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# On the geodesic connectedeness of Lorentzian manifolds

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## 1 Introduction and statement of the results

In this paper we study the geodesic connectedeness of Lorentzian manifolds. We recall that a Lorentzian manifold $(\mathcal{M}, g)$ is a smooth connected finite dimensional manifold $\mathcal{M}$, equipped with a smooth (0, 2) symmetric tensor field $g$, such that for every $z \in \mathcal{M}$, $g(z)[\cdot, \cdot]$ is a nondegenerate bilinear form, having exactly one negative eigenvalue.

Geodesic curves play an important role in the study of the global geometry of a Lorentzian manifold. We recall that a smooth curve $\gamma\colon ]a, b[ \to M$ is said **geodesic** if

$$\nabla_s \dot{\gamma} = 0 ,$$

where $\dot{\gamma}$ is the tangent vector field along $\gamma$, and $\nabla_s \dot{\gamma}$ is the covariant derivative of $\dot{\gamma}$ along $\gamma$ induced by the metric.

It is well known that if $\gamma$ is a geodesic, there exists a constant $E(\gamma)$ such that

$$E(\gamma) = g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] \quad \forall s \in ]a, b[ .$$

Then $\gamma$ is said

**timelike** if $E(\gamma) < 0$ ;
**lightlike** if $E(\gamma) = 0$ ;
**spacelike** if $E(\gamma) > 0$ .

This classification is called **causal character of geodesics**.

The terminology of the causal character comes from general relativity. Indeed four dimensional Lorentzian manifolds, which are called space-times, describe gravitational fields of general relativity. The points of a space-time are called events.

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In particular a timelike geodesic represents the trajectory of a free falling particle in a space-time. Lightlike geodesics represent the light rays. The spacelike geodesics have a more subtle interpretation: for a suitable local observer, they represent "Riemannian" geodesics consisting of simultaneous events.

In global Lorentzian geometry, a problem which arises naturally is the geodesic connectedeness. A Lorentzian manifold (or also a Riemannian manifold) is said **geodesically connected** if every couple of points may be joined by a geodesic.

This problem is quite different and more difficult than in the Riemannian case. Indeed, the Hopf–Rinow Theorem guarantees that

*Every geodesically complete Riemannian manifold is geodesically connected.*

We recall that a Riemannian (or Lorentzian) manifold is said **geodesically complete** if every geodesic can be extended to a geodesic defined in $\mathbb{R}$. For example every compact Riemannian or Lorentzian manifold is geodesically complete.

In the Lorentzian case, the geodesic completeness does not imply the geodesic connectedeness (see [Pe, p. 7]), not even when the manifold is compact (see [BP]).

Some results on the geodesic connectedeness have been obtained for stationary Lorentzian manifolds, using variational methods (see [BF1, BF2, BFG] and the survey paper [BFM]). Another result on the geodesic connectedeness for a manifold with a linear connection has been obtained in [BP], without using variational tools.

Another interesting global problem in Lorentzian geometry is the connectedeness of two points by a timelike geodesic.

An important result was proved by Avez and Seifert in the class of the globally hyperbolic Lorentzian manifolds, which were introduced by Leray in the study of the well posedness of the Cauchy Problem (see [ON, p. 412] for the definition a globally hyperbolic Lorentzian manifold). Avez and Seifert proved (see [Av, Se]) that

*Every couple of points causally related in a globally Lorentzian manifold, can be joined by a timelike geodesic.*

We recall that two points are **causally related** if there is a timelike curve joining them.

In this paper we study the geodesic connectedeness of Lorentzian manifolds. We consider a connected manifold

$$\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}\,, \tag{1.1}$$

equipped with the Lorentz metric

$$g(z)[\zeta, \zeta] = g(z)[(\xi, \tau), (\xi, \tau)] = \langle \alpha(z)\xi, \xi \rangle - \beta(z)\tau^2\,, \tag{1.2}$$

for all $z=(x,t)\in\mathcal{M}$, $\zeta=(\xi,\tau)\in T_z\mathcal{M}\equiv T_x\mathcal{M}_0\times\mathbb{R}$, where $\langle\cdot,\cdot\rangle$ is a Riemannian metric on $\mathcal{M}_0$, $\alpha(z)=\alpha(x,t)$ is a positive linear operator on $T_x\mathcal{M}_0$ which depends smoothly on $z$, and $\beta$ is a smooth positive function on $\mathcal{M}$.

These assumptions are not too restrictive. Indeed Geroch has proved that every globally hyperbolic time-oriented Lorentzian manifold satisfy (1.1) and (1.2) (see [Ge] and [U]).

The main result of this paper is the following theorem on the geodesic connectedeness of $(\mathcal{M}, g)$.

**Theorem 1.1**
*Let $(\mathcal{M}, g)$ be a Lorentzian manifold which satisfies* (1.1) *and* (1.2). *Assume that:*

($A_1$) *$(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is a complete Riemannian manifold;*

($A_2$) *there exists a constant $\lambda>0$, such that for every $z=(x,t)\in\mathcal{M}$ and for every $\xi\in T_x\mathcal{M}_0$:*

$$\lambda\langle\xi,\xi\rangle\leqq\langle\alpha(x,t)\xi,\xi\rangle\ ;$$

($A_3$) *there exists two constants $0<\nu\leqq C$, such that*

$$\nu\leqq\beta(z)\leqq C\ ;$$

*we set for simplicity $C=1$.*

($A_4$)
$$\sup_{z\in\mathcal{M}}\{|\beta_t(z)|,\ \|\alpha_t(z)\|\}<+\infty\ ,$$

*where $\beta_t(z)$ and $\alpha_t(z)$ denote the partial derivatives of $\beta$ and $\alpha$;*

($A_5$)
$$\text{(i)}\ \limsup_{t\to+\infty}\langle\alpha_t(x,t)\xi,\xi\rangle\leqq 0\ ,$$

$$\text{(ii)}\ \liminf_{t\to-\infty}\langle\alpha_t(x,t)\xi,\xi\rangle\geqq 0\ ,$$

*uniformly in $x\in\mathcal{M}_0$ and $\xi\in T_x\mathcal{M}_0$.*

*Then $(\mathcal{M}, g)$ is geodesically connected.*

*Remark 1.2* Suppose that assumptions ($A_1$)–($A_5$) hold. Moreover, suppose that $\mathcal{M}_0$ is not contractible in itself. Then, for every couple of points of $\mathcal{M}$, there exists a sequence $(\gamma_m)_{m\in\mathbb{N}}$ of geodesics joining them, such that

$$\lim_{m\to\infty} E(\gamma_m)=+\infty\ .$$

This result can be proved by using the estimates of Sect. 4 and the techniques developed in [BF] in order to get infinitely geodesics joining two points of a stationary Lorentzian manifold. So, we omit the proof.

*Remark 1.3* It is not difficult to see that assumptions ($A_1$)–($A_3$) imply that $(\mathcal{M}, g)$ is globally hyperbolic. However global hyperbolicity is not sufficient to guarantee the geodesic connectedeness. For example consider $\mathbb{R}^2$ equipped with the Lorentzian metric

$$ds^2=(1+t^2)dx^2-\frac{1}{1+t^2}dt^2 . \tag{1.3}$$

Notice that after the change of variables

$$\begin{cases} x=x \\ t=\operatorname{tg}\tau , \end{cases}$$

the metric (1.3) becomes

$$ds^2=\frac{dx^2-d\tau^2}{\cos^2\tau} , \tag{1.4}$$

with $x\in\mathbb{R}$ and $\tau\in]-\frac{\pi}{2},\frac{\pi}{2}[$. It can be seen that $]-\frac{\pi}{2},\frac{\pi}{2}[\times\mathbb{R}$, equipped with (1.4) is globally hyperbolic. Nevertheless it is not geodesically connected (see [Pe, p. 7]).

