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Thetanullwerte and stable modular forms for Hecke groups

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0 Introduction

In this paper we follow the study of the relations among homogeneous polynomials in the Thetanullwerte and theta series.

In particular we try to extend the results obtained in [11] for the full modular group Γ_g to the Hecke groups $\Gamma_{g,0}(q)$. We will be successful putting some "natural" restriction on the weight. To be more precise let us recall some definitions. For any commutative ring R, we denote by $\mathrm{Sp}(g, R)$ the symplectic group and we write its elements σ in 4 blocks of g by g matrices, i.e.

(1)
$$\sigma=\begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

We shall write Γ_g for $\mathrm{Sp}(g, \mathbb{Z})$.

Let q be any positive integer we shall denote by $\Gamma_g(q)$ the kernel of the surjective homomorphism mod q

(2)
$$\mathrm{Sp}(g, \mathbb{Z}) \to \mathrm{Sp}(g, \mathbb{Z}/q\mathbb{Z}) .$$

If we add the conditions $\mathrm{diag}(A^t B) \equiv \mathrm{diag}(C^t D) \equiv (\mathrm{mod}\, 2q)$ we get Igusa's congruence subgroup $\Gamma_g(q, 2q)$.

The Hecke group $\Gamma_{q,0}(q)$ is the subgroup of $\mathrm{Sp}(g, \mathbb{Z})$ defined by $C \equiv 0 \bmod q$.

In general a subgroup Γ of $\mathrm{Sp}(g, \mathbb{R})$ is called a congruence subgroup if it contains some $\Gamma_g(q)$ as a subgroup of finite index.

Let τ be a point of the Siegel upper half space of degree g $\mathbb{H}_g$, $\mathrm{Sp}(g, \mathbb{R})$ acts biholomorphically on $\mathbb{H}_g$ by

(3)
$$\sigma \cdot \tau = (A\tau + B)(C\tau + D)^{-1} .$$

Let v be a character of Γ and k be a positive integer, then a holomorphic function f defined on $\mathbb{H}_g$, satisfying

$$f(\sigma \cdot \tau) = v(\sigma) \det(c\tau + d)^k f(\tau) \quad \forall \sigma \in \Gamma , \tag{4}$$

is called for $g > 1$, a modular form of weight k, relative to Γ and v. In the case $g = 1$ we add the usual condition at the cusps. The space of such functions is finite dimensional and it will be denoted by $[\Gamma, k, v]$, we shall write $[\Gamma, k]$ if v is trivial. We set $\underline{e}(t) = \exp(2\pi\sqrt{-1}t)$, let Q be a $2k$ by $2k$ symmetric, positive-definite, even (i.e. diag $Q \equiv 0 \bmod 2$) integral matrix, then the theta series

$$\vartheta_Q(\tau) = \sum_{G \in M_{g,2k}(\mathbb{Z})} \underline{e}(1/2 \operatorname{tr}(Q[{}^tG]\tau)) \tag{5}$$

belongs to $[\Gamma_{g,0}(q), k, \chi_Q^g]$. Here q is the level of Q, i.e. qQ^{-1} is even and integral.

We shall denote by $B_k^g(q, \chi^g)$ the space spanned by all theta series with $\chi^g = \chi_Q^g$. In [3] Freitag has shown that for $g > 2k$ $[\Gamma_{g,0}(q), k, \chi^g] = B_k^g(q, \chi^g)$.

Let m be a column vector of $\mathbb{Q}^{2g}$, and denote by m' and m'' its first g, respectively last g entries, then the series.

$$\vartheta_m(\tau) = \sum_{p \in \mathbb{Z}^g} \underline{e}(1/2\,{}^t(p+m')\tau(p+m') + {}^t(p+m')m'') \tag{6}$$

defines a Thetanullwert of characteristic m.

For any $n \in \mathbb{Z}^{2g}$ we have

$$\vartheta_{m+n}(\tau) = \underline{e}({}^tm'n'')\,\vartheta_m(\tau) . \tag{7}$$

Therefore we shall consider m in $(\mathbb{Q}/\mathbb{Z})^{2g}$ and normalize the entries putting them equal to a/s, $0 \leqq a \leqq (s-1)$ providing that $sm \equiv 0 \bmod 1$.

