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Failure of weak holomorphic averaging on multiple connected domains

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0 Introduction

In translating theorems about H^∞ functions on the upper half plane $\mathbf{H}$ into theorems about H^∞ functions on an annulus A, we have a very useful theorem.

Theorem 0 *If $F: A \to \mathbf{C}$ is a multiple-valued holomorphic function whose collective values $\{F(z)\}$ are bounded on compact subsets of A, then there is a single-valued holomorphic function $f: A \to \mathbf{C}$ such that*

$$f(z) \in \operatorname{cch}\{F(z)\} \tag{0}$$

for all $z \in A$ ("cch" means "closed convex hull of"). We say that such an f "holomorphically averages" F.

Our use of the phrase "multiple-valued function" is explained in the remark at the end of this introduction. For a simple proof of Theorem 0 see the introduction to [2]. This theorem allows one to "pull back" questions about $\mathrm{H}^\infty(A)$ to questions about $\mathrm{H}^\infty(\mathbf{H})$, answer them in the latter, simpler setting, and then "push the answers forward" to $\mathrm{H}^\infty(A)$ via holomorphic averaging, with no increase in H^∞ norm. Unfortunately, as Barrett [2] has shown, the theorem does not generalize to domains of connectivity greater than two.

For many applications, we do not need the full strength of Theorem 0. We could settle for an f satisfying Eq. (0) only at points near the boundary of the multiply-connected domain in question. The maximum principle would then ensure satisfactory control of f in the interior of the domain. There are several questions one can ask about the existence of this latter, weaker sort of averaging function. In all three of the following questions, $F: \Omega \to \mathbf{C}$ is a multiple-valued holomorphic function whose values are bounded on compact subsets of the open Riemann surface Ω.

Question A. Does there necessarily exist a single-valued holomorphic function f averaging F in the weaker sense described above?

Question B. Does the non-existence of a holomorphic average for F (in the sense of Theorem 0) imply the non-existence of a weak holomorphic average for F?

Question C. If there exists no weak holomorphic average for F, how might one further loosen the definition of holomorphic average to obtain useful single-valued holomorphic functions associated with F? Would it help to allow the average to have a pole in Ω? Would it help to require the average to take values that are only near the closed convex hull of the values of F, rather than actually in the closed convex hull?

We show in this paper that the answer to Question A is no whenever Ω is topologically more complicated than a disk or an annulus. Our approach is to examine a more concrete version of Barrett's counterexample using Cauchy's Theorem. We thus obtain restrictions on the admissible boundary behavior of a holomorphic averaging function. As corollaries to our work we show in addition that neither a theorem of Alexander and Wermer about polynomial hulls, nor one of Berndtsson and Ransford about bounded solutions to the $\bar{\partial}$-equation generalize to all multiply-connected domains.

We do not yet know the answer to Question B. The papers of Forelli [7] and Earle and Marden [8] offer an answer to Question C. As Barrett points out in [2], however, their results are not optimal for all domains Ω.

Remark about notation. We shall refer frequently in this paper to multiple-valued functions $F: \Omega \to \mathbf{C}$ where Ω is some open Riemann surface. Such functions are more correctly thought of as single-valued functions defined on the universal cover $\tilde{\Omega}$ of Ω, but in this paper, it will suit our purposes better to abuse notation slightly and think of F as being defined on Ω. Covering maps will all be holomorphic here, so there will be no confusion about what it means for a multiple-valued function to be holomorphic (or real analytic, or smooth, etc). When we speak of the values $\{F(z)\}$ of F at a point $z \in \Omega$, we will mean the set $\{F(w)\colon w \in \tilde{\Omega}$ is a preimage of $z\}$. And when we speak of a branch of F on Ω, we will mean the restriction of F to an open subset of $\tilde{\Omega}$ on which the covering map is injective.