*Remark 1.4* If $|t_1-t_0|$ is sufficiently large (in dependence of $x_0$, $x_1$ and $t_0$), the geodesic found in Theorem 1.1 is timelike. We recall that the existence of a timelike geodesic joining two causally related points has been proved in [A, Se].

## 2 The functional framework

In this section we introduce the functional framework in order to prove Theorems 1.1 and 1.4.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold which satisfies (1.1) and (1.2). By the Nash imbedding theorem, the Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is isometric to a submanifold of the euclidean space $\mathbb{R}^N$, with $N=\frac{n}{2}(n+1)(3n+11)$, $n=\dim\mathcal{M}_0$.

Let $\varphi:\mathcal{M}_0\to\mathbb{R}^N$ be an isometric imbedding and $\tilde{\mathcal{M}}_0=\varphi(\mathcal{M}_0)$. Moreover, we set $\tilde{\mathcal{M}}=\tilde{\mathcal{M}}_0\times\mathbb{R}$. We consider on $\mathcal{M}$ the Lorentzian metric $\tilde{g}$, which is the pull-back metric of $g$ for the diffeomorphism

$$(x,t)\in\tilde{\mathcal{M}}\to(\psi(x),t)\in\mathcal{M} .$$

It is easy to see that $g$ has the form (1.2) and the Riemannian metric on $\mathcal{M}_0$ is the Euclidean one. Hence, without loss of generality, we may suppose that $\mathcal{M}_0$ is a submanifold of $\mathbb{R}^N$ and $\langle\cdot,\cdot\rangle$ is the usual Euclidean metric.

Now we introduce the functional manifolds in which we shall work. We set $I=[0,1]$.

Let $z_0=(x_0,t_0)$ and $z_1=(x_1,t_1)$ be two points of $\mathcal{M}$. We set

$$\begin{aligned}\Omega^1&=\Omega^1(\mathcal{M}_0,x_0,x_1)\\ &=\{x:I\to\mathcal{M}_0 \mid x\in H^{1,2}(I,\mathbb{R}^N), x(0)=x_0, x(1)=x_1\} .\end{aligned} \tag{2.1}$$

We recall that for every $k \in \mathbb{N}$, $H^{1,2}(I, \mathbb{R}^k)$ is the Sobolev space of absolutely continous curves in $\mathbb{R}^k$, whose derivative is square summable. It is a Hilbert space with norm

$$\|x\|_1^2 = \|x\|^2 + \|\dot{x}\|^2 ,$$

where $\dot{x}$ is the derivative of $x$, and $\|\cdot\|$ is the usual norm of $L^2(I, \mathbb{R}^k)$.

It is well known, see for instance [Pa], that $\Omega^1$ is a Riemannian submanifold of $H^{1,2}(I, \mathbb{R}^N)$. For every $x \in \Omega^1$, the tangent space is

$$T_x\Omega^1 = \{\xi: I \to \mathbb{R}^N \mid \xi \in H_0^{1,2}(I, \mathbb{R}^N),\ \xi(s) \in T_{x(s)}\mathcal{M}_0 \forall s \in I\} ,$$

where, for every $k \in \mathbb{N}$,

$$H_0^{1,2}(I, \mathbb{R}^k) = \{\xi \in H^{1,2}(I, \mathbb{R}^k) \mid \xi(0) = \xi(1) = 0\} .$$

Now, we set

$$H^{1,2}(t_0, t_1) = \{t \in H^{1,2}(I, \mathbb{R}) \mid t(0) = t_0,\ t(1) = t_1\} .$$

$H^{1,2}(t_0, t_1)$ is a closed linear submanifold of $H^{1,2}(I, \mathbb{R})$. Indeed, let $\bar{t}(s) = t_0 + s(t_1 - t_0)$ be the segment joining $t_0$ and $t_1$. Then

$$H^{1,2}(t_0, t_1) = \bar{t} + H_0^{1,2}(I, \mathbb{R}) .$$

Finally we set

$$\mathscr{Z} = \Omega^1 \times H^{1,2}(t_0, t_1) .$$

$\mathscr{Z}$ is the manifold of the curves in $H^{1,2}(I, \mathbb{R}^{N+1})$ joining $z_0$ and $z_1$ in $\mathcal{M}$. The tangent space to a curve $z = (x, t)$ in $\mathscr{Z}$, is given by

$$T_z\mathscr{Z} = T_x\Omega^1 \times H_0^{1,2}(I, \mathbb{R}) .$$

In $\mathscr{Z}$ we consider the *action integral*

$$f(z) = f(x, t) = \frac{1}{2}\int_0^1 g(z(s))[\dot{z}(s), \dot{z}(s)]\,ds$$

$$= \frac{1}{2}\int_0^1 [\langle \alpha(z)\dot{x}, \dot{x}\rangle - \beta(z)\dot{t}^2]\,ds . \tag{2.2}$$

It is easy to see that $f$ is a smooth functional on $\mathscr{Z}$. Moreover, its critical points are the geodesics joining $z_0$ and $z_1$.

The action integral $f$ is indefinite both from below and from above, also modulo compact perturbations. This fact creates difficulties in searching its critical points.

In order to overcome this problem, we introduce a Galerkin approximation argument in the variable $t$. For every $k \in \mathbb{N}$, we set

$$H_{0,k}^{1,2} = \text{span}\{\sin(\pi l s),\ 1 \leq l \leq k\} .$$

Notice that $H^{1,2}_{0,k}$ is a finite dimensional subspace of $H^{1,2}_{0}(I, \mathbb{R})$. Moreover, we set

$$H^{1,2}_{0,k}(t_0, t_1) = \bar{t} + H^{1,2}_{0,k} \ ,$$

which is a finite dimensional submanifold of $H^{1,2}(t_0, t_1)$.

Finally we set

$$\mathscr{Z}_k = \Omega^1 \times H^{1,2}_{0,k}(t_0, t_1) \ . \tag{2.3}$$

Now we state a critical point theorem, which is a particular case of the Rabinowitz saddle point theorem.

Let $\bar{x} \in \Omega^1$ fixed and $\bar{t}$ the segment joining $t_0$ and $t_1$. We set

$$S = \{(x, \bar{t}), x \in \Omega^1\} \subseteq \mathscr{Z} \ . \tag{2.4}$$

Moreover, for every $k \in \mathbb{N}$, $R > 0$, we set

$$Q(R) = \{(\bar{x}, t) \in \mathscr{Z} \mid \|t - \bar{t}\|_1 < R\} \ , \tag{2.5}$$

$$Q_k(R) = \{(\bar{x}, t) \in \mathscr{Z}_k \mid \|t - \bar{t}\|_1 < R\} \ . \tag{2.6}$$

Notice that $Q$ and $Q_k$ are Hilbertian submanifolds of $\mathscr{Z}$ and $\mathscr{Z}_k$ respectively, whose boundaries are

$$\partial Q(R) = \{(\bar{x}, t) \in \mathscr{Z} \mid \|t - \bar{t}\|_1 = R\} \ , \tag{2.7}$$

$$\partial Q_k(R) = \{(\bar{x}, t) \in \mathscr{Z}_k \mid \|t - \bar{t}\|_1 = R\} \ . \tag{2.8}$$

Now we recall the well known Palais–Smale (PS) condition.