Le us denote by Σ_s the subset of $(\mathbb{Q}/\mathbb{Z})^{2g}$ consisting of the above elements; then when s is even we know that

$$\vartheta_m \vartheta_n \in [\Gamma_g(s^2, 2s^2), 1, \;] . \tag{8}$$

For any $\Gamma_{g,0}(q)$ containing $\Gamma_g(s^2, 2s^2)$ we shall denote by $[\Gamma_{g,0}(q), \Sigma_s, k, \chi^g]$ the subspace of $[\Gamma_{g,0}(q), k, \chi^g]$ spanned by homogeneous polynomials of degree $2k$ in the Thetanullwerte.

The main result of this paper is the inclusion of $[\Gamma_{g,0}(q), \Sigma_s, k, \chi^g]$ in $B_k^g(q, \chi^g)$. We shall prove the equality when q is a power of 2 and s is large enough.

These results are consequence of the surjectivity of the Φ operator restricted to $[\Gamma_{g,0}(q), \Sigma_s, k, \chi^g]$.

The results of this paper are a generalization of those of [11], since the method will be the same, we often will refer to them.

Moreover we shall give an application to coding theory, characterizing in terms of thetanullwerte the theta series coming from codes of type A.

1

In this section we recall some basic facts from [11], we refer to this paper for the notations.

Let $M=(m_1 \dots m_r)$ be a sequence of elements of Σ_s, i.e. $M \in M_{2g,r}(1/s\ \mathbb{Z}/\mathbb{Z})$, then for every σ in Γ_g we get

(9) $$\sigma \cdot M = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \begin{bmatrix} M' \\ M'' \end{bmatrix} + 1/2 \begin{bmatrix} \operatorname{diag}_r(C^t D) \\ \operatorname{diag}_r(A^t B) \end{bmatrix};$$

this defines an action of Γ_g on $M_{2g,r}(1/s\ \mathbb{Z}/\mathbb{Z})$ that induces the action of $\mathrm{Sp}(g, \mathbb{Z}/s\mathbb{Z})$. We shall write $\sigma \circ M$ for the unique element with normalized element congruent to $\sigma \cdot M \bmod 1$.

Let $s = p_0^{l_0} \dots p_t^{l_t} = s_0 \dots s_t$ be the factorization of s in primes with $p_0 = 2$ and assume $l_0 \geqq 2$.

We have that

(10) $$\mathrm{Sp}(g, \mathbb{Z}/s\mathbb{Z}) \cong \mathrm{Sp}(g, \mathbb{Z}/s_0\mathbb{Z}) \times \dots \mathrm{Sp}(g, \mathbb{Z}/s_t\mathbb{Z});$$

moreover a similar decomposition holds for the module $M_{2g,r}(1/s\mathbb{Z}/\mathbb{Z})$. This induces a decomposition of the action described in (9) in such a way that on the first factor it remains non homogeneous and it become homogeneous on the others. We remark that the condition $l_0 \geqq 2$ is not a real restriction since $M_{2g,r}((1/s)\ \mathbb{Z}/\mathbb{Z})$, s even, can be considered as a submodule of $M_{2g,r}((1/2s)\ \mathbb{Z}/\mathbb{Z})$. Clearly on this submodule the kernel of the surjective homomorphism from $\mathrm{Sp}(g, \mathbb{Z}/2s\,\mathbb{Z})$ to $\mathrm{Sp}(g, \mathbb{Z}/s\mathbb{Z})$ acts trivially. For any M in $M_{2g,r}(1/s\mathbb{Z}/\mathbb{Z})$ we shall denote by $G(M)$ the subgroup of Σ_s spanned by the columns of M. Moreover we say that

(11) $$\tilde{M} = \begin{bmatrix} M'' \\ 0 \\ M'' \\ 0 \end{bmatrix}$$

is the extension of M of degree $\tilde{g}$; we put

(12) $$|M| = \max_{\sigma \in \mathrm{Sp}(\tilde{g}, \mathbb{Z}/s\mathbb{Z})} |G(\sigma \circ \tilde{M})|$$

with $\tilde{g} \geqq \max\{g, r\}$.