1 Statement of the main theorem

In what follows let $\hat{\Omega}$ be a compact Riemann surface and $\Omega \subset \hat{\Omega}$ be a domain whose complement consists of $n > 0$ disjoint, simply-connected, closed sets. We also require that none of the components of $\hat{\Omega} \setminus \Omega$ be single points. Then we have the following

Theorem 1 (Failure of weak holomorphic averaging) *If $\pi_1(\Omega)$ is non-abelian, then there exists a multiple-valued holomorphic function $F\colon \Omega \to \mathbf{C}$ such that*

(i) *the sets $\{F(z)\}$ are bounded on compact subsets of Ω*

(ii) *F has no "weak holomorphic average." That is, given any $K \subset\subset \Omega$, there is no holomorphic, single-valued $f\colon \Omega \to \mathbf{C}$ satisfying Eq. (0) for all $z \in \Omega \setminus K$.*

It will suffice to prove this theorem for the case when $b\Omega$ (boundary of Ω) co nsists of n real-analytic, simple, closed curves, since Ω will at least be biholomorphic to such a domain (see [9], for example). As corollaries to this theorem, we will see that there are many domains Ω – in particular, those Ω whose boundaries consist of disjoint Jordan curves – for which the function F in Theorem 1 can be taken to be continuous or better up to the boundary of Ω and bounded on all of $\bar{\Omega}$. In this case, it is even impossible to find a bounded holomorphic f whose boundary values satisfy (0) almost everywhere on $b\Omega$.

2 Proof of Theorem 1

To facilitate the proof of Theorem 1, we define some more notation. Notice that a domain satisfying the hypothesis of the theorem will necessarily be hyperbolic. Let $G\colon \mathbf{H}\to\Omega$ be a holomorphic covering map and $g\colon \Omega\to\mathbf{H}$ be its multiple-valued inverse. We will use the variable z to refer to points in $\hat{\Omega}$ and the variable w to refer to points in $\mathbf{H}$ (and occasionally to points in $\mathbf{C}$). The holomorphic cotangent bundle of any open Riemann surface is trivial (see [8, Chap. 3]). Consequently, we can choose a point p in the complement of $\bar{\Omega}$ and a non-vanishing holomorphic (1, 0)-form ω on $\hat{\Omega}\setminus\{p\}$. In particular, the form is defined and non-vanishing on a neighborhood of $\bar{\Omega}$. We define multiple-valued, holomorphic functions g', g'' by

$$\partial g = g'\omega\ ,$$
$$\partial g' = g''\omega\ .$$

Note that $\partial\log g' = (g''/g')\omega$. Furthermore, since g is the inverse of a covering map, g' is never equal to zero. In fact, since we assume $b\Omega$ to be real-analytic, we may also assume that all branches of g extend past $b\Omega$ with the same non-vanishing condition on the derivative.

From the expression for the Poincaré metric on $\mathbf{H}$ and the formula for the pullback of a metric, one can readily compute that the Poincaré metric on Ω is $R|\omega|$ where R is a positive real-valued function given by

$$R = \frac{|g'|}{2\operatorname{Im} g}$$

for any branch of the function g. While g is not single-valued, R certainly is. Hence, the complex-valued function c, which we obtain from the connection form for the Poincaré metric

$$\partial\log R = \left(\frac{g''}{2g'} + \frac{ig'}{2\operatorname{Im} g}\right)\omega = c\omega\ ,$$

is also single-valued. Evidently then, all values of the multiple-valued function $F = g''/2g'$ evaluated at $z\in\Omega$ are taken on the circle with center $c(z)$ and radius $R(z)$. R and c are both bounded on compact subsets of Ω, so F satisfies (i) of

Theorem 1. This construction seems more natural once one realizes that the surface $S=\{(z,w)\in\Omega\times\mathbf{C}: |w-c(z)|=R(z)\}$ is Levi-flat. From this point of view, the graph of F appears as a leaf in the foliation of S by complex submanifolds. It was actually the study of Levi-flat surfaces with circular cross-sections – inspired by the papers [4] and [5] – that led us to this example. Also, Kumagai [10] gives a more complete description of the relationship between Levi-flat surfaces with circular cross-sections and constant negative curvature metrics.

