\partial t}\Phi=\dot{\Phi}\left(\frac{\partial}{\partial t}A\right)$$

which implies Eq. 3.9). □

Example. For the harmonic mean curvature flow we have

$$\frac{\partial}{\partial t}\Phi=\Phi^2(\bar{\Delta}\Phi+n\Phi)$$

which is related to certain porous medium equations. The associated elliptic operator for this flow is $\bar{\mathscr{L}}=\Phi^2\bar{\Delta}$.

I will conclude this section with a simple result which allows us to transform between evolution equations in the Gauss map parametrization and the corresponding equations in the standard parametrization:

Lemma 3.10 *Let $Q:M^n+[0,T)\to\mathbb{R}$ and $\bar{Q}:S^n\times[0,T)\to\mathbb{R}$ be related by the equation*

$$\bar{Q}(\nu(x,t),t)=Q(x,t)$$

for all x in M^n and t in $[0,T)$. Then the following equation relates the time derivatives of Q and $\bar{Q}$:

$$\left(\frac{\partial}{\partial t}Q\right)_{\text{standard param.}}=\left(\frac{\partial}{\partial t}\bar{Q}\right)_{\text{Gauss param.}}+\mathit{II}^{-1}(\nabla F,\nabla Q)\,, \tag{3.11}$$

where II^{-1} is the element of $TM^n\otimes TM^n$ which is the inverse of II.

Proof. Differentiation of Eq. (2.10) gives the following expression for the evolution of the immersion under the Gauss map parametrization:

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(z,t)&=\Phi(z,t)z+\bar{\nabla}\Phi(z,t)\\&=-F(\mathscr{W},\nu)\nu-T\varphi(\mathscr{W}^{-1}\nabla F)\,,\end{aligned} \tag{3.12}$$

using the Eq. (3.4) which relates Φ and F, and Eq. (2.12) which relate the metrics g and $\bar{g}$. This differs from (1.1) only in the extra tangential vector field on the right hand side, which introduces a gradient term into the evolution equation of any scalar quantity, as in (3.11). □

4 Examples

In this section I will give some examples of functions $F(\mathscr{W},\nu)$ which may serve as speeds in the Eq. (1.1), together with the dual functions $\Phi(A,z)$ which give the speed in terms of the support function. I will also note some convexity properties which will be relevant to the main results proved in the next section.

4.1 Isotropic Flows

For isotropic flows the speed takes the form $F(\mathscr{W})=f(\lambda)=-\phi(\kappa)$, where $\lambda=(\lambda_1,\ldots,\lambda_n)$ are the principal curvatures, and $\kappa=(\kappa_1,\ldots,\kappa_n)$ are the principal radii of curvature: $\kappa_i=\lambda_i^{-1}$. ϕ is a symmetric function defined on a symmetric domain $\mathscr{C}$ in $\mathbb{R}^n$, and f is the dual function defined by $f(\lambda_1,\ldots,\lambda_n)=-\phi(\lambda_1^{-1},\ldots,\lambda_n^{-1})$. The domain of definition Ω of the function Φ is given by the set

$$\Omega=\{\mathscr{Z}\in T^*S^n\otimes TS^n:\kappa(\mathscr{Z})\in\mathscr{C}\}\ .$$

For flows of convex hypersurfaces we usually choose $\mathscr{C}=\Gamma_+\subset\mathbb{R}^n$, the **positive cone** consisting of those points with all coordinates positive.

Homogeneous examples. It is often natural to consider flows which are invariant under dilations of space, in the sense that a solution remains a solution under this operation, up to a rescaling of time. This criterion leads us to consider speed functions which are homogeneous of some degree in the principal curvatures (or the principal radii of curvature). The flows in [Hu1, Ts, Ch1–2, A1] are isotropic, and homogeneous of positive degree in the principal curvatures (negative in the principal radii of curvature).