We know that $|M|$ is equal to $|G(M)|$ or $2|G(M)|$ cf. [8].

From now on we shall consider $\tilde{M}$ as the extension of M of degree $(g+1)$. Let

$$\bar{N} = \begin{bmatrix} N' \\ 0 \\ N \\ v \end{bmatrix}$$

be in $M_{2(g+1),r}(1/s\mathbb{Z}/\mathbb{Z})$ and conjugate to $\tilde{M}$ under the action of $\Gamma_{g+1,0}(q)$, then as in [11], where it was considered the case of the full modular group Γ_{g+1}, we have the following

Proposition 1 *Let* $\tilde{M}$ *and* $\tilde{N}$ *as above with q dividing s, then M and N are conjugate under the action of* $\Gamma_{g,0}(q)$ if and only if $|M|=|N|$.

Proof. The first part of it is equal to that of Proposition 1 in [8], then we omit it and assume $\tilde{M}$ and $\tilde{N}$ conjugate. We proceed as in Proposition 2 of [8]. First we use the decomposition (10) and consider the action on each factor. We shall discuss only the first case, that is the more difficult, the other cases are similar.

Let $\tilde{N}$ and $\tilde{M}$ on $M_{2(g+1),r}(1/s_0\mathbb{Z}/\mathbb{Z})$, then there exist σ in $\Gamma_{g+1,0}(q)/\Gamma_g(s_0)$ such that

(13)
$$\sigma\cdot\tilde{N}\equiv\tilde{M}\equiv\begin{bmatrix} D_{11} & D_{12} & -C_{11} & -C_{12}\\ D_{21} & D_{22} & -C_{21} & -C_{22}\\ -B_{11} & -B_{12} & A_{11} & A_{12}\\ -B_{21} & -B_{22} & A_{21} & A_{22}\end{bmatrix}\begin{bmatrix} N'\\ 0\\ N''\\ 0\end{bmatrix}$$
$$+1/2\begin{bmatrix}\mathrm{diag}_r(C_{11}{}^tD_{11}+C_{12}{}^tD_{12})\\ \mathrm{diag}_r(C_{21}{}^tD_{21}+C_{22}D_{22})\\ \mathrm{diag}_r(A_{11}{}^tB_{11}+A_{12}{}^tB_{12})\\ \mathrm{diag}_r(A_{21}{}^tB_{21}+A_{22}B_{22})\end{bmatrix}.$$

If D_{22} belongs to $(\mathbb{Z}/s_0\mathbb{Z})^*$, then we set

(14)
$$\sigma_2=\begin{bmatrix} 1_g & .{}^tD_{21} & 0 & 0\\ 0 & D_{22} & 0 & 0\\ 0 & -{}^tC_{21} & 1_g & 0\\ -D_{22}^{-1}C_{21} & -C_{22} & -D_{22}^{-1}D_{21} & D_{22}^{-1}\end{bmatrix}$$

(15)
$$\sigma_1=\sigma\cdot\sigma_2=\begin{bmatrix}\bar{A}_{11} & 0 & \bar{B}_{11} & 0\\ * & 1 & * & *\\ \bar{C}_{11} & 0 & \bar{D}_{11} & *\\ 0 & 0 & 0 & 1\end{bmatrix}$$

then $\sigma_2^{-1}\circ\tilde{N}=\tilde{N}\bmod 1$; and

(16)
$$\bar{\sigma}_1=\begin{bmatrix}\bar{A}_{11} & \bar{B}_{11}\\ \bar{C}_{11} & \bar{D}_{11}\end{bmatrix}$$

belongs to $\Gamma_{g,0}(q)/\Gamma_g(s_0)$ and $\bar{\sigma}_1\circ N=M$.

A similar argument holds if A_{22} belongs to $(\mathbb{Z}/s_0\mathbb{Z})^*$.