Now suppose that the multiple-valued function F, has a weak holomorphic average f. Then we may write

$$f=c+vR\,,$$

where $v\colon \Omega\to\mathbf{C}$ satisfies $|v(z)|\leqq 1$ for all $z\in\Omega\setminus K$ (K is the compact subset referred to in the statement of Theorem 1). Since f is holomorphic, Cauchy's theorem tells us that

$$\int_\Gamma f\omega=0$$

for all curves Γ homologous to $b\Omega$. Expanding this, we get

$$\frac{1}{2}\int_\Gamma \partial\log g'+i\int_\Gamma\frac{1+u}{2\operatorname{Im} g}\partial g=0\,, \tag{1}$$

where $|u|=|v|$.

For the remainder of this section we devote ourselves to computing the possible values of the integrals in Eq. (1). First we prove a technical lemma that will provide us with the "right" curves for our residue computation. The approximate content of the lemma is that given $\varepsilon>0$ and a component of $b\Omega$, there exists a smooth homologous curve that is ε-close to, but also ε-distant from, this component.

Lemma 2 *Given small enough $\varepsilon>0$ and a component $\Gamma\colon[a,b]\to\hat{\Omega}$ of $b\Omega$, let $\tilde{\Gamma}\colon[a,b]\to\mathbf{H}$ be a lift of Γ to the real axis in $\mathbf{C}$ via G. Then we can choose a smooth loop $\Gamma_\varepsilon\colon[a,b]\to\Omega$ homologous to Γ and with a lift $\tilde{\Gamma}_\varepsilon\colon[a,b]\to\mathbf{H}$ such that*

(i) $|\tilde{\Gamma}_\varepsilon(t)-\tilde{\Gamma}(t)|<k_1\varepsilon$
(ii) $|\tilde{\Gamma}'_\varepsilon(t)-\tilde{\Gamma}'(t)|<k_2\varepsilon$
(iii) $\operatorname{Im}\tilde{\Gamma}_\varepsilon(t)>k_3\varepsilon$,

for all $t\in[a,b]$ and constants k_1, k_2, and k_3 independent of ε.

Proof. With no loss of generality, we may assume that $\tilde{\Gamma}(t)=t+i0$ and $[a,b]=[0,1]$. For $t\in(-0.1,0.1)$ we set $\tilde{\Gamma}_\varepsilon(t)=t+i\varepsilon$, and $\tilde{\Gamma}_\varepsilon(1+t)=g_1(G(\tilde{\Gamma}_\varepsilon(t)))$ where g_1 is the branch of g satisfying $g_1(G(0))=g_1(G(1))=1$. For small enough ε, we will have that

$$|\tilde{\Gamma}_\varepsilon(1)-1|<2|(g_1\circ G)'(0)|\varepsilon<k_1\varepsilon$$

for some constant k_1. Also, since $(g_1 \circ G)'(t+i0)$ is real, positive, and bounded away from zero on a neighborhood of $t=0$,

(2) $$\operatorname{Im} \tilde{\Gamma}_\varepsilon(t) > k_3 \varepsilon .$$

and

(3) $$|\arg \tilde{\Gamma}_\varepsilon'(t)| = |\arg \tilde{\Gamma}_\varepsilon'(t-1) + \arg(g_1 \circ G)'(\tilde{\Gamma}_\varepsilon(t-1))| < k_2 \varepsilon$$

for all t near 1. If we pick $k_3 < 1$, then (2) and (3) hold for all t near 0 as well. Clearly, we will be able to define $\tilde{\Gamma}_\varepsilon(t)$ for $t \in [0.1, 0.9]$ such that the result will be a smooth path $\tilde{\Gamma}_\varepsilon$: $(-0.1, 1.1) \to \mathbf{H}$ satisfying (2) and (3) for all $t \in [0, 1]$. Restricting this path to $[0, 1]$ will give us the preimage of a smooth loop Γ_ε: $[0, 1] \to \Omega$ homologous to Γ. To ensure that $\tilde{\Gamma}_\varepsilon$ satisfies (i) and (ii) of the lemma (it already satisfies (iii)), we reparametrize as follows: Let $L = \operatorname{Re}(\tilde{\Gamma}_\varepsilon(1) - \tilde{\Gamma}_\varepsilon(0))$. Then define $\tilde{\Gamma}_\varepsilon(t)$ to be the point on the curve given by $\operatorname{Re} \tilde{\Gamma}_\varepsilon(t) = \frac{t}{L}$. The new parametrization is well-defined because of (3). Several moments thought shows that this parametrization also satisfies (i) and (ii) – with perhaps slightly more generous constants k_1 and k_2. □