The **elementary symmetric functions** are defined by:

$$e^{[k]}(x)=\frac{1}{\binom{n}{k}}\sum_{i_1<\cdots<i_k}x_{i_1}\ldots x_{i_k}\quad\text{for } k=1,\ldots,n\ .$$

This gives the mean curvature if $f=e_1$, and the Gauss curvature if $f=e_n$. From these a wider class can be defined by taking powers and ratios:

$$e^{[k,l,\alpha]}(x)=\operatorname{sgn}\alpha\left(\frac{e^{[k]}(x)}{e^{[l]}(x)}\right)^{\alpha}\quad\text{for } 0\leqq l<k\leqq n \text{ and } \alpha\in\mathbb{R}\setminus\{0\}\ .$$

Here $e^{[0]}(x)=1$). If $f=e^{[k,l,\alpha]}$, then $\phi=e^{[n-l,n-k,-\alpha]}$. The following result is useful:

Lemma. *The functions* $e^{\left[k,l,\frac{1}{k-l}\right]}$, *for* $0\leqq l<k\leqq n$, *are concave*:

$$e^{\left[k,l,\frac{1}{k-l}\right]}(x+y)\geqq e^{\left[k,l,\frac{1}{k-l}\right]}(x)+e^{\left[k,l,\frac{1}{k-l}\right]}(y)\ .$$

Proof. See [BMV, p. 306]. □

Other interesting examples in this class are the scalar curvature ($f=e^{[2,0,1]}$) and the harmonic mean curvature ($f=e^{[n,n-1,1]}$, $\phi=e^{[1,0,-1]}$).

The **power means** provide another class of examples in this category. They are defined as follows for $x=(x_1,\ldots,x_n)$ in Γ_+:

$$H_r(x)=\begin{cases}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^r\right)^{\frac{1}{r}} & \text{for } r\neq 0\,;\\ e^{\left[n,0,\frac{1}{n}\right]} & \text{for } r=0\,.\end{cases}$$

If $f=H_r$, then $\phi=-H_{-r}^{-1}$. The functions H_r are concave for $r\leqq 1$, and convex for $r\geqq 1$.

More generally, any symmetric function f which is homogeneous of degree one and increasing with respect to each argument gives rise to a family of examples f_r^α, defined as follows for any r and any nonzero α:

$$f_r^\alpha(x)=\begin{cases}\operatorname{sgn}\alpha\left[\frac{1}{f(1,\ldots,1)}f(x^r)\right]^{\frac{\alpha}{r}} & \text{for } r\neq 0\,;\\ e^{\left[n,0,\frac{\alpha}{n}\right]} & \text{for } r=0\,.\end{cases}$$

Homogeneous flows are normally divided into two subsets: The *contraction flows*, for which f is homogeneous of positive degree, and the *expansion flows*, for which f is homogeneous of negative degree.

Non-homogeneous examples. There are many ways to produce functions which are non-homogeneous satisfying the required conditions; most of these are of little interest. However, there are a few examples which have some applications.

The **quasi-arithmetic means** are defined as follows: Suppose $\Psi\colon I\subset\mathbb{R}\to\mathbb{R}$ is a strictly increasing smooth function on some (possibly infinite) interval I. Define a symmetric function Ψ_n, with $\mathscr{C}=I^n$, by $\Psi_n(x)=\Psi^{-1}\left(\frac{1}{n}\sum_{i=1}^n \Psi(x_i)\right)$. Many of the examples already discussed are of this form.

A further useful example is the following: For $k\in\mathbb{R}$, and f a homogeneous example as in the previous section, define $\tilde{f}(x)=f(x-(k,\ldots,k))$, with domain $\mathscr{C}'=(k,\infty)^n$. These have some geometrical applications, which are the subject of [A2].

4.2 Anisotropic flows

Taylor et al. [CHT] have considered some classes of flows which are anisotropic, as a model for crystal growth phenomena. The simplest of these flows take the form $F=-\mu(v)H$, where H is the mean curvature, and μ is a 'mobility function'; this is usually given as the support function of some fixed convex hypersurface W, called the **Wulff shape.** More generally, one might generalise this to flows of the form $F=-\mu(\nu)f(\lambda)$, for any of the function f of the preceding sections.

Another class of flows considered in [CHT] is given as follows: Let W be a fixed, strictly convex Wulff shape containing the origin, with support function μ. The Weingarten map $\mathscr{Y}$ of W is then given by (2.11):

$$\mathscr{Y}^{-1}=\bar{g}^*(\mathrm{Hess}_{\bar{\nabla}}\mu+\mu\bar{g})\ .$$

Now consider flows with $F(\mathscr{W},\nu)=\mu(\nu)\mathrm{tr}(\mathscr{Y}^{-1}\circ\mathscr{W})$. These are natural anisotropic generalisations of the mean curvature flow. More general flows related to this, referred to as **relative curvature flows**, have speeds given by $\mu(\nu)\bar{F}(\mathscr{Y}^{-1}\circ\mathscr{W})$, where $\bar{F}$ may be any of the examples from the isotropic case considered above. These flows have the desirable property that the Wulff shape evolves trivially – the hypersurfaces at different times are identical up to a scaling factor. Some other examples of natural anisotropic flows are important in [A3]. Very general anisotropic flows are also considered in [AG, GG1, GG2].