**Definition 2.1** *Let $f: \mathscr{M} \to \mathbb{R}$ be a functional defined on the Riemannian manifold $\mathscr{M}$. $f$ satisfies* **(PS) condition** *if for every sequence $(x_m)_{m \in \mathbb{N}}$, such that*

(i) $(f(x_m))_{m \in \mathbb{N}}$ *is bounded;*
(ii) $\|\nabla f(x_m)\| \xrightarrow[m \to \infty]{} 0$,

*there exists a converging subsequence.*

We recall that $\nabla f(x)$ is the gradient of $f$ in $x$, with respect to the Riemannian structure of $\mathscr{M}$.

We have the following slight variant of the saddle point theorem (see [R, BF1]).

**Theorem 2.2** *Let $I: \mathscr{Z} \to \mathbb{R}$ be a $C^1$ functional and $I_k$ the restriction of $I$ to $\mathscr{Z}_k$. Assume that*

a) $I_k$ *satisfies* (PS) *condition for every* $k \in \mathbb{N}$;
b) $\exists R > 0$, *such that:*

$$\text{(i)} \ \sup I(Q(R)) < +\infty \ ;$$

$$\text{(ii)} \ \sup I(\partial Q(R)) < \inf I(S) \ .$$

*Define, for every $k \in \mathbb{N}$:*

$$c_k = \inf_{h \in \Gamma_k} \sup I(h(Q_k(R))) ,$$

*where*

$$\Gamma_k = \{h \in C(\mathscr{X}_k, \mathscr{X}_k) \mid h(z) = z \text{ for every } z \in \partial Q_k(R)\} .$$

*Then every $c_k$ is well defined, $c_k \in ]\inf I(S), \sup I(Q(R))]$, and is a critical value of $I_k$.*

## 3 The (PS) for the penalized functional

In Theorem 2.2 the (PS) plays a basic role. Unfortunately we are not able to prove the (PS) condition for the action integral and actually we think that it does not satisfy this condition.

In order to avoid this difficulty, we introduce a penalization argument. We consider the function $\psi$: $\mathbb{R}_+ \to \mathbb{R}_+$, defined in the following way:

$$\psi(s) = \begin{cases} 0 & 0 \leqq s \leqq 1 \\ (s-1)^2 & s \geqq 1 . \end{cases} \tag{3.1}$$

It is clear that $\psi$ is $C^1$.

Now, for every $\delta > 0$, we consider the penalized functional $f_\delta$: $\mathscr{X} \to \mathbb{R}$, such that for all $z \in \mathscr{X}$

$$f_\delta(z) = f(z) + \psi\left(\frac{\delta}{2} \lambda \|\dot{x}\|^2\right) - \psi\left(\frac{\delta}{2} \|\dot{t}\|^2\right), \tag{3.2}$$

where $\lambda$ is defined in assumption $(A_2)$ of Theorem 1.1. We set $f_0 = f$.

By the choice of the penalization, it is easy to prove the following

**Lemma 3.1** *For every $\delta > 0$, let $z_\delta = (x_\delta, t_\delta)$ be a critical point of $f_\delta$, such that*

$$\sup_{\delta > 0} \{\|\dot{x}_\delta\|, \|\dot{t}_\delta\|\} < +\infty . \tag{3.3}$$

*Then, if $\delta$ is small enough, $z_\delta$ is a critical point of $f$.*

We want to prove the (PS) condition for the penalized functional $f_\delta$. To this end, we need the following lemma, whose proof is contained in [BF2].

**Lemma 3.2** *Let $(x_m)_{m \in \mathbb{N}}$ be a sequence in $\Omega^1$ such that*

$$x_m \to x, \quad \text{weakly in } H^{1,2}(I, \mathbb{R}^N) .$$

*Then there exist two sequences $(\xi_m)_{m \in \mathbb{N}}$ and $(v_m)_{m \in \mathbb{N}}$ in $H^{1,2}(I, \mathbb{R}^N)$, such that:*

$$x_m - x = \xi_m + v_m ; \tag{3.4}$$

$$\xi_m \in T_{x_m}\Omega^1, \; v_m \in H_0^{1,2}(I, \mathbb{R}^N) ; \tag{3.5}$$

$$\xi_m \xrightarrow[m \to \infty]{} 0 \text{ weakly in } H^{1,2}(I, \mathbb{R}^N) ; \tag{3.6}$$

$$v_m \xrightarrow[m \to \infty]{} 0 \text{ strongly in } H^{1,2}(I, \mathbb{R}^N) . \tag{3.7}$$

In the following we shall assume for simplicity that

$$\langle\alpha(z)\xi,\xi\rangle=\alpha(z)\langle\xi,\xi\rangle\,,$$

i.e. $\alpha$ is a smooth function on $\mathscr{M}$. Moreover we assume that the constant $\lambda$ in assumption ($A_2$) of Theorem 1.1 is equal to 1. The same calculation can be carried out also in the general case, with some more technicality.

Next proposition shows that the penalized functional $f_\delta$ satisfies (PS) condition.

**Proposition 3.3** *Assume that* ($A_1$)–($A_4$) *of Theorem* 1.1 *hold and fix* $\delta>0$. *Let* $(z_m)=(x_m,t_m)_{m\in\mathbb{N}}$ *be a sequence in* $Z$ *such that*

$$f_\delta(z_m)\leqq c\,; \tag{3.8}$$

$$\|f'_\delta(z_m)\|\xrightarrow[m\to\infty]{}0\,. \tag{3.9}$$

*Then* $(z_m)_{m\in\mathbb{N}}$ *contains a converging subsequence.*

*Proof.* The Frechet differential of $f_\delta$ at $z=(x,t)\in\mathscr{Z}$ and $\zeta=(\xi,\tau)\in T_Z\mathscr{Z}$ is given by

$$\begin{aligned}f'(z)[\zeta]=&\int_0^1\left[\alpha(z)\langle\dot x,\dot\xi\rangle+\frac{1}{2}\langle\dot x,\dot x\rangle\langle\nabla\alpha(z),\xi\rangle+\frac{1}{2}\langle\dot x,\dot x\rangle\alpha_t(z)\tau\right]ds+\\&-\int_0^1\left[\beta(z)\dot t\dot\tau+\frac{1}{2}\dot t^2\langle\nabla\beta(z),\xi\rangle+\frac{1}{2}\dot t^2\beta_t(z)\tau\right]ds\\&+\delta\psi'\left(\frac{\delta}{2}\|\dot x\|^2\right)\int_0^1\langle\dot x,\dot\xi\rangle ds-\delta\psi'\left(\frac{\delta}{2}\|\dot t\|^2\right)\int_0^1\dot t\dot\tau ds\,,\end{aligned}$$

where $\nabla\alpha(z)$ and $\nabla\beta(z)$ denote respectively the gradient of $\alpha$ and $\beta$ with respect to the Riemannian structure of $\mathscr{M}_0$.