Lemma 3 *Given $\delta > 0$ there is a union $\Gamma_\delta \subset \Omega \setminus K$ of simple, closed curves homologous to $b\Omega$ such that*

(4) $$\operatorname{Im}\left(\frac{1}{2} \int_{\Gamma_\delta} \partial \log g' + i \int_{\Gamma_\delta} \frac{1+u}{2 \operatorname{Im} g} \partial g\right) > -\delta + \operatorname{Im} \frac{1}{2} \int_{b\Omega} \partial \log g' .$$

Remark. Since g is multiple-valued, the meaning of each individual integral in (4) is a priori ambiguous. In order to work with the integrals, we will choose a lift of each component of $b\Omega$ to $\mathbf{H}$ via G. This will determine the meaning of quantities related to g and eliminate the ambiguity in the meaning of the integrals. However, we note that taken together, the two integrals on the left side of (4) have a meaning in terms of the functions R and c described earlier in this section that is independent of the lifts that we choose. In Lemma 4 we will see that the imaginary part of the integral on the right side of (4) is also independent of our choice of lifts.

Proof. Denote the n boundary components of $b\Omega$ by Γ_j for all $j = 1, \ldots, n$. For each of these, choose a lift $\tilde{\Gamma}_j$. Since these lifts are all segments $[a_j, b_j]$ of the real axis, we will identify them with their parametrizations. Let $\Gamma_\delta = \bigcup_{j=1}^n \Gamma_{j,\varepsilon}$ are the curves given by Lemma 2 corresponding to each of the Γ_j. Γ_δ will not intersect K if we choose ε small enough. We will work with each integral on the left side of (4) separately. In order to reduce the amount of notation, we will occasionally suppress composition with G or with one of the Γ's. For instance, when integrating in $\mathbf{H}$, we will write u instead of $u \circ G$.

First we estimate

$$\left|\int_{\Gamma_\delta} \partial \log g' - \int_{b\Omega} \partial \log g'\right| \leqq \sum_j \left| \int_{\tilde{\Gamma}_{j,\varepsilon}} \frac{g''}{(g')^2} dw - \int_{\tilde{\Gamma}_j} \frac{g''}{(g')^2} dw \right|$$

$$= \sum_j \int_{a_j}^{b_j} \left| \frac{g'' \circ G \circ \tilde{\Gamma}_{j,\varepsilon}}{(g' \circ G \circ \tilde{\Gamma}_{j,\varepsilon})^2} \tilde{\Gamma}'_{j,\varepsilon} - \frac{g'' \circ G \circ \tilde{\Gamma}_j}{(g' \circ G \circ \tilde{\Gamma}_j)^2} \tilde{\Gamma}'_j \right| dt .$$

But any branch of g extends analytically past $b\Omega$ with $|g'|$ bounded away from 0. Therefore any fixed branch of $g''/(g')^2$ along with its derivatives will be uniformly continuous on the compact set $b\Omega$. This fact, together with properties (i) and (ii) from Lemma 2 give us that the entire last expression is $O(\varepsilon)$. In particular we can choose ε small enough so that

$$\text{(5)} \qquad \operatorname{Im} \frac{1}{2} \int_{\Gamma_\delta} \partial \log g' > -\frac{\delta}{2} + \operatorname{Im} \frac{1}{2} \int_{b\Omega} \partial \log g' .$$

Now we pull the second integral in (4) back to $\mathbf{H}$.