Assume that B_{22} or C_{22} belongs to $(\mathbb{Z}/s_0\mathbb{Z})^*$, then if q is odd we multiply σ by the matrix.

$$(17) \qquad E_{g,1}(q)=\begin{bmatrix} 1_g & 0 & 0 & 0 \\ 0 & (-q+1) & 0 & -q \\ 0 & 0 & 1_g & 0 \\ 0 & q & 0 & q+1 \end{bmatrix}$$

and obtain one of the previous cases.

When q is even, B_{22} belongs to $(\mathbb{Z}/s_0\mathbb{Z})^*$, then as in [11] we consider σ_3 with $C=0$,

$$A=\begin{bmatrix} 1_g & 0 \\ A_{21} & 1 \end{bmatrix}={}^tD^{-1}, \quad B=\begin{bmatrix} 0.{}^tB_{21} \\ B_{21}, A_{22}B_{22} \end{bmatrix}$$

then $\sigma_3 \circ \tilde{N}=\tilde{N}$ and the element in the bottom of $\sigma \cdot \sigma_3$ is invertible.

When $A_{22}\equiv B_{22}\equiv C_{22}\equiv D_{22}\equiv 0 \bmod 2$ we can apply the same σ_3.

We recall that for M in $M_{2g,r}(1/s\mathbb{Z}/\mathbb{Z})$, r even, the congruences

$$(18) \qquad sM{}^tM\equiv 0 \bmod 1 \quad \text{and} \quad s\,\mathrm{diag}(M{}^tM)\equiv 0 \bmod 2$$

are preserved under the action of $\mathrm{Sp}(g, \mathbb{Z}/s\mathbb{Z})$, cf. [5].

2

Let Q be the matrix of an integral positive definite, even quadratic form of level q and r, even, variables.

Then, as stated in the introduction $\vartheta_Q(\tau)$ belongs to $B_k^g(q, \chi^g)$ with $\chi^g=\chi_Q^g$ and $2k=r$. Here χ_Q^g is trivial if $q=1$ and $\chi_Q^g(\sigma)=\chi_Q(\det D)$ if $q>1$, where χ_Q is a real Dirichlet character modulo q satisfying $\chi_Q(-1)=(-1)^k$, $\chi_Q(p)=\left(\frac{(-1)^k \det Q}{p}\right)$ if p is an odd prime with p not dividing q and $\chi_Q(2)=2^{-k}\sum_{x\in(\mathbb{Z}/2\mathbb{Z})^r} \underline{e}(1/4Q[x])$ if q is odd cf. [1]. Assuming $rq\equiv 0 \bmod 8$ and $\det Q$ equal to a square number we have that the Dirichlet character depends only from q and we shall write χ_q^g for it, then we have $\chi_q^g(\sigma)=\chi_q(\det D)$. χ_q becomes always trivial except that when $r\equiv 2 \bmod 4$; in this case we have

$$(19) \qquad \chi_q(-1)=-1 \quad \text{and} \quad \chi_q(p)=\left(\frac{-1}{p}\right).$$

For any M in $M_{2g,r}(1/s\ \mathbb{Z}/\mathbb{Z})$ we put

$$P(M)(\tau)=\vartheta_{m_1}(\tau)\ldots\vartheta_{m_r}(\tau).$$

Let us assume $Q=\lambda{}^tHH$ with H in $M_r(1/s\mathbb{Z})$ and λ a natural number then we learned from [9, p. 218] the following formula relating theta series and

Thetanullwerte

$$\vartheta_Q(\tau)=d^{-1}\sum_{M'\in K_1,\, M''\in K_2} P(M)(\lambda\tau)\,, \tag{20}$$

where $K_1=M_{g,r}(\mathbb{Z})^tH/M_{g,r}(\mathbb{Z})^tH\cap M_{g,r}(\mathbb{Z})$, $K_2=M_{g,r}(\mathbb{Z})H^{-1}/M_{g,r}(\mathbb{Z})H^{-1}\cap M_{g,r}(\mathbb{Z})$ and $d=|K_2|$.