For every $m\in\mathbb{N}$, we set

$$\tau_m=t_m-\bar t\,. \tag{3.10}$$

Denoting by $o(1)$ an infinitesimal sequence, from ($A_4$) and (3.9), we have:

$$\begin{aligned}o(1)\|\tau_m\|_1&=f'_\delta(z_m)[(0,\tau_m)]\\&=\frac{1}{2}\int_0^1\alpha_t(z_m)\langle\dot x_m,\dot x_m\rangle\tau_m ds-\int_0^1\beta(z_m)\dot t_m\dot\tau_m ds+\\&\quad-\frac{1}{2}\int_0^1\beta_t(z_m)\dot t_m^2\tau_m ds-\delta\psi'\left(\frac{\delta}{2}\|\dot t_m\|^2\right)\int_0^1\dot t_m\dot\tau_m ds\\&\leqq M\|\tau_m\|_\infty[\|\dot x_m\|^2+\|\dot t_m\|^2]-\int_0^1\beta(z_m)\dot t_m^2 ds+\int_0^1\beta(z_m)\dot t_m\dot{\bar t}ds+\\&\quad-\delta\psi'\left(\frac{\delta}{2}\|\dot t_m\|^2\right)\int_0^1\dot t_m^2 ds+\delta\psi'\left(\frac{\delta}{2}\|\dot t_m\|^2\right)\int_0^1\dot t_m\dot{\bar t}_m ds\,,\end{aligned}$$

where $\|\tau_m\|_\infty=\sup_{t\in I}|\tau_m(s)|$.

Putting $\Delta=t_1-t_0$, we have:

$$o(1)\|\tau_m\|_1 \leqq M\|\tau_m\|_\infty[\|\dot{x}_m\|^2+\|\dot{t}_m\|^2]-\int_0^1 \beta(z_m)\dot{t}_m^2 ds+$$
$$-\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)\|\dot{t}_m\|^2+\Delta\|\dot{t}_m\|+\delta\Delta^2\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right). \quad (3.11)$$

Now, from (3.8) and (3.11) we get

$$o(1)\|\tau_m\|_1$$
$$\leqq M\|\tau_m\|_\infty[\|\dot{x}_m\|^2+\|\dot{t}_m\|^2]-\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)\|\dot{t}_m\|^2+\Delta\|\dot{t}_m\|$$
$$+\delta\Delta^2\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)+2c-2\psi\left(\frac{\delta}{2}\|\dot{x}_m\|^2\right)+2\psi\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right). \quad (3.12)$$

So we have:

$$2\psi\left(\frac{\delta}{2}\|\dot{x}_m\|^2\right)-2\psi\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)\|\dot{t}_m\|^2$$
$$\leqq M\|\tau_m\|_\infty[\|\dot{x}_m\|^2+\|\dot{t}_m\|^2]+\Delta\|\dot{t}_m\|+\delta\Delta^2\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)$$
$$+c-o(1)\|\tau_m\|_1 . \quad (3.13)$$

From (3.13) we deduce that the sequences $(\|\dot{x}_m\|)_{m\in\mathbb{N}}$ and $(\|\dot{t}_m\|)_{m\in\mathbb{N}}$ are bounded. Indeed, for every $m\in\mathbb{N}$ such that $\frac{\delta}{2}\|\dot{x}_m\|^2\geqq 2$ and $\frac{\delta}{2}\|\dot{t}_m\|^2\geqq 2$, we have:

$$2\psi\left(\frac{\delta}{2}\|\dot{x}_m\|^2\right)-2\psi\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)\|\dot{t}_m\|^2$$
$$\leqq M\|\tau_m\|_\infty[\|\dot{x}_m\|^2+\|\dot{t}_m\|^2]+\Delta\|\dot{t}_m\|+2\delta\Delta^2\left[\frac{\delta}{2}\|\dot{t}_m\|^2-1\right]$$
$$+2c-o(1)\|\tau_m\|_1 .$$

Moreover, straightforward calculations show that for every $s\geqq 1$

$$s\psi'(s)-\psi(s)=s^2-1 , \quad (3.14)$$

so we get

$$2\psi\left(\frac{\delta}{2}\|\dot{x}_m\|^2\right)+2\left[\frac{\delta^2}{4}\|\dot{t}_m\|^4-1\right]$$
$$\leqq M\|\tau_m\|_\infty[\|\dot{x}_m\|^2+\|\dot{t}_m\|^2]+\Delta\|\dot{t}_m\|+2\delta\Delta^2\left[\frac{\delta}{2}\|\dot{t}_m\|^2-1\right]$$
$$+2c-o(1)\|\tau_m\|_1 .$$

Finally, since

$$\psi(s) \geqq c_1 s^2 - c_2$$

for suitable positive constants $c_1$ and $c_2$, we get:

$$c_1 \cdot \frac{\delta^2}{2} \|\dot{x}_m\|^4 + \frac{\delta^2}{2} \|\dot{t}_m\|^4$$

$$\leqq M \|\tau_m\|_\infty [\|\dot{x}_m\|^2 + \|\dot{t}_m\|^2] + \Delta \|\dot{t}_m\| + 2\delta\Delta^2 \left[\frac{\delta}{2} \|\dot{t}_m\|^2 - 1\right]$$

$$+ 2c - o(1)\|\tau_m\|_1 + 2 + 2c_2 ,$$

from which we deduce that the sequences $(\|\dot{x}_m\|)_{m\in\mathbb{N}}$ and $(\|\dot{t}_m\|)_{m\in\mathbb{N}}$ are bounded.

Hence $(x_m)_{m\in\mathbb{N}}$ and $(t_m)_{m\in\mathbb{N}}$ weakly converges respectively to $x \in H^{1,2}(I, \mathbb{R}^N)$ and $t \in H^{1,2}(I, \mathbb{R})$.

By $(A_1)$, $\Omega^1$ is a complete submanifold of $H^{1,2}(I, \mathbb{R}^N)$, so $x \in \Omega^1$. Moreover $t \in H^{1,2}(t_0, t_1)$, because it is a closed linear submanifold of $H^{1,2}(I, \mathbb{R})$.

Now we prove that

$$x_m \xrightarrow[m\to\infty]{} x \quad \text{strongly in } H^{1,2}(I, \mathbb{R}^N) , \tag{3.15}$$

$$t_m \xrightarrow[m\to\infty]{} t \quad \text{strongly in } H^{1,2}(I, \mathbb{R}) . \tag{3.16}$$

Let $(\xi_m)_{m\in\mathbb{N}}$ and $(v_m)_{m\in\mathbb{N}}$ be two sequences which satisfy (3.4), (3.5), (3.6) and (3.7) (see Lemma 3.2). Moreover, let

$$\tau_m = t_m - t .$$

From (3.9) we have:

$$o(1) = f'(z_m)[(\xi_m, -\tau_m)]$$

$$= \int_0^1 \left[\alpha(z_m)\langle \dot{x}_m, \dot{\xi}_m\rangle - \frac{1}{2}\langle \dot{x}_m, \dot{x}_m\rangle \alpha_t'(z_m)\tau_m\right.$$

$$\left. + \frac{1}{2}\langle \dot{x}_m, \dot{x}_m\rangle\langle \nabla\alpha(z_m), \xi_m\rangle\right] ds$$

$$+ \int_0^1 \left[\beta(z_m)\dot{t}_m\dot{\tau}_m + \frac{1}{2}\beta_t'(z_m)\dot{t}_m^2\tau_m + \frac{1}{2}\dot{t}_m^2\langle \nabla\beta(z_m), \xi_m\rangle\right] ds$$

$$+ \delta\psi'\left(\frac{\delta}{2}\|\dot{x}_m\|^2\right)\int_0^1 \langle \dot{x}_m, \dot{\xi}_m\rangle ds + \delta\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)\int_0^1 \dot{t}_m\dot{\tau}_m ds . \tag{3.17}$$