$$\text{(6)} \quad \operatorname{Im}\left(i \int_{\Gamma_\delta} \frac{1+u}{2 \operatorname{Im} g} \partial g \right) = \operatorname{Re}\left(\int_{\tilde{\Gamma}_\delta} \frac{1+u}{2y} dw \right)$$

$$\geqq \sum_j \int_{a_j}^{b_j} \frac{(1-|\operatorname{Re} u|)(\operatorname{Re} \tilde{\Gamma}'_{j,\varepsilon}) - |(\operatorname{Im} u)(\operatorname{Im} \tilde{\Gamma}'_{j,\varepsilon})|}{2 \operatorname{Im} \tilde{\Gamma}_{j,\varepsilon}} dt$$

$$\geqq \sum_j \int_{a_j}^{b_j} \frac{1-|\operatorname{Re} u| - k_2 \varepsilon |\operatorname{Im} u|}{2 \operatorname{Im} \tilde{\Gamma}_{j,\varepsilon}} \operatorname{Re} \tilde{\Gamma}'_{j,\varepsilon} dt$$

$$= \sum_j \int_{\tilde{\Gamma}_{j,\varepsilon}} \frac{1-|\operatorname{Re} u| - k_2 \varepsilon |\operatorname{Im} u|}{2y} dx .$$

The second inequality in (6) follows from (ii) of Lemma 2 and the fact that $\tilde{\Gamma}'_j = 1 + i0$. For points in $\Omega \setminus K$, we know that $|\operatorname{Re} u|^2 + |\operatorname{Im} u|^2 \leqq 1$, so a computation reveals

$$|\operatorname{Re} u| + k_2 \varepsilon |\operatorname{Im} u| \leqq \sqrt{1 + k_2^2 \varepsilon^2} < 1 + k_2^2 \varepsilon^2 .$$

Also, (iii) of Lemma 2 tells us that $y > k_3 \varepsilon$ for points on $\tilde{\Gamma}_{j,\varepsilon}$. Plugging both of these pieces of information into (6) gives

$$\text{(7)} \qquad \operatorname{Im}\left(i \int_{\Gamma} \frac{1+u}{2 \operatorname{Im} g} \partial g \right) > -\sum_j \frac{k_2 \varepsilon^2}{2 k_3 \varepsilon} \int_{\tilde{\Gamma}_{j,\varepsilon}} dx > -\frac{\delta}{2}$$

for small enough ε. (5) and (7) combine to establish the lemma. □

Lemma 4 *Let κ: $b\Omega \to \mathbf{C}$ be the (positively oriented) geodesic curvature and ds the arc-length element of $b\Omega$ with respect to the metric $|\omega|$. Then*

$$\operatorname{Im} \int_{b\Omega} \partial \log g' = -\int_{b\Omega} \kappa \, ds .$$

Proof. First note that g' (and thus, $\partial \log g'$) remains invariant under local biholomorphism. That is, if h: $U \subset \mathbf{C} \to \Omega$ is a local biholomorphism, and we define $(g \circ h)'$ by $\partial(g \circ h) = (g \circ h)' h^* \omega$, then $(g \circ h)' = g' \circ h$. Also, geodesic curvature and the arc-length element are local invariants of the metric and the curve. So for the purpose of computing κ locally, we may assume that we are working on a curve in $\mathbf{C}$ and that $\omega = dz$. $b\Omega$ is parametrized locally by the covering map G restricted to a small open interval I on the real axis. Since $b\Omega$ is real-analytic, G extends conformally to the negative imaginary side of I. If we interpret it as a complex number, the tangent vector to $b\Omega$ at $z = G(w) = G(x + i0)$ will be given by

$$T(z) = \frac{G'(w)}{|G'(w)|} .$$

We deduce the value of κ from the formula

$$\kappa N = \frac{dT}{ds} = \frac{dT}{dx} \cdot \frac{dx}{ds} ,$$

where $N = iT$ is inward pointing normal vector to $b\Omega$. $dx/ds = 1/|G'|$, so expanding the right side of the last equation shows

$$\begin{aligned} \kappa N &= \frac{G'}{|G'|} \frac{G''\bar{G}' - \bar{G}''G'}{2|G'|^3} \\ &= \left(\frac{1}{|G'|} \operatorname{Im} \frac{G''}{G'}\right) N . \end{aligned}$$