We remark that the coefficients of M' and M'' are not normalized, however this can be easily done using (7). In the next section we shall prove a converse theorem for the formula (20).

We conclude this section with the following

Lemma 1 *Let q be a power of 2 and* $\det Q$ *a square then $\vartheta_Q(\tau)$ can always be expressed as linear combination of monomials in the Thetanullwerte.*

Proof. Clearly it is enough to prove that there exists some

$$H \text{ in } M_r(1/2^h\mathbb{Z}) \text{ such that } Q={}^tHH\,.$$

This fact is an immediate consequence of the statements on p. 243 and p. 247 of [6]. We conclude this section, remarking, in general that if $Q={}^tHH$ with H in $M_r((1/s)\mathbb{Z})$ then $H^{-1}=Q^{-1}{}^tH$ belongs to $M_r((1/sq)\mathbb{Z})$.

3

We recall some basic fact about the transformation formula of Thetanullwerte, for details we refer to [6] or [11].

We have for any M in $M_{2g,r}(\Sigma_s)$.

$$P(\sigma\circ M)(\sigma\cdot\tau)=k(\sigma)^r\underline{e}\left(\sum_{i=1}^{r}\varphi_{m_i}(\sigma)\right)\underline{e}(-\operatorname{tr}((\sigma\circ M)'(\sigma\cdot M-\sigma\circ M)'')) \tag{21}$$
$$\times\det(c\tau+d)^k P(M)(\tau)\,.$$

Here $k(\sigma^2)$ is a character of $\Gamma_g(1,2)$, is trivial on $\Gamma_g(4)$ and $k(\sigma)^4=(-1)^{\operatorname{tr}(B^tC)}$ for every σ in Γ_g cf. [6] and [7].

Moreover we have

$$\varphi_m(\sigma)=(-1/2)({}^tbd[m']-2{}^tm'{}^tbcm''+{}^tac[m'']-{}^t\operatorname{diag}(a^tb)(dm'-cm''))\,. \tag{22}$$

We know that $P(M)(\tau)$ belongs to $[\Gamma_g(s),k]$ if and only the matrix M satisfies (18) cf. [5]; moreover the elements of $[\Gamma_{g,0}(q),\Sigma_s,k,\chi_q^g]$ are symmetrizations of the previous elements, in the following sense

$$f_M(\tau)=\sum_{\sigma\in\Gamma_{g,0}(q)\,|\,\Gamma_g(s)}\overline{\chi_q^g(\det D)}\chi(\sigma,M)P(\sigma^{-1}\circ M)(\tau) \tag{23}$$

where

$$\chi(\sigma,M)=k(\sigma)^r\underline{e}\left(\sum_{i=1}^{m}\varphi_{n_i}(\sigma)\right)\underline{e}(\operatorname{tr}({}^t(\sigma^{-1}\circ M)'(N-\sigma^{-1}\circ M)''))\,. \tag{24}$$

Here we have set $n_i=\sigma^{-1}\cdot m_i$ and $N=\sigma^{-1}\cdot M$.

As an immediate consequence of the transformation formula we have

(25) $$\chi(\sigma_1 \cdot \sigma_2, M) = \chi(\sigma_1, M)\chi(\sigma_2, \sigma_1^{-1} \circ M) .$$

For any M in $M_{2g,r}(\Sigma_s)$ we shall denote by $\bar{S}(M)$ the subgroup of $\Gamma_{g,0}(q)$ that stabilizes M. Clearly $\Gamma_g(s) \subseteq \bar{S}(M)$.

Moreover assuming M satisfying (18) we have that $\chi(\sigma, M)=1$ for every σ in $\Gamma_g(s)$, therefore from now on we often shall consider $S(M)$ as subgroup of $\Gamma_{g,0}(q)/\Gamma_g(s)$.