5 Harnack inequalities

The previous sections have set up the tools necessary to prove the main results of this paper. The Gauss map parametrization reduces the main result to the following short calculation:

Lemma 5.1 *Suppose φ is a solution to* (1.1) *for which all the hypersurfaces $\varphi_t(M^n)$ are strictly locally convex. In the Gauss map parametrization, the following evolution equation holds for the quantity $P=\frac{\partial}{\partial t}\Phi$, where we denote $Q=\frac{\partial}{\partial t}A$:*

$$\frac{\partial}{\partial t}P=\bar{\mathscr{L}}P+\dot{\Phi}(\mathrm{Id})P+\ddot{\Phi}(Q,Q) \tag{5.2}$$

where $\ddot{\Phi}(\mathscr{Z})\in(TS^n\otimes T^*S^n)\otimes(TS^n\otimes T^*S^n)$ is defined by:

$$\ddot{\Phi}(\mathscr{Z})(\mathscr{B},\mathscr{C})=\frac{\partial}{\partial b}\frac{\partial}{\partial c}\Phi(\mathscr{Z}+b\mathscr{B}+c\mathscr{C})\Big|_{b=c=0} \tag{5.3}$$

for every $\mathscr{B}$ and $\mathscr{C}$ in $T^*S^n\otimes TS^n$.

Proof. Note that $P=\dot{\Phi}(Q)$. Differentiation of Eq. (3.9) yields the result immediately, since the metric $\bar{g}$ and the connection $\bar{\nabla}$ are independent of time. □

Example. For the harmonic mean curvature flow this calculation is as follows:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\Phi\right)=\frac{\partial}{\partial t}(\Phi^2(\bar{\Delta}\Phi+n\Phi))$$

$$=\Phi^2\left(\bar{\Delta}\left(\frac{\partial}{\partial t}\Phi\right)+n\frac{\partial}{\partial t}\Phi\right)+2\Phi\left(\frac{\partial}{\partial t}\Phi\right)(\bar{\Delta}\Phi+n\Phi)$$

$$=\bar{\mathscr{L}}\left(\frac{\partial}{\partial t}\Phi\right)+n\Phi^2\left(\frac{\partial}{\partial t}\Phi\right)+\frac{2}{\Phi}\left(\frac{\partial}{\partial t}\Phi\right)^2.$$

If a function $\Phi: T^*S^n \otimes TS^n \to \mathbb{R}$ satisfies the condition $\ddot{\Phi}(\mathscr{Z})(\mathscr{A},\mathscr{A})\leqq 0$ for all $\mathscr{Z}$ in Ω and all $\mathscr{A}$ in $T^*S^n \otimes TS^n$, then Φ will be called **concave**; if the reverse inequality holds, Φ will be called **convex**. If $\Phi=\operatorname{sgn}\alpha\cdot B^\alpha$, where B is positive and concave (convex), then Φ is called **α-concave (α-convex)**. α-concavity is equivalent to the inequality:

$$\ddot{\Phi}\leqq\frac{\alpha-1}{\alpha\Phi}\dot{\Phi}\otimes\dot{\Phi}. \tag{5.4}$$

These conditions become considerably more complicated when written in terms of the principal curvatures and the function F. For example, concavity of Φ becomes:

$$\ddot{F}(X,X)+2\dot{F}(X\circ\mathscr{W}^{-1}\circ X)\geqq 0 \tag{5.5}$$

for every symmetric map X in $T^*M\otimes TM$.

Theorem 5.6 *Suppose φ is a strictly convex solution to* (1.1).