So $\kappa \circ G = \left(\frac{1}{|G'|} \operatorname{Im} \frac{G''}{G'}\right)$. We rewrite G'' and G' in terms of g' and g'' and get $\kappa = -|g'| \operatorname{Im} \frac{g''}{(g')^2}$. Let $\widetilde{b\Omega}$ be a lift of $b\Omega$ to $\mathbf{C}$ via G. Then

$$\int_{b\Omega} \kappa \, ds = \int_{\widetilde{b\Omega}} \kappa \circ G \frac{ds}{dx} dx = -\operatorname{Im} \int_{\widetilde{b\Omega}} \frac{g'' \circ G}{(g' \circ G)^2} dw$$

since dy contributes nothing to an integral along the real axis. Finally we move our path of integration from $\mathbf{C}$ to $\hat{\Omega}$. Here we use the fact that when we change coordinates dw becomes $dg = \partial g = g' \omega$.

$$\int_{b\Omega} \kappa \, ds = -\operatorname{Im} \int_{b\Omega} \frac{g''}{g'} \omega = -\operatorname{Im} \int_{b\Omega} \partial \log g' . \quad \square$$

Proof of Theorem 1 Let $\delta > 0$. Using Lemmas 3 and 4, we choose a union $\Gamma \subset \Omega \setminus K$ of simple closed curves, homologous to $b\Omega$, such that

$$\operatorname{Im}\left(\frac{1}{2} \int_{\Gamma} \partial \log g' + i \int_{\Gamma} \frac{1+u}{2 \operatorname{Im} g} \partial g\right) > -\frac{1}{2} \int_{b\Omega} \kappa \, ds - \delta .$$

Since ω is holomorphic, the metric $|\omega|$ is flat – that is, the Gaussian curvature of $|\omega|$ is identically zero. In this case, the Gauss–Bonnet theorem gives us

$$\int_{b\Omega} \kappa\, ds = 2\pi\chi(\Omega) = 2\pi(2-2\,\mathrm{genus}\,\Omega - n)\,,$$

where $\chi(\Omega)$ is the Euler characteristic of Ω. Putting these last two formulae together, we see that

$$\text{(8)}\qquad \mathrm{Im}\int_{\Gamma} f\omega = \mathrm{Im}\left(\frac{1}{2}\int_{\Gamma} \partial \log g' + i\int_{\Gamma} \frac{1+u}{2\,\mathrm{Im}\, g}\partial g\right)$$
$$> \pi(2\,\mathrm{genus}\,\Omega + n - 2) - \delta\,.$$

The left side of (8) must be zero by Cauchy's theorem. Therefore, (8) can only be true for small δ if genus $\Omega = 0$ and $n = 0$, 1 or 2 (i.e. Ω is a sphere, a disk, or an annulus), or genus $\Omega = 1$ and $n = 0$ (Ω is a torus). These are exactly the cases where $\pi_1(\Omega)$ is abelian. Actually, our computation is invalid for the sphere and the torus anyhow, because neither of these surfaces is hyperbolic and neither has a boundary. In all other cases, we have a contradiction to Cauchy's theorem, and f cannot exist. This concludes the proof of Theorem 1. □

Remark 1 Using similar methods this proof can be modified to allow for single point components (punctures) as well as real-analytic curves in the boundary of Ω. Some changes need to be made, however. The deck transformation corresponding to traveling clockwise about a puncture point will be parabolic, and hence conjugate via some $M \in \mathrm{Aut}\,\mathbf{H}$ to the map $T(w) = w - 1$. So we first transform the left side of Eq. (4) to the analogous equation with "g" replaced by "$M \circ g$", incurring an error term in the process. We then pull the integration back (via $G \circ M^{-1}$) to $\mathbf{H}$, g $\tilde{\Gamma}_\delta = \{t + iy\colon t \in [0, 1]\}$ as our path of integration. For large enough y we are able to control the error term, and prove Lemma 3 with "$b\Omega$" replaced by "Γ_δ". Finally, we modify Lemma 4 to allow for κ and ds to be the curvature and arc-length of our path of integration, rather than of $b\Omega$.

Remark 2 As we noted in Sect. 1, the object of concern in this theorem is really the complex-analytic foliation of a Levi-flat surface in $\Omega \times \mathbf{C}$ with circular cross-sections. We defined this surface solely in terms of the Poincaré metric on Ω. It is possible to prove Theorem 1 by developing an asymptotic expression for the metric near the boundary of Ω. Such a proof does not refer to the holomorphic covering map of Ω. Therefore, one might hope that the techniques used in this proof would also help to answer questions about domains in $\mathbf{C}^n$ where covering maps are scarce but metrics plentiful.