Lemma 2 *Let M in $M_{2g,r}(\Sigma_s)$, $rq \equiv 0 \bmod 8$ and $f_M(\tau)$ doesn't vanish identically, then we have*

(a) *M satisfies* (18)
(b) $\chi(\sigma, M) = \chi_q^g(\det D)$ *for every σ in $\bar{S}(M)$*
(c) $\chi(\sigma, \tilde{M}) = \chi_q^{g+1}(\det D)$ *for every σ in $\bar{S}(\tilde{M})$.*

Proof. It is similar to that of Lemma 7 in [8]. (a) and (b) are trivial. We write

(26) $$\chi(\sigma, \tilde{M}) = k(\sigma)^r \psi(\sigma, \tilde{M}) .$$

As cited in the reference it can be proved that $\psi(\sigma, \tilde{M}) = 1$.

We have to prove that $k(\sigma)^r = \chi_q^{g+1}(\det D)$.

Using the decomposition (10) we have that if $\sigma \equiv 1_{2(g+1)} \bmod s_0$. Then

(27) $$k(\sigma)^r = \chi_q^{g+1}(\det D) = 1 \quad \text{since } \sigma \text{ belongs to } \Gamma_{g+1}(4) .$$

Thus we assume σ in $\mathrm{Sp}(g+1, \mathbb{Z}/s_0\mathbb{Z})$. In this case we still get (27) except that when $r \equiv 2 \bmod 4$. We remark that we are in the same situation of (13), once we set $\tilde{M} = \tilde{N}$, then we have to compute $k(\sigma)^2$ for all "σ" appearing in Proposition 1. To be more precise we need some ρ in $\Gamma_{g+1,0}(q)$ with $\rho \equiv \sigma \bmod s_0$ and then compute $k(\rho)^2$.

We know that there exists $\rho_1 \equiv \sigma_1 \bmod s_0$ of the form (15) without congruences then it is a well known fact that in this case $k(\rho_1)^2 = k(\bar{\rho}_1)^2 = \chi_q^g(\det D_{11}) = \chi_q(\det D)$, the second equality is a consequence of (b).

From (14) we obtain that ρ_2 belongs to $\Gamma_{g+1}(1, 2)$ and $B \equiv 0 \bmod 2$, then confronting the transformation formula for $\vartheta_{2 \cdot 1_2}(\tau)$ and $\vartheta_0^2(\tau)$ we obtain $k(\sigma)^2 = \chi_q^{g+1}(\det D)$. Moreover $E_{g,1}(q)$ belongs to $\Gamma_{g+1}(4)$ and ρ_3 can be taken with $C=0$ and in this case the result follows from [6, p. 181].

For any holomorphic function f. We define the Siegel operator Φ as

(28) $$\Phi f(\tau') = \lim_{\lambda \to +\infty} f\begin{pmatrix} \tau' & 0 \\ 0 & i\lambda \end{pmatrix}, \quad \tau' \text{ in } \mathbb{H}_g ,$$

then the Φ operator is a linear map from $[\Gamma_{g+1,0}(q), k, \chi_q^{g+1}]$ to $[\Gamma_{g,0}(q), k, \chi_q^g]$. In particular let M^0 be the matrix obtained from M deleting the last row of M' and M'', then we have

(29) $$\Phi P(M)(\tau') = \begin{cases} P(M^0)(\tau') & \text{if the } g\text{-th row is } (0, \ldots, 0) \\ 0 & \text{otherwise,} \end{cases}$$

and

(30) $$(\Phi\vartheta_Q)(\tau')=\vartheta_Q(\tau')\ .$$

Theorem 1 *Let M in $M_{2g,r}(\Sigma_s)$ such that $f_M(\tau)$ is not an identically zero element of $[\Gamma_{g,0}, \Sigma_s, k, \chi_q^g]$, then there exists g in $[\Gamma_{g+1,0}(q), \Sigma_s, k, \chi_q^{g+1}]$ such that $\Phi(g)(\tau)=f_M(\tau)$, i.e. the Φ operator restricted to these spaces is surjective.*

The proof is by induction on $|M|=2a$.

When $a=1$, $M=(m, m, \ldots, m)$, with $2m$ integral.