Since the weak convergence in $H^{1,2}(I, \mathbb{R}^N)$ implies the uniform convergence, from (3.6), (3.10) and (3.17), we have:

$$o(1)=\int_0^1\left[\alpha(z_m)+\delta\psi'\left(\frac{\delta}{2}\|\dot{x}_m\|^2\right)\right]\langle\dot{x}_m, \dot{\xi}_m\rangle ds$$
$$+\int_0^1\left[\beta(z_m)+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)\right]\dot{t}_m\dot{\tau}_m ds\,.$$

Finally, from (3.4), (3.7) and (3.10) we get

$$o(1)=\int_0^1\left[\alpha(z_m)+\delta\psi'\left(\frac{\delta}{2}\|\dot{x}_m\|^2\right)\right]\langle\dot{\xi}_m, \dot{\xi}_m\rangle ds$$
$$+\int_0^1\left[\beta(z_m)+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_m\|^2\right)\right]\dot{\tau}_m^2 ds\,.$$

Since $\psi$ is nondecreasing, we deduce that

$$\|\dot{\xi}_m\|^2 \xrightarrow[m\to\infty]{} 0\,,$$
$$\|\dot{\tau}_m\|^2 \xrightarrow[m\to\infty]{} 0\,,$$

from which we deduce (3.15) and (3.16).

*Remark 3.4* For every $k\in\mathbb{N}$, let $f_{\delta,k}$ be the restriction of $f_\delta$ to the submanifold $\mathscr{Z}_k$ of $\mathscr{Z}$, defined in (2.3). With the same proof of Proposition 3.3, we have that $f_{\delta,k}$ satisfies (PS) condition, for every $k\in\mathbb{N}$ and $\delta>0$.

## 4 Some apriori estimates on the critical points of the penalized functional

In this section we shall prove some estimates on the critical points of the penalized functional, defined in (3.2).

Let $z=(x,t)$ be a critical point of $f_\delta$, $\delta\geqq 0$. Straightforward calculations show that $z$ is smooth and satisfies the following system of differential equations:

$$\begin{cases}\nabla_s(\alpha(z)\dot{x})-\frac{1}{2}\langle\dot{x},\dot{x}\rangle\nabla\alpha(z)+\frac{1}{2}\dot{t}^2\nabla\beta(z)+\delta\psi'\left(\frac{\delta}{2}\|\dot{x}\|^2\right)\nabla_s\dot{x}=0\\ \frac{d}{ds}(\beta(z)\dot{t})+\frac{1}{2}\langle\dot{x},\dot{x}\rangle\alpha_t(z)-\frac{1}{2}\dot{t}^2\beta_t(z)+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\ddot{t}=0\,,\end{cases}$$

where $\nabla_s$ is the covariant derivative along $x$, induced by the Riemannian metric $\langle\cdot,\cdot\rangle$.

Multiplying the first equation by $\dot{x}$, the second by $\dot{t}$ and subtracting, we get

$$\frac{d}{ds}\left[\alpha(z)\langle\dot{x},\dot{x}\rangle-\beta(z)\dot{t}^2+\delta\psi'\left(\frac{\delta}{2}\|\dot{x}\|^2\right)\langle\dot{x},\dot{x}\rangle-\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}^2\right]=0\ ,$$

hence there exists a constant $E_\delta(z)$, such that

$$E_\delta(z)=\alpha(z)\langle\dot{x},\dot{x}\rangle-\beta(z)\dot{t}^2+\delta\psi'\left(\frac{\delta}{2}\|\dot{x}\|^2\right)\langle\dot{x},\dot{x}\rangle-\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}^2\ .$$

The main result of this section is the following apriori estimate on the critical points of the functional $f_\delta$, $\delta\geqq 0$.

**Theorem 4.1** *Let $c\in\mathbb{R}$ and $\delta_0>0$, then there exists a constant $K>0$, such that for every critical point $z=(x,t)$ of $f_\delta$, $\delta\in[0,\delta_0]$, with $E_\delta(z)\leqq c$:*

$$\|\dot{x}\|_\infty\leqq K\ , \tag{4.1}$$

$$\|\dot{t}\|_\infty\leqq K\ , \tag{4.2}$$

*where*

$$\|\dot{x}\|_\infty=\sup_{s\in I}|\langle\dot{x}(s),\dot{x}(s)\rangle|\ ,$$

$$\|\dot{t}\|_\infty=\sup_{s\in I}|\dot{t}(s)|\ .$$

*Remark 4.2* Set

$$\tilde{E}_\delta(z)=\alpha(z)\langle\dot{x},\dot{x}\rangle-\beta(z)\dot{t}^2-\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}^2\ .$$

Since $\tilde{E}_\delta(z)\leqq E_\delta(z)$, we may prove Theorem 4.1 assuming that

$$\tilde{E}_\delta(z)\leqq c\ . \tag{4.3}$$

*Remark 4.3* Assume that (4.2) holds. Then (4.1) holds, too. Indeed, if for some constant $\tilde{K}$

$$\|\dot{t}\|_\infty\leqq\tilde{K}\ ,$$

from $(A_2)$ and (4.3) we also get (4.1).

In order to get (4.2), we first prove the following lemma.

**Lemma 4.4** *Let $c\in\mathbb{R}$ and $\delta_0>0$, then there exists a positive constant $K'$ such that for every critical point $z=(x,t)$ of $f_\delta$, $\delta\in[0,\delta_0]$, with $\tilde{E}_\delta(z)\leqq c$:*

$$\|t\|_\infty\leqq K'\ ,$$

*where*

$$\|t\|_\infty=\sup_{s\in I}|t(s)|\ .$$

*Proof.* By assumption $(A_5)$, for every $\varepsilon>0$ there exists a constant $M_\varepsilon>0$, that we may choose big enough, such that:

$$\forall t>M_\varepsilon,\ \forall x\in\mathcal{M}_0: \quad \alpha_t(x,t)\leqq\varepsilon\alpha(x,t)\,, \tag{4.4}$$

$$\forall t<-M_\varepsilon,\ \forall x\in\mathcal{M}_0: \quad \alpha_t(x,t)\geqq-\varepsilon\alpha(x,t)\,. \tag{4.5}$$

Moreover, by assumptions $(A_2)$, $(A_3)$ and $(A_4)$, we may choose a positive constant $\Lambda>0$, such that for every $z\in M$

$$\frac{|\beta_t(z)|}{\beta(z)}\leqq\Lambda\,, \tag{4.6}$$

$$\frac{|\alpha_t(z)|}{\alpha(z)}\leqq\Lambda\,. \tag{4.7}$$

Now, choose

$$\varepsilon<\min\left\{1,\left(\frac{\nu}{1+|c|+\Lambda}\right)^2\right\} \tag{4.8}$$

and $M_\varepsilon$ in order that $M_\varepsilon>\{|t_0|,|t_1|\}$. We recall that $\nu$ is defined in $(A_3)$.