(1) *Suppose Φ is α-concave for $\alpha<1$. Then the following estimate holds in the Gauss map parameterization for positive times t, as long as the solution exists:*

$$\frac{\partial}{\partial t}\Phi+\frac{\alpha\Phi}{(\alpha-1)t}\leqq 0. \tag{5.7}$$

(2) *If Φ is positive and concave, then the following weaker estimate holds:*

$$\sup_t\left(\frac{\partial}{\partial t}\ln\Phi\right) \quad \textit{is decreasing}. \tag{5.8}$$

(3) *Suppose Φ is α-convex, for $\alpha>1$. Then the following holds at every x in M^n and every $t>0$:*

$$\frac{\partial}{\partial t}\Phi+\frac{\alpha\Phi}{(\alpha-1)t}\geqq 0. \tag{5.9}$$

(4) *If Φ is positive and convex, then the following holds:*

$$\inf_{t_2}\left(\frac{\partial}{\partial t}\ln\Phi\right) \quad \textit{is increasing}. \tag{5.10}$$

Proof. I will prove the first two cases: The last term in Eq. (5.2) can be estimated as follows, since Φ is α-concave:

$$\ddot{\Phi}(X, X) \leqq \frac{\alpha-1}{\alpha\Phi}(\dot{\Phi}(X))^2$$

for any $\mathscr{B}$ in $T^*S^n \otimes TS^n$. For the case $\alpha = 1$, the following inequality holds for the quantity $R = \frac{\partial}{\partial t}\ln\Phi$:

$$\frac{\partial}{\partial t}R \leqq \bar{\mathscr{L}}R + \frac{2}{\Phi}\dot{\Phi}(\bar{g}^*(\bar{\nabla}\Phi \otimes \bar{\nabla}R)) .$$

The result (5.8) follows immediately from the parabolic maximum principle, since the first term is an elliptic operator, and the second a gradient term. In the case where $\alpha < 1$, one can estimate as follows, where $R = t\frac{\partial}{\partial t}\Phi + \frac{\alpha\Phi}{\alpha-1}$:

$$\frac{\partial}{\partial t}R \leqq \bar{\mathscr{L}}R + \left(\frac{\alpha-1}{\alpha\Phi}\frac{\partial}{\partial t}\Phi + \dot{\Phi}(\mathrm{Id})\right)R .$$

Since $t\frac{\partial}{\partial t}\Phi + \frac{\alpha\Phi}{\alpha-1}$ is initially negative, the parabolic maximum principle implies that it remains so as long as the solution exists. The proof for the convex case is similar. □

This calculation can easily be transferred to the standard parametrization, by writing the various quantities in terms of the metric and connection on the hypersurface. This is most easily done by considering the change in the evolution equations coming from the modified parametrization. Here we denote by II^{-1} the map inverse to II in the following sense: II is an element of $T^*M \otimes T^*M$, so we can consider it as a map from TM to T^*M. II^{-1} is then a map from T^*M to TM, and is therefore an element of $TM \otimes TM$.

Corollary 5.11 *Suppose φ is a strictly convex solution to* (1.1).

(1) *If Φ is α-concave for some $\alpha < 1$, the following inequality holds in the standard parametrization*:

$$\frac{\partial}{\partial t}F - \mathit{II}^{-1}(\nabla F, \nabla F) + \frac{\alpha F}{(\alpha-1)t} \geqq 0 . \tag{5.12}$$

(2) *If Φ is concave and positive, then the following holds*:

$$\sup_{S^n}\left(\frac{\partial}{\partial t}\ln|F| - F\mathit{II}^{-1}(\nabla\ln|F|, \nabla\ln|F|)\right) \quad \textit{is decreasing} . \tag{5.13}$$

(3) *If Φ is a α-convex for $\alpha > 1$, then*:

$$\frac{\partial}{\partial t}F - \mathit{II}^{-1}(\nabla F, \nabla F) + \frac{\alpha F}{(\alpha-1)t} \leqq 0 . \tag{5.14}$$

(4) *If Φ is convex and positive, then*:

$$(5.15)\qquad \sup_{S^n}\left(\frac{\partial}{\partial t}\ln|F| - F\Pi^{-1}(\nabla\ln|F|, \nabla\ln|F|)\right) \quad \textit{is increasing}\ .$$

Proof. Recall the Eq. (3.11) which relates the time derivatives of functions in the Gauss map and standard parametrizations. Applying this to the function F gives the following:

$$(5.16)\qquad \left(\frac{\partial}{\partial t}F\right)_{\text{Gauss}} = \left(\frac{\partial}{\partial t}F\right)_{\text{standard}} - \Pi^{-1}(\nabla F, \nabla F)\ .$$