In general we know that there are two Γ_g orbits. Σ_2^+ and Σ_2^- (even and odd characteristics accordingly as $\underline{e}(2{}^t m'm'')=+1$ or -1).

It is a well known fact that $\vartheta_m(\tau)\equiv 0$ if and only if m is an odd characteristic.

Let us consider the orbits for the $\Gamma_{g,0}(q)$ action on Σ_2^+.

We have one orbit if q is odd, two orbits G_0 and G_1 if q is even

(31) $$G_0=\{m \mid m'=0\}, \quad G_1=\{m \mid m'\neq 0\}\ .$$

Thus when q is odd we have

(32) $$f_M=\sum_{m\ \text{even}} \vartheta_m^{8t}, \quad r=8t\ .$$

If $r\equiv 0 \bmod 4$ it is easy to prove that we have

(33) $$f_{M_0}=\sum_{m\in G_0} \vartheta_m^{4t}, \quad r=4t \quad \text{and} \quad f_{M_1}=\sum_{m\in G_1} \vartheta_m^{8t}, \quad r=8t\ .$$

If $r\equiv 2 \bmod 4$, then we

$$f_{M_0}=\sum_{m\in G_0} \vartheta_m^{2t}, \quad r=2t\ .$$

It is immediate that for all then we have

(34) $$(\Phi f_{\tilde{M}})(\tau)=2f_M(\tau)\ .$$

Now the proof follows as in Theorem 2 of [8].

From the surjectivity of the Φ operator we deduce the following

Corollary. (a) *$[\Gamma_{g,0}(q), \Sigma_s, k, \chi_q^g] \subseteq B_k^g(\chi_q^g)$ for all g and $qk\equiv 0 \bmod 4$.*
(b) *$[\Gamma_{g,0}(q), \Sigma_s, k, \chi_q^g]$ is spanned by the Theta series $\vartheta_Q(\tau)$, such that $Q={}^tHH$, with H and H^{-1} in $M_r(1/s\mathbb{Z})$.*
(c) *When q is a power of 2, and s is large enough then in* (a) *the equality holds.*

Proof. It is an immediate consequence of Freitag's result [2], of the above theorem and of Lemma 1.

4

In this section we shall consider some relation with coding theory. We recall that a binary code is a subspace C of $\mathbb{F}_2^n$ of dimension k. Let d denote the minimal weight

$$d = \min_{\alpha \in C \setminus \{0\}} \{{}^t\alpha \cdot \alpha\} . \tag{35}$$

A linear code of length n, dimension k and minimal distance d is said to be a $[n, k, d]$ code.

A linear code C may be specified by a generator matrix.

It is an $n \times k$ matrix M such that the row of M are a basis of C. For more details we refer to [2] and [10].

We say that two codes are equivalent if there is a permutation S_n that maps one code in the other.

It is always possible to find in each class a code having generator matrix of the form

$$M = \begin{bmatrix} 1_k \\ A \end{bmatrix} \quad A \in M_{n-k,k}(\mathbb{F}_2) . \tag{36}$$

The dual code C^* is the orthogonal subspace to C.

A code is self dual if $C = C^*$, this implies n even and $\dim C = \frac{n}{2}$. A code double even if the weight of each codeword is divisible by 4. In particular if C is doubly even then $C \subseteq C^*$. Self dual doubly even codes exists if and only if 8 divides n.

We shall associate lattices to codes, we consider lattices coming from construction A of [2].

Let $\psi : \mathbb{Z}^n \to \mathbb{F}_2^n$ the canonical morphism, then we put

$$\Lambda(C) = \frac{1}{\sqrt{2}} \psi^{-1}(C) \subseteq \mathbb{R}^n . \tag{37}$$

If C has generator matrix (36), then

$$\frac{1}{\sqrt{2}} T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1_k & 0 \\ A & 2_{n-k} \end{bmatrix} \tag{38}$$

is a generator matrix for $\Lambda(C)$.

Given a lattice Λ we define the dual lattice

$$\Lambda^* = \{x \in \mathbb{R}^n : {}^t x \cdot y \in \mathbb{Z} \text{ for all } y \in \Lambda\} .$$

We say that Λ is integral if $\Lambda \subseteq \Lambda^*$.