We claim that for every critical point $z=(x,t)$ of $f_\delta$, $\delta\in[0,\delta_0]$ (we set for simplicity $\delta_0=1$), such that with $\tilde{E}_\delta(z)\leqq c$, we have:

$$\|t\|_\infty\leqq M_\varepsilon+1\,. \tag{4.9}$$

Arguing by contradiction, suppose that there exists a critical point $z=(x,t)$ of some $f_\delta$, $\delta\in[0,1]$, such that

$$\|t\|_\infty>M_\varepsilon+1\,.$$

First suppose that $\|t\|_\infty$ is achieved in a point $s_1\in]0,1[$, and

$$\|t\|_\infty=t(s_1)>M_\varepsilon+1\,. \tag{4.10}$$

Let

$$A=\{s\in I\,|\,t(s)>M_\varepsilon\}\,.$$

Then $A$ is an open subset of $I$, and $s_1$ is an internal point of $A$.
Since $z$ is a critical point of $f_\delta$, $t$ satisfies the equation

$$\frac{d}{ds}\left[\beta(z)\dot{t}+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}\right]+\frac{1}{2}\langle\dot{x},\dot{x}\rangle\alpha_t(z)-\frac{1}{2}\dot{t}^2\beta_t(z)=0\,. \tag{4.11}$$

Now, we set

$$u(s)=\beta(z)\dot{t}+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}\,.$$

From (4.3) and (4.4), for every $s\in A$, we have:

$$\frac{1}{2}\langle\dot{x},\dot{x}\rangle\alpha_t(z)\leqq\varepsilon\alpha(z)\langle\dot{x},\dot{x}\rangle$$

$$\leqq\varepsilon c+\varepsilon\beta(z)\dot{t}^2+\varepsilon\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}^2\leqq\varepsilon|c|+\varepsilon u\dot{t}\,. \tag{4.12}$$

Hence, from (4.11) and (4.12), we get for every $s \in A$:

$$\dot{u} = -\frac{1}{2}\langle \dot{x}, \dot{x}\rangle \alpha_t(z) + \frac{1}{2}\dot{t}^2 \beta_t(z) \geqq$$

$$\geqq -\varepsilon|c| - \varepsilon u\dot{t} + \frac{1}{2}\dot{t}^2 \beta_t(z)\,, \tag{4.13}$$

and, from (4.6) and (4.13),

$$-\dot{u} \leqq \Lambda \dot{t}^2 + \varepsilon|c| + \varepsilon u\dot{t}\,. \tag{4.14}$$

Now, let $B$ be the maximal component of $A$ containing $s_1$, and $\bar{s}$ be the infimum of $B$, so that

$$\bar{s} < s_1\,,$$

$$t(\bar{s}) = M_\varepsilon\,.$$

By the mean value theorem, there exists $r \in B$, $\bar{s} < r < s_1$, such that $\dot{t}(r) \geqq 1$.

So there exists also a point $s_0 \in B$, $s_0 < s_1$, such that

$$\dot{t}(s_0) = \sqrt{\varepsilon}\,, \tag{4.15}$$

$$0 \leqq \dot{t}(s) \leqq \sqrt{\varepsilon}, \quad \text{for every } s \in [s_0, s_1]\,.$$

Indeed, we have $s_0 = \sup\{r \leqq s \leqq s_1 \mid \dot{t}(s) = \sqrt{\varepsilon}\}$

Integrating (4.14), we get

$$u(s_0) - u(s_1) \leqq \Lambda\varepsilon + \varepsilon|c| + \varepsilon \int_{s_0}^{s_1} u\dot{t}\,ds$$

$$= \Lambda\varepsilon + \varepsilon|c| + \varepsilon \int_{s_0}^{s_1} \left[\beta(z)\dot{t}^2 + \delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}^2\right] ds$$

$$\leqq (\Lambda + |c|)\varepsilon + \varepsilon^2 \left[1 + \delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\right]. \tag{4.16}$$

On the other hand, since $\dot{t}(s_1) = 0$, we have

$$u(s_0) - u(s_1) = \beta(z(s_0))\dot{t}(s_0) + \delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}(s_0)\,. \tag{4.17}$$

Hence, from (4.16) and (4.17), we have:

$$\beta(z(s_0))\dot{t}(s_0) + \delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)\dot{t}(s_0)$$

$$\leqq (\Lambda + |c| + 1)\varepsilon + \varepsilon^2\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right),$$

and from (4.15)

$$\beta(z(s_0))\sqrt{\varepsilon} \leqq (\Lambda + |c| + 1)\varepsilon + (\varepsilon^2 - \sqrt{\varepsilon})\delta\psi'\left(\frac{\delta}{2}\|\dot t\|^2\right) \leqq (\Lambda + |c| + 1)\varepsilon .$$

Finally we get

$$\sqrt{\varepsilon} \geqq \frac{v}{\Lambda + |c| + 1} ,$$

which is in contradiction with (4.8).

The lemma is proved when (4.9) holds. In the other case, i.e.

$$\|t\|_\infty = t(s_1) < -(M_\varepsilon + 1) ,$$

the proof is similar. Notice that in this case we use (4.5), which is obtained from ii) of assumption $A_5$).

*Proof of Theorem* 4.1 Let $z = (x, t)$ as in the assumptions of Theorem 4.1. Then, by Remark 4.3, it is enough to prove (4.2).

Let $u(s)$ as in the previous lemma; from (4.6), (4.7) and (4.11), we have:

$$\begin{aligned} \dot u &= -\frac{1}{2}\langle \dot x, \dot x\rangle \alpha_t(z) + \frac{1}{2}\dot t^2 \beta_t(z) \leqq \Lambda\alpha(z)\langle \dot x, \dot x\rangle + \Lambda\beta(z)\dot t^2 \\ &\leqq \Lambda\left[\beta(z)\dot t^2 + \delta\psi'\left(\frac{\delta}{2}\|\dot t\|^2\right)\dot t^2 + |c|\right] + \Lambda\beta(z)\dot t^2 \leqq 2\Lambda u\dot t + \Lambda|c| . \end{aligned} \tag{4.18}$$

We take for simplicity $\Lambda = 1$.

Consider the set

$$B = \{s \in I \mid |\dot t(s)| > 1\} .$$

Let $]s_0, s_1[$ be a maximal component of $B$ and suppose that $\dot t(s) > 0$ in $]s_0, s_1[$ (the other case is similar).

Dividing (4.18) by $u$, and integrating from $s_0$ to $s$, $s \in ]s_0, s_1[$, we get:

$$\begin{aligned} \log u(s) - \log u(s_0) &\leqq |c| \int_{s_0}^{s} \frac{1}{u(r)}\, dr + 2[t(s) - t(s_0)] \\ &\leqq \frac{|c|}{v} + 4\|t\|_\infty \leqq \frac{|c|}{v} + 4M , \end{aligned}$$

where $M$ is an upper bound for $\|t\|_\infty$.

Moreover, recalling the definition of $u$, we have:

$$\begin{aligned} &\log\left[\beta(z(s))\dot t(s) + \delta\psi'\left(\frac{\delta}{2}\|\dot t\|^2\right)\dot t(s)\right] \\ &\leqq \log\left[\beta(z(s_0))\dot t(s_0) + \delta\psi'\left(\frac{\delta}{2}\|\dot t\|^2\right)\dot t(s_0)\right] + \frac{|c|}{v} + 4M \\ &= \log\left[\beta(z(s_0)) + \delta\psi'\left(\frac{\delta}{2}\|\dot t\|^2\right)\right] + \frac{|c|}{v} + 4M , \end{aligned} \tag{4.19}$$

because $\dot{t}(s_0)=1$. Taking the exponentials, from ($A_3$), and (4.19), we get

$$\nu\dot{t}(s)\leqq\beta(s))\dot{t}(s)\leqq\tilde{M}+\delta\psi'\left(\frac{\delta}{2}\|\dot{t}\|^2\right)[\tilde{M}-\dot{t}(s)]\ , \tag{4.20}$$

where $\tilde{M}$ is a fixed constant $\left(\text{the exponential of } \frac{|c|}{\nu}+4M\right)$.