This expression immediately gives the results above from Theorem 5.6. □

Remark. It is possible to perform the calculations of Lemma 5.1 entirely in the standard parametrization – this was done in the special cases proved in [Ha 3] and [Ch 3]. The calculations are then much messier, since the connection and metric are time-dependent, and there are extra gradient terms in the quantities of interest. Furthermore, these calculations were carried out only for isotropic flows; in the case of anisotropic flows the calculations rapidly become unmanageable. This extra complication in the calculations made the simplicity of the results rather mysterious – particularly since the equations are fully nonlinear. The results here are made easier because the evolution equations have a very nice form in the Gauss map parametrization. This parametrization of the hypersurfaces seems more geometrically natural for this situation: For example, solutions for which the hypersurfaces evolve by pure scaling (homothetic solutions) have a very simple description in Gauss map parametrization, but not in the the standard parametrization. The deep relationship between these homothetic solutions and Harnack inequalities has been noted before [Ha 3, Ch 3].

We can now obtain a Harnack inequality for many cases of interest:

Theorem 5.17 *Suppose φ is a strictly convex solution of an equation of form* (1.1). *The following inequalities apply in the standard parametrization for the cases described, for any points x_1 and x_2 in M^n, any times $t_2 > t_1 > 0$, and any curve γ joining (x_1, t_1) to (x_2, t_2)*:

(1) *Φ α-concave, $\alpha < 0$*:

$$(5.18)\qquad \frac{F(x_2, t_2)}{F(x_1, t_1)} \geqq \left(\frac{t_1}{t_2}\right)^{\frac{\alpha}{\alpha-1}} \exp\left(-\frac{1}{4}\int_\gamma F^{-1}\Pi(\dot\gamma, \dot\gamma)\,dt\right).$$

(2) *Φ α-convex, $\alpha > 1$*:

$$(5.19)\qquad \frac{F(x_2, t_2)}{F(x_1, t_1)} \geqq \left(\frac{t_1}{t_2}\right)^{\frac{\alpha}{\alpha-1}} \exp\left(-\frac{1}{4}\int_\gamma |F|^{-1}\Pi(\dot\gamma, \dot\gamma)\,dt\right).$$

(3) *Φ convex and positive*:

$$(5.20) \qquad \frac{F(x_2,t_2)}{F(x_1,t_1)} \geqq e^{-C_1(t_2-t_1)} \exp\left(-\frac{1}{4}\int_\gamma |F|^{-1} |I\!I(\dot\gamma,\dot\gamma)\, dt\right).$$

where $C_1 = \sup_{t=0}\left(\frac{\partial}{\partial t}\ln|F| - F I\!I^{-1}(\nabla \ln|F|, \nabla\, ln\,|F|)\right)$.

Proof. I will give the proof only for one case – the other calculations are similar. Consider the case where Φ is α-concave for $\alpha<0$: Along a curve γ,

$$D_{\dot\gamma} \ln F = \frac{\partial}{\partial t} \ln F = \langle \dot\gamma, \nabla \ln F\rangle .$$

This can be estimated using (5.12) and the Cauchy–Schwarz inequality:

$$\begin{aligned} D_{\dot\gamma} \ln F &\geqq F I\!I^{-1}(\nabla \ln F, \nabla \ln F) + \langle \dot\gamma, \nabla \ln F\rangle - \frac{\alpha}{(\alpha-1)t} \\ &\geqq -\frac{1}{4} F^{-1} I\!I(\dot\gamma,\dot\gamma) - \frac{\alpha}{(\alpha-1)t} . \end{aligned}$$

Integrating along the curve γ yields (5.18). □

Example. For the mean curvature flow we have $F^{-1} I\!I \leqq 1$, and hence the estimate (5.18) becomes

$$\frac{H(x_2,t_2)}{H(x_1,t_1)} \geqq \left(\frac{t_1}{t_2}\right)^{\frac{1}{2}} \exp\left(-\frac{d^2}{4(t_2-t_1)}\right)$$

where d is the distance from x_1 to x_2 with respect to the metric g at time t_1.

The estimates in Theorem 5.17 have been obtained by Hamilton [Ha 3] for the mean curvature flow, and by Chow [Ch3] for flows by positive powers of the Gauss curvature. It should be noted that the integrals on the right hand side of these inequalities are in general difficult to estimate – for the mean curvature flow, there is a useful estimate as noted above, but for flows which are homogeneous with powers other than one, more natural estimates can be found by integrating the inequalities from (5.11) in a different way. The following theorem summarises these results in the special case where the second fundamental form can be controlled in terms of an appropriate powers of the speed. This is automatically the case, for example, for speeds given by powers of the power means H_r with $r>0$, or for powers of functions for which $\frac{H}{f}$ is bounded above on the positive cone $[0,\infty)^n$.