Corollary 5 *Let $\hat{\Omega}$ and Ω (not necessarily smoothly bounded) be as in Theorem* 1. *For any $K \subset\subset \Omega$, there exists a smoothly bounded domain $\Omega_K \subset\subset \Omega$ that contains K and a multiple-valued, holomorphic function $F\colon \Omega \to \mathbf{C}$ such that*

(i) *F is uniformly bounded on all of $\bar{\Omega}_K$*

(ii) *There exists no single-valued holomorphic function $f \in \mathrm{H}^\infty(\Omega_K)$ whose boundary values satisfy* (0) *for* a.e. $z \in b\Omega_K$.

Proof. Again, it is enough to consider the case where Ω has real-analytic boundary. For small enough ε, the curves Γ_ε constructed in Lemma 2 will lie in $\Omega \setminus K$, and the curves corresponding to different boundary components will not intersect. In this case, we let Ω_K be the relatively compact region in Ω bounded by Γ. The entire computation performed above in this section goes through to establish that f cannot exist. □

In fact, with a slight additional restriction we can improve this corollary even further to the following cleaner statement.

Corollary 6 *Suppose $\hat{\Omega}$ and Ω are as in Theorem 1 with $b\Omega$ consisting of n disjoint Jordan curves. Then there is a multiple-valued function F: $\Omega \to \mathbf{C}$ holomorphic in Ω and continuous on $\bar{\Omega}$ such that*

(i) *F is uniformly bounded on all of $\bar{\Omega}$,*

(ii) *There is no single-valued holomorphic function $f \in H^\infty(\Omega)$ whose boundary values satisfy* (0) for a.e. $z \in b\Omega$.

Sketch of proof. We reduce to the case where $b\Omega$ is real-analytic, noting that a conformal map of Ω onto such a domain extends continuously to the boundary. We wish to apply Corollary 6 using Ω as our smaller domain and some other domain with real-analytic boundary as our larger domain. That is, Ω becomes Ω_K, and we look for a new larger Ω. The problem is that we will need to justify the computations performed in Lemmas 2 through 4 with Γ_δ set equal to $b\Omega$. Careful scrutiny of these lemmas will reveal that it is enough to be able to choose a strictly larger domain Ω_0 with real-analytic boundary and a covering map G_0 that is close to the covering map G of Ω in the following sense: we can analytically extend every branch of $g = G^{-1}$ to $\bar{\Omega}_0$, and any fixed branch of g along with finitely many of its derivatives is as close as we choose (the choice being made before selecting Ω_0) to a corresponding branch and derivatives of g_0 on all of $\bar{\Omega}$. The existence of these nearby larger domains can be shown by carefully applying Schwarz reflection to G and then using the Riemann mapping theorem to produce the holomorphic covering maps for larger domains. Once this is done, the proof is finished. □

Remark. If $b\Omega$ is real-analytic (or smooth), then we can take our multiple-valued function to be likewise on $\bar{\Omega}$.

3 Applications of Theorem 1

The shortest path to our applications of Theorem 1 will involve stating several previously known theorems and referring the reader to proofs occurring in other papers. To begin with, let $\Omega \subset \mathbf{C}$ be a domain. In this case, it is most natural to choose $\omega = dz$. Let R: $\Omega \to (0, \infty)$ and c: $\Omega \to \mathbf{C}$ be twice continuously differentiable functions.

Theorem 7 *Given twice continuously differentiable functions R: $\Omega \to (0, \infty)$ and c: $\Omega \to \mathbf{C}$, $S = \{(z, w) \in \Omega \times \mathbf{C}$: $|w - c(z)| = R(z)\}$ is Levi-flat if and only if*

(i) $(\log R)_{z\bar{z}} = \dfrac{|c_{\bar{z}}|^2}{R^2}$

(ii) $Rc_{z\bar{z}} = 2R_z c_{\bar{z}}$.

Proof. See [2].