Now, if $\dot{t}(s)>\tilde{M}$, from (4.20), we get:

$$\dot{t}(s)\leqq\frac{\tilde{M}}{\nu}\ .$$

Hence, we have proved that

$$\|\dot{t}\|_\infty\leqq\max\left\{1,\tilde{M},\frac{\tilde{M}}{\nu}\right\},$$

from which we deduce (4.2). By Remark 4.3, also (4.1) is true and the proof of Theorem 4.1 is complete.

## 5 Existence of a critical point of the penalized functional

In this section, we shall prove the existence of a critical point of the penalized functional $f_\delta$, using Theorem 2.2. We have the following

**Lemma 5.1** *With the notations of Sect.* 2 (*see* (2.4), (2.5), (2.6), (2.7) *and* (2.8)), *we have*:

a) $\inf f(S)>-\infty$;
b) $\sup f(Q(R))<+\infty,\ \forall R>0$;
c) $\exists\bar{R}>0$, *such that*

$$\sup f(\partial Q(\bar{R}))<\inf f(S)\ .$$

*Proof.* a) By virtue of ($A_5$), for every $z=(x,\bar{t})\in S$, we have:

$$f(z)=\frac{1}{2}\int_0^1[\alpha(z)\langle\dot{x},\dot{x}\rangle-\beta(z)\Delta^2]\,ds\geqq-\Delta^2\ ,$$

where $\Delta=t_1-t_0$.

b) By assumption ($A_4$), there exists a positive constant $M$ and a continuous function $a(x)$ on $\mathcal{M}_0$, such that for every $z=(x,t)\in\mathcal{M}$:

$$\alpha(x,t)\leqq a(x)+M|t|\ . \tag{5.1}$$

Hence, for every $z=(\bar{x},t)\in Q(R)$, we get:

$$f(z)\leqq c_1\int_0^1 a(\bar{x})ds+c_1M\int_0^1|t|ds-\nu\int_0^1\dot{t}^2ds\ , \tag{5.2}$$

where $c_1$ is a suitable constant. From (5.2) we deduce b).

c) From (5.2) we have for every $z \in \partial(Q(R))$:

$$f(z) \leqq c_2 + c_3 \|\dot{t}\| - \nu \|\dot{t}\|^2 \to -\infty \text{ as } \|\dot{t}\| \to +\infty ,$$

so we get c) for $\bar{R}$ sufficiently large.

Similar calculations show that this lemma holds for the penalized functionals $f_\delta$, too. Next lemma gives more precise results if $\delta$ is sufficiently small.

**Lemma 5.2** a) $\exists \delta_0 > 0$, *such that* $\forall \delta < \delta_0$:

$$\inf f(S) \leqq \inf f_\delta(S) ;$$

b) $\exists \delta_1 > 0$ *such that* $\forall \delta < \delta_1$, $\forall R > 0$:

$$\sup f_\delta(Q(R)) \leqq \sup f(Q(R)) ,$$

$$\sup f_\delta(\partial Q(R)) \leqq \sup f(\partial Q(R)) ;$$

c) *If* $\bar{R}$ *is the number defined in* c) *of Lemma* 5.1 *and* $\delta < \min \{\delta_0, \delta_1\}$, *we have*:

$$\sup f_\delta(\partial Q(\bar{R})) < \inf f_\delta(S) . \tag{5.3}$$

*Proof.* a) For every $z = (x, t) \in S$, we have:

$$f_\delta(z) = f(z) + \psi\left(\frac{\delta}{2} \|\dot{x}\|^2\right) - \psi\left(\frac{\delta}{2} \|\dot{t}\|^2\right) = f(z) + \psi\left(\frac{\delta}{2} \|\dot{x}\|^2\right) - \psi\left(\frac{\delta}{2} \Delta^2\right)$$

$$= f(z) + \psi\left(\frac{\delta}{2} \|\dot{x}\|^2\right) \geqq f(z) .$$

Hence, we get a) if $\delta < \frac{2}{\Delta^2}$ (if $\Delta = 0$, we may choose $\delta$ arbitrarily). In the same way we have b). Finally c) is consequence of a), b) and of c) of Lemma 5.1.

From these two lemmas we get the existence of a critical point for the functional $f_{\delta,k}$, for all $k \in \mathbb{N}$ and $\delta$ small.

**Theorem 5.3** *For every* $\delta$ *small enough and* $k \in \mathbb{N}$, *the number*

$$c_{\delta,k} = \inf_{h \in \Gamma_k} \sup f(h(Q_k(\bar{R})) ,$$

*where*

$$\Gamma_k = \{h \in C(\mathscr{Z}_k, \mathscr{Z}_k) | h(z) = z, \ \forall z \in \partial Q_k(\bar{R})\} ,$$

*is a critical value of* $f_{\delta,k}$, *such that*

$$\inf f(S) \leqq c_{\delta,k} \leqq \sup f(Q(\bar{R})) . \tag{5.4}$$

The proof follows from Theorem 2.2, Remark 3.4 and Lemma 5.2.

Now, let $(z_{\delta,k}, t_{\delta,k})$ be a critical point of $f_{\delta,k}$ at level $c_{\delta,k}$. Our aim now is to show that the sequence $\{(z_{\delta,k}, t_{\delta,k})\}_{k \in \mathbb{N}}$ converges, up to a subsequence, to a critical point of $f_\delta$.

We need the following approximation result, whose proof is contained in [BF2].

**Lemma 5.4** *Let $(x_m)_{m\in\mathbb{N}}$ be a sequence in $\Omega^1$ which converges weakly to $x$ in $H^{1,2}(I, \mathbb{R}^N)$. Let $\xi \in T_x\Omega^1$ and let $\xi_m$ be the orthogonal projection of $\xi$ on $T_{x_m}\mathcal{M}_0$.*

*Then the sequence $(\xi_m)_{m\in\mathbb{N}}$ contains a subsequence weakly convergent at $\xi$ in $H^{1,2}(I, \mathbb{R}^N)$.*

**Theorem 5.5** *Let $z_{\delta,k}$ be a critical point of $f_{\delta,k}$ at level $c_{\delta,k}$. Then there exists a subsequence of $(z_{\delta,k})_{k\in\mathbb{N}}$ converging to a critical point $z_\delta$ of $f_\delta$, such that:*

$$\inf f(S) \leqq f_\delta(z_\delta)) \leqq \sup f(Q(\bar{R})) .$$

*Proof.* Arguing as in the proof of Proposition 3.3, the sequence $(z_{\delta,k})_{k\in\mathbb{N}}$ converges, up to a subsequence, to $z_\delta \in \mathscr{Z}$. We claim that $z_\delta$ is a critical point of $f_\delta$. Indeed, let $\zeta = (\xi, \tau) \in T_{z_\delta}\mathscr{Z}$, $\xi_k$ the orthogonal projection of $\xi$ on $T_{x_{\delta,k}}\mathcal{M}_0$, and $\tau_k$ the orthogonal projection of $\tau$ on $H^{1,2}_{0,k}(I, \mathbb{R})$. Then, up to a subsequence, we may suppose that $\xi_k$ converges weakly to $\xi$, by virtue of Lemma 5.5. Moreover, $\tau_k$ converges to $\tau$ in $H^{1,2}_0(I, \mathbb{R})$. Putting $\zeta_k = (\xi_k, \tau_k)$, we get

$$f'_\delta(z_\delta)[\zeta] = f'_\delta(z_\delta)[\zeta - \zeta_k] + (f'_\delta(z_\delta) - f'_\delta(z_{\delta,k}))[\zeta_k] .$$

Since $\zeta_k$ weakly converges to $\zeta$, we have:

$$f'_\delta(z_\delta)[\zeta - \zeta_k] = o(1) .$$

Moreover, since $(z_{\delta,k})$ converges strongly to $z$, we have:

$$(f'_\delta(z_\delta) - f'_\delta(z_{\delta,k}))[\zeta_k] = o(1) ,$$

so $f'_\delta(z_\delta)[\zeta] = 0$. Therefore, $z_\delta$ is a critical point of $f_\delta$.