Examples. In [A1] it is shown that solutions to a wide class of flows with speeds F homogeneous of degree one satisfy a condition $\frac{H}{F} \leqq C$, where C depends on the initial hypersurface. Hence all these solutions satisfy the hypotheses of the following theorem.

Theorem 5.21 *Suppose φ is a strictly convex solution to an equation of the form (1.1), and the speed satisfies $\Phi = \operatorname{sgn}\alpha B^{\alpha}$, for some homogeneous degree 1 function B. Assume further that $B(A)A^{-1} \leqq C_2 \mathrm{Id}$ on the solution φ, for some constant C_2. Then the following estimates hold for any times $t_2 > t_1 > 0$ and any points x_1 and x_2 in M^n:*

(1) *If B is concave and $\alpha < -1$:*

$$t_2^{\frac{\alpha+1}{\alpha-1}} F(x_2, t_2)^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} F(x_1, t_1)^{\frac{\alpha+1}{\alpha}} \leqq \frac{(\alpha+1) C_2 d^2}{2\alpha(1-\alpha)\left[t_2^{\frac{2}{1-\alpha}} - t_1^{\frac{2}{1-\alpha}}\right]} \tag{5.22}$$

where d is the distance from x_1 to x_2 with respect to the metric g at time t_1.

(2) *If B is concave and $\alpha = -1$:*

$$\frac{F(x_2, t_2)}{F(x_1, t_1)} \geqq \left(\frac{t_1}{t_2}\right)^{\frac{1}{2}} \exp\left(-\frac{C_2 d^2}{4(t_2 - t_1)}\right) \tag{5.23}$$

where d is the same as in (1)

(3) *If B is concave and $-1 < \alpha < 0$:*

$$t_2^{\frac{\alpha+1}{\alpha-1}} F(x_2, t_2)^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} F(x_1, t_1)^{\frac{\alpha+1}{\alpha}} \leqq \frac{|1+\alpha| C_2 d^2}{2|\alpha(1-\alpha)|\left[t_2^{\frac{2}{1-\alpha}} - t_1^{\frac{2}{1-\alpha}}\right]} \tag{5.24}$$

where d is the same as in (1).

(4) *If B is convex and $\alpha > 1$:*

$$t_2^{\frac{\alpha+1}{\alpha-1}} \left|F(x_2, t_2)\right|^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} \left|F(x_1, t_1)\right|^{\frac{\alpha+1}{\alpha}} \geqq -\frac{(\alpha+1) C_2 d^2}{2\alpha(1-\alpha)\left[t_1^{\frac{2}{\alpha-1}} - t_2^{\frac{2}{\alpha-1}}\right]} \tag{5.25}$$

where d is the distance between x_1 and x_2 with respect to the metric g at time t_2.

Example. Flows with $F = H^{\alpha}$ satisfy the required conditions with $C_2 = 1$.

Proof. Consider the case (1). The estimate (5.12) can be written as follows:

$$\frac{\partial}{\partial t}\left(t^{\frac{\alpha+1}{\alpha-1}} F^{\frac{\alpha+1}{\alpha}}\right) \geqq \frac{\alpha t^{\frac{\alpha+1}{1-\alpha}}}{\alpha+1} (B \cdot \mathit{II})^{-1}\left(\nabla\left(t^{\frac{\alpha+1}{\alpha-1}} F^{\frac{\alpha+1}{\alpha}}\right), \nabla\left(t^{\frac{\alpha+1}{\alpha-1}} F^{\frac{\alpha+1}{\alpha}}\right)\right).$$

This can be used in the same manner as in the proof of Theorem 5.17 to yield the following inequality:

$$t_2^{\frac{\alpha+1}{\alpha-1}} F(x_2, t_2)^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} F(x_1, t_1)^{\frac{\alpha+1}{\alpha}} \geqq -\frac{C_2(\alpha+1)}{4\alpha} \int_{t_1}^{t_2} t^{\frac{\alpha+1}{\alpha-1}} g_{t_1}(\dot{\gamma}, \dot{\gamma})\, dt\,.$$

The result follows by minimising the integral on the right hand side over all paths joining the two points. The calculations for the other cases are similar. □

In some instances where Harnack inequalities are obtained, one also obtains an integral estimate or entropy estimate by integrating the evolution equation from Lemma 5.1 over the whole manifold M^n. This is true, for example, in the case of the Gauss curvature flow [Ch3]. In this next theorem I describe an entropy estimate which holds for a class of flows including the Gauss curvature and the harmonic mean curvature flow. This estimate was pointed out to me by Gerhard Huisken in the case of the harmonic mean curvature flow.