It is a consequence of the maximum principle that if R and c are continuous up to the boundary of Ω, then the region $\bar{S} = \{(z, w) \in \bar{\Omega} \times \mathbf{C}: |w - c(z)| \leqq R(z)\}$ is a subset of the $A(\Omega \times \mathbf{C})$-hull of $S \cap (b\Omega \times \mathbf{C})$. See [5] for further details about this fact. Now we take R and c to be defined as in Sect. 2. The statement that the Poincaré metric has constant Gaussian curvature equal to -1 translates to the partial differential equation $(\log R)_{z\bar{z}} = R^2$. From this fact, one can calculate straightforwardly that R and c satisfy (i) and (ii) of Theorem 6. We can now rewrite Corollary 6 in the slightly stronger form

Corollary 6a *Let $\hat{\Omega}$ and Ω (not necessarily smoothly bounded) be as in Corollary 6. Then there exists a Levi-flat surface $S = \{(z, w) \in \Omega \times \mathbf{C}: |w - c(z)| = R(z)\}$, extending continuously to $\bar{\Omega}$, and such that*

(i) *The circular cross-sections $S(z)$ of S are uniformly bounded for all $z \in \bar{\Omega}$*

(ii) *There is no single-valued holomorphic function $f \in \mathrm{H}^\infty(\Omega)$ whose boundary values satisfy*

$$f(z) \in \operatorname{cch} S(z)$$

for a.e. $z \in b\Omega$.

In the language of [5], $\bar{S}$ has no "analytic selector." Corollary 6a provides us with ample counterexamples to the following two theorems when we replace Δ = "the unit disk" in each of them with Ω = "a general multiply-connected domain." A stronger version of the first theorem was proved by Alexander and Wermer, and independently by Slodkowski. A proof may be found in [1].

Theorem 8 *Let $K \subset\subset b\Delta \times \mathbf{C}$ be such that $K(z)$ is convex for each $z \in b\Delta$ and the $A(\Delta \times \mathbf{C})$-hull of K is non-trivial. Then there exists a single-valued holomorphic function $f \in \mathrm{H}^\infty(\Delta)$ whose boundary values satisfy $f(z) \in K(z)$ for* a.e. $z \in b\Delta$.

Counterexample in the multiply-connected case. Pick any domain in $\mathbf{C}$ satisfying the hypothesis of Corollary 6. Let S be the Levi-flat surface with circular sections, guaranteed by Corollary 6a. As we noted above, the $A(\Omega \times \mathbf{C})$-hull of $\bar{S} \cap b\Omega$ contains $\bar{S}$. However, Corollary 6a rules out the existence of the corresponding holomorphic function f. □

In [5], Berndtsson and Ransford used Alexander and Wermer's result to prove a theorem about bounded solutions to the $\bar{\partial}$-problem on the unit disk. They show that their theorem implies another such theorem proved by Wolff, and that it also implies the Corona theorem. Before we provide a counterexample to Berndtsson and Ransford's theorem on multiply-connected domains, we note that it is not clear to us what implications the counterexample has for Wolff's theorem or for the Corona theorem in this more general setting. Clearly it does not disprove the Corona theorem – which is known to hold for finitely-connected domains in **C**. However, it may be the case that the counterexample we offer here yields a lower bound for the constant appearing in either Wolff's theorem or the Corona theorem.

Theorem 9 *Given functions u and a continuous on $\bar{\Delta}$, twice differentiable on Δ, and satisfying*

$$u_{z\bar{z}} \geqq |a|^2 e^{-2u} + |a_z + 2u_z a| e^{-u},$$

there is a solution to $\bar{\partial} g = a$ satisfying $|g| \leqq e^u$ for a.e. $z \in b\Delta$.

Counterexample in the multiply-connected case. Let Ω and S be the same as in the last counterexample. Set $u = \log R$ and $a = c_{\bar{z}}$, and note that these two functions satisfy the differential inequality in Theorem 8. If g existed, then $f = c - g$ would violate Corollary 6a. □

Remark. The referee for this paper has kindly (and correctly) pointed out that Lemmas 3 and 4 are a roundabout way of saying that if Γ_δ is chosen properly, then the geodesic curvature of Γ_δ in the Poincaré metric is greater than $1 \bmod \delta$. It is possible to prove Theorem 1 in a way that uses this fact more explicitly.

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