## 6 Proof of Theorem 1.1

In this section we shall prove Theorem 1.1.

*Proof of Theorem* 1.1 In order to prove Theorem 1.1, we have to show that the action integral $f$ has a critical point.

In the last section, we have proved that for every $\delta > 0$ sufficiently small, the penalized functional $f_\delta$ has a critical point $z_\delta = (x_\delta, t_\delta)$, such that for every $\delta$ sufficiently small

$$f_\delta(z_\delta) \leqq \sup f(Q(\bar{R})) , \tag{6.1}$$

where $\bar{R}$ is defined in c) of Lemma 5.1.

By virtue of Lemma 3.1 and Theorem 4.1, it suffices to prove that there exists a real constant $c$, such that if $\delta$ is small enough,

$$E_\delta(z_\delta) \leqq c , \tag{6.2}$$

(see Sect. 4 for the definition of $E_\delta$).

For every $\delta$ we have:

$$E_\delta(z_\delta)=\alpha(z_\delta)\langle\dot{x}_\delta,\dot{x}_\delta\rangle-\beta(z_\delta)\dot{t}_\delta^2+\delta\psi'\left(\frac{\delta}{2}\|\dot{x}_\delta\|^2\right)\langle\dot{x}_\delta,\dot{x}_\delta\rangle-\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_\delta\|^2\right)\dot{t}_\delta^2,$$

and integrating

$$\begin{aligned}E_\delta(z_\delta)=&\int_0^1[\alpha(z_\delta)\langle\dot{x}_\delta,\dot{x}_\delta\rangle-\beta(z_\delta)\dot{t}_\delta^2]ds\\&+\delta\psi'\left(\frac{\delta}{2}\|\dot{x}_\delta\|^2\right)\|\dot{x}_\delta^2\|-\delta\psi'\left(\frac{\delta}{2}\|\dot{t}_\delta\|^2\right)\|\dot{t}_\delta^2\|\\&=2f_\delta(z_\delta)+2\left[\frac{\delta}{2}\|\dot{x}_\delta\|^2\psi'\left(\frac{\delta}{2}\|\dot{x}_\delta\|^2\right)-\psi\left(\frac{\delta}{2}\|\dot{x}_\delta\|^2\right)\right]+\\&-2\left[\frac{\delta}{2}\|\dot{t}_\delta\|^2\psi'\left(\frac{\delta}{2}\|\dot{t}_\delta\|^2\right)-\psi\left(\frac{\delta}{2}\|\dot{t}_\delta\|^2\right)\right].\end{aligned}$$

Put, for every $s\in\mathbb{R}_+$, $\rho(s)=s\psi'(s)-\psi(s)$, and put for simplicity $s_x=\frac{\delta}{2}\|\dot{x}_\delta\|^2$ and $s_t=\frac{\delta}{2}\|\dot{t}_\delta\|^2$. Then, in order to prove (6.2), we have to show that for $\delta$ small enough

$$\rho(s_x)-\rho(s_t)\text{ is bounded above independently on }\delta\,. \tag{6.3}$$

We distinguish some cases:

a) If $s_x\leqq s_t$, then $\rho(s_x)-\rho(s_t)\leqq 0$, because the function $\rho$ is nondecreasing (indeed it is equal to 0 if $s\in[0,1]$ and is equal to $s^2-1$ if $s>1$, see (3.1)).

In particular if $\psi(s_x)=0$, then $\rho(s_x)-\rho(s_t)\leqq 0$.

b) If $\int_0^1[\alpha(z_\delta)\langle\dot{x}_\delta,\dot{x}_\delta\rangle-\beta(z_\delta)\dot{t}_\delta^2]ds\leqq 0$, then assumptions $(A_2)$ and $(A_3)$ gives $s_x\leqq s_t$ (recall that we have set $\lambda=1$), then (6.3) is true.

Then we have to consider the cases in which $\psi(s_x)>0$ and $\int_0^1[\alpha(z_\delta)\langle\dot{x}_\delta,\dot{x}_\delta\rangle-\beta(z_\delta)\dot{t}_\delta^2]>0$. Notice that in this case, by (6.1) there exists a constant $c$ independent on $\delta>0$, such that $\psi(s_x)-\psi(s_t)\leqq c$.

c) Assume that $\psi(s_x)>0$, $\int_0^1[\alpha(z_\delta)\langle\dot{x}_\delta,\dot{x}_\delta\rangle-\beta(z_\delta)\dot{t}_\delta^2]>0$, and $\psi(s_t)=0$. In this case, we have $\psi(s_x)\leqq c$, hence by the definition of $\psi$

$$(s_x-1)^2\leqq c\,,$$

from which we get

$s_x\leqq c'=c^{1/2}-1$. Then we have

$$\rho(s_x)-\rho(s_t)=\rho(s_x)\leqq\rho(c')\,,$$

because $\rho$ is monotone.

Finally, we have to consider the last case

d) $\psi(s_x)>0$, $\psi(s_t)>0$, $s_x>s_t$, $\int_0^1 [\alpha(z_\delta)\langle\dot{x}_\delta, \dot{x}_\delta\rangle-\beta(z_\delta)\dot{t}_\delta^2]>0$.
From (6.1), we have:

$$(s_x-1)^2-(s_t-1)^2=\psi(s_x)-\psi(s_t)\leqq c\ ,$$

where $c$ is a suitable constant, independent on $\delta>0$. So we have:

$$s_x^2-s_t^2-2s_x+2s_t=(s_x-s_t)(s_x+s_t-2)\leqq c\ .$$

On the other hand (6.3) is equivalent to show that there exists a constant $c'$, independent on $\delta$, such that

$$\rho(s_x)-\rho(s_t)=s_x^2-s_t^2\leqq c'\ .$$

Now, if $(s_x-s_t)\geqq 1$, we get $s_x+s_t\leqq c+2$, hence $s_x\leqq c+2$, so

$$\rho(s_x)-\rho(s_t)=s_x^2-s_t^2\leqq s_x^2\leqq (c+2)^2\ .$$

If $(s_x-s_t)<1$, since $s_x^2-s_t^2-2s_x+2s_t\leqq c$, we have:

$$\rho(s_x)-\rho(s_t)=s_x^2-s_t^2\leqq c+2(s_x-s_t)\leqq c+2\ .$$

The proof of (6.3) is complete. Then by Lemma 3.1, Theorem 5.5 and (6.3), the proof of Theorem 1.1 is complete.

*Remark 6.1* From Lemma 3.1 and (6.4), we get a critical point $z=(x, t)$ of $f$, such that

$$f(z)\leqq \sup f(Q(\bar{R}))\ . \qquad (6.4)$$

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