In [A4], it is shown that entropy estimates are intimately related to the Aleksandrov–Fenchel inequalities for convex bodies. Entropy inequalities are proved, by a different method, for a much wider class of flows (including anisotropic flows). Some applications of these results are also given.

Theorem 5.26 *Suppose that Φ is α-concave for some $\alpha \neq 0$, and also that the map $\bar{\nabla}(\Phi^{-2}\dot{\Phi}): TS^n \otimes T^*S^n \otimes TS^n \to \mathbb{R}$ satisfies the following condition:*

$$\bar{\nabla}(\Phi^{-2}\dot{\Phi})(\mathrm{Id} \otimes u) = 0 \tag{5.27}$$

for all $u \in TS^n$. For any strictly convex solution $\{M_t\}$, the following evolution equation holds under an isotropic flow (1.1) with speed $\Phi(A)$:

$$\frac{\partial}{\partial t} \int_{S^n} \frac{\partial}{\partial t}(\ln|\Phi|)\, d\mu \geqq \frac{\alpha+1}{\alpha} \int_{S^n} \left(\frac{\partial}{\partial t} \ln|\Phi|\right)^2 d\mu\,. \tag{5.28}$$

Consequently the integral can be estimated in terms of the maximum interval of existence T of the solution:

$$\int_{S^n} \left(\frac{\partial}{\partial t} \ln|\Phi|\right) d\mu \leqq \frac{\alpha}{(\alpha+1)|S^n|T} \tag{5.29}$$

where $|S^n|$ is the volume of the manifold S^n.

The condition (5.27) says that the trace of $\bar{\nabla}(\Phi^{-2}\dot{\Phi})$ over the first two arguments is identically zero – in local coordinates, $\bar{\nabla}_i(\Phi^{-2}\dot{\Phi}^i_j) = 0$.

The entropy estimate (5.29) amounts to a kind of Poincaré inequality for the speed – for example, in the case of the harmonic mean curvature flow:

$$n \int_{S^n} (H_{-1})^2\, d\mu \leqq \frac{1}{2|S^n|T} + \int_{S^n} |\bar{\nabla} H_{-1}|^2\, d\mu\,. \tag{5.30}$$

The only reference to the flow in this inequality is in the time of existence T. For a compact convex initial hypersurface this can be computed exactly for these special flows. This calculation is carried out in [A3].

The only isotropic homogeneous speeds which satisfy the condition (5.27) are the following special functions:

Corollary 5.31 *Suppose $\phi = e^{[k,0,-1]}$ (and hence also $f = e^{[n,n-k,1]}$) for $k = 1, \ldots, n$. Then Theorem 5.26 holds with $\alpha = k$.*

Remark. For $k=1$ this is the harmonic mean curvature flow; $k=n$ gives the Gauss curvature flow.

Proof of Theorem 5.26 Integrating using Eq. (5.2) yields the following:

$$(5.32)\quad \frac{\partial}{\partial t}\int_{S^n}\frac{\partial}{\partial t}\ln|\Phi|\,d\mu=\int_{S^n}\Phi\left(\Phi^{-2}\dot{\Phi}\bar{g}^*\left(\mathrm{Hess}_{\bar{\nabla}}\frac{\partial}{\partial t}\Phi+\bar{g}\frac{\partial}{\partial t}\Phi\right)\right)d\mu$$
$$+\int_{S^n}\Phi^{-1}\ddot{\Phi}(\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}\Phi+\mathrm{Id}\,\Phi,\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}\Phi+\mathrm{Id}\,\Phi)\,d\mu$$
$$-\int_{S^n}\left(\Phi^{-1}\frac{\partial}{\partial t}\Phi\right)^2 d\mu\,.$$

Now integration by parts yields the following, after using the α-concavity condition and Eq. (5.27):

$$(5.33)\qquad \frac{\partial}{\partial t}\int_{S^n}\left(\frac{\partial}{\partial t}\ln|\Phi|\right)d\mu\geqq\frac{\alpha+1}{\alpha}\int_{S^n}\left(\frac{\partial}{\partial t}\ln|\Phi|\right)^2 d\mu\,.$$

The result follows by applying Hölder's inequality and comparing with the ordinary differential equation

$$\frac{\partial}{\partial t}x=\frac{\alpha+1}{\alpha}x^2\,,$$

since the integral must remain finite as long as the solution exists. □

Acknowledgement. The author is pleased to thank Gerhard Huisken for his expert guidance and advice, and also Klaus Ecker, Richard Hamilton, Steven Altschuler, and Lang-Fang Wu for discussion and encouragement. This work forms a part of the author's doctoral thesis for the Australian National University.