We recall from [2] the following well known

Lemma. *Let C be a linear code of dimension k, then*

(i) $\det \Lambda(C) = 2^{n-2k}$
(ii) $\Lambda(C^*) = \Lambda(C)^*$
(iii) *$\Lambda(C)$ is integral if and only if $C \subseteq C^*$*
(iv) *$\Lambda(C)$ is even if and only if C is doubly even.*

For each code C of type A let $Q=1/2{}^tT\,T$ be the positive definite matrix associate to C.

Then the theta series $\vartheta_Q(r)$ belongs to $\left[\Gamma_g(2,4),\frac{n}{2},\chi\right]$ cf. [4]. From now on we shall assume n even, to simplifying our argument. Since T is the form (38) we have that $2T^{-1}$ is integral and the same is true for $2Q^{-1}$.

If C is doubly even then $\vartheta_Q(\tau)$ belongs to $\left[\Gamma_{g,0}(4),\frac{n}{2},\chi_4^g\right]$ and it belongs to $\left[\Gamma_{g,0}(2),\frac{n}{2}\right]$ if and only if $\operatorname{diag}(A^tA+1_{n-k})\equiv 0 \bmod 2$. In particular if C is self dual ϑ_Q belongs to $\left[\Gamma_g,\frac{n}{2}\right]$.

We put $\vartheta\begin{bmatrix}\sigma\\0\end{bmatrix}(\tau)=\vartheta_{\left[\begin{smallmatrix}\sigma\\0\end{smallmatrix}\right]}(2\tau)$, then applying the formula (20) for (39) $Q={}^t\left(\frac{T}{2}\right)2\left(\frac{T}{2}\right)$ we obtain that $\vartheta_Q(\tau)$ belongs to $\mathbb{C}\left[\vartheta\begin{bmatrix}\sigma\\0\end{bmatrix}(\tau)\right]$. It is a well known fact that the space spanned by $\vartheta\begin{bmatrix}\sigma\\0\end{bmatrix}\cdot\vartheta\begin{bmatrix}\rho\\0\end{bmatrix}$, $\sigma,\rho\in(1/2\mathbb{Z})^g/\mathbb{Z}^g$ is equal to that spanned by ϑ_m^2, $m\in(1/2\mathbb{Z})^{2g}/\mathbb{Z}^{2g}$.

Let $[\Gamma_{g,0}(q),\Sigma_2^4,k,\chi_q^g]$ the subspace of $[\Gamma_{g,0}(q),\Sigma_4,k,\chi_q^g]$ spanned by homogeneous polynomials of degree k in the ϑ_m^2, m half integral. It is stable under ϕ operator.

Clearly when C is doubly even ϑ_Q belongs to this subspaces. In [10] it has been proved that $\left[\Gamma_g,\Sigma_2^2,\frac{n}{2}\right]$ is generated by theta series coming by a self dual code. More generally as a consequence of the above discussion and of Theorem 1 we have.

Corollary. (a) $\left[\Gamma_{g,0}(4),\Sigma_2^2,\frac{n}{2},\chi_4^g\right]$ *is generated by theta series associated to doubly even codes.*

(b) $\left[\Gamma_{g,0}(2),\Sigma_2^2,\frac{n}{2},\right]$, $n\equiv 0 \bmod 4$, *spanned by theta series associated to doubly even codes A in* (36) *satisfying* $\operatorname{diag}(A^tA)\equiv 1_{n-k} \bmod 2$.

(c) $\left[\Gamma_g,\Sigma_2^2,\frac{n}{2}\right]$ $n\equiv 0 \bmod 8$ *is spanned by theta series associated to doubly even, self dual codes.*

Proof. Let f be a modular form in one of these spaces then by Theorem 1 we can assume $g=n$, moreover we have $f(\tau)=\sum a_Q\vartheta_Q(\tau)$, now considering the Fourier's coefficient of $f(\tau)$ as in [11] we have that if a_Q is different from 0, then Q is necessarily of the form (39).

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