References

[A1] Andrews, B.: Contraction of convex hypersurfaces in Euclidean space. Preprint no. MR 20-92, C.M.A., Australian National University 1992, 35 pages Calculus of Variations and P.D.E. (to appear)
[A2] Andrews, B.: Contraction of convex hypersurfaces in Riemannian spaces. J. Differ. Geom. **39**, 407–431 (1994)
[A3] Andrews, B.: Curvature Flows and the Aleksandrov-Fenchel inequalities. (to appear)
[A4] Andrews, B.: Entropy estimates for evolving hypersurfaces. Commun. Anal. Geom. **2**, 53–64 (1994)
[AG] Angenent, S., Gurtin, M.: Multiphase thermodynamics with interfacial structure 2. Evolution of an isothermal interface. Arch. Ration. Mech. Anal. **108**, 323–391 (1989)
[BMV] Bullen, P.S., Mitrinovič, D.S.: Vasič, P.M. (Eds.). Means and their inequalities. Mathematics and its applications series. D. Reidel, 1987
[CHT] Taylor, J.E., Cahn, J.W., Handwerker, C.A.: Geometric models of crystal growth. Acta Metall. Mater. **40**, 1443–1474 (1992)
[Ch1] Chow, B.: Deforming convex hypersurfaces by the *n*th root of the Gaussian curvature. J. Differ. Geom. **23**, 117–138 (1985)
[Ch2] Chow, B.: Deforming hypersurfaces by the square root of the scalar curvature. Invent. Math. **87**, 63–82 (1987)
[Ch3] Chow, B.: On Harnack's Inequality and Entropy for the Gaussian curvature flow. Comm. Pure Appl. Math. **44**, 469–483 (1991)

[Ch4] Chow, B.: The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature Comm. Pure Appl. Math. **45**, 1003–1014 (1992)

[G] Gerhardt, C.: Flow of nonconvex hypersurfaces into spheres. J. Differ. Geom. **32**, 199–314 (1990)

[GG1] Giga, Y., Goto, S.: Motion of hypersurfaces and geometric equations. J. Math. Soc. Japan **44**, 99–111 (1992)

[GG2] Giga, Y., Goto, S.: Geometric evolution of phase-boundaries. On the evolution of phase boundaries, IMA volumes in mathematics and its applications **43**. Springer, 1992, pp. 51–65

[Ha1] Hamilton, R.S.: The Ricci flow on surfaces. Mathematics and General Relativity, Contemporary Mathematics 71, American Mathematical Society, Providence, RI

[Ha2] Hamilton, R.S.: Heat equations in geometry. Lecture notes, Hawaii, 1990

[Ha3] Hamilton, R.S.: The Harnack Estimate for the Ricci Flow. J. Differ. Geom. **37**, 225–243 (1993)

[Hu1] Huisken, G.: Flow by mean curvature of convex hypersurfaces into spheres. J. Differ. Geom. **20**, 237–268 (1984)

[Hu2] Huisken, G.: On the expansion of convex hypersurfaces by the inverse of symmetric curvatures functions. (to appear)

[LY] Li, P., Yau, S.T.: On the parabolic kernel of the Schrödinger operator. Acta Math. 156, 153–201 (1986)

[M] Moser, J.: A Harnack Inequality for Parabolic Differential Equations. Commun. Pure Appl. Math **17**, 101–134 (1964)

[Ts] Tso, K.: Deforming a hypersurface by its Gauss–Kronecker curvature. Commun. Pure Appl. Math. **38**, 867–882 (1985)

[U1] Urbas, J.I.E.: An expansion of convex surfaces. J. Differ. Geom. **33** (1991), 91–125

[U2] Urbas, J.I.E.: On the expansion of star-shaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z. **205** (1990), 355–372