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On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas*

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1 Introduction

This paper is devoted to the study of the asymptotic behavior of smooth solutions to initial boundary value problems in the dynamics of a one-dimensional, viscous, heat-conducting gas. The equations describing the motion of a one-dimensional gas (in Lagrangean coordinates) are those of balance of mass, balance of momentum and balance of energy

$$
\begin{aligned}
u_t - v_x &= 0 ,\\
v_t - \sigma_x &= 0 ,\\
\left(e + \frac{v^2}{2}\right)_t - (\sigma v)_x + q_x &= 0 ,
\end{aligned}
\tag{1.1}
$$

while the second law of thermodynamics is expressed by the Clausius–Duhem inequality

$$\eta_t + \left(\frac{q}{\theta}\right)_x \geqq 0 . \tag{1.2}$$

Here u, v, σ, e, η, θ and q denote the specific volume, the velocity, the stress, the specific internal energy, the specific entropy, the temperature and the heat flux, respectively. Note that u, θ and e may only take positive values.

We shall consider the system (1.1) in the region $\{0 \leqq x \leqq 1, t \geqq 0\}$ under the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x) \quad x \in [0,1] . \tag{1.3}$$

As boundary conditions we consider

$$q(0,t) = q(1,t) = 0, \quad t \geqq 0 \tag{1.4}$$

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and

$$v(0,t)=v(1,t)=0, \quad t\geqq 0\,, \tag{1.5.a}$$

or

$$\sigma(0,t)=\sigma(1,t)=0, \quad t\geqq 0\,, \tag{1.5.b}$$

or

$$\sigma(0,t)=v(0,t), \quad \sigma(1,t)=-v(1,t), \quad t\geqq 0\,. \tag{1.5.c}$$

The condition (1.4) implies that the ends are thermally insulated. (1.5.a) means that the gas is confined to a fixed tube with impermeable ends, and (1.5.b) describes that the gas is put in a vacuum, while (1.5.c), boundary damping, implies that the ends are connected to some sort of dash pot.

When the gas is polytropic ideal, i.e.

$$e=c\theta, \quad \sigma=-R\frac{\theta}{u}+\mu\frac{v_x}{u}, \quad q=-\kappa\frac{\theta_x}{u} \tag{1.6}$$

with suitable positive constants c, R, μ and κ, the existence and the asymptotic behavior of smooth solutions to (1.1) and (1.6) have been investigated by some authors, e.g. see [7, 10–17] on initial boundary value problems and the Cauchy problem.

Within moderate ranges of θ and u, a real gas is well approximated by an ideal gas. At high temperatures and densities, however, the specific heat, the conducticity and the viscosity vary with the temperature and the density, the constitutive Eqs. (1.6) become inadequate. Here we consider a more realistic model than (1.6) (or Newtonian fluid)

$$\sigma(u,\theta,v_x)=-p(u,\theta)+\frac{\mu(u,\theta)}{u}v_x \tag{1.7}$$

satisfying the Fourier law of the heat flux

$$q(u,\theta,\theta_x)=-\frac{\kappa(u,\theta)}{u}\theta_x\,, \tag{1.8}$$

where the internal energy e and the pressure p are interrelated by

$$e_u(u,\theta)=-p(u,\theta)+\theta p_\theta(u,\theta) \tag{1.9}$$

to comply with (1.2). We refer to [1, 20] for an exposition of such models.

We assume that e, p, σ and κ are twice continuously differentiable on $0<u<\infty$ and $0\leqq\theta<\infty$. We impose upon $e(u,\theta)$, $p(u,\theta)$ and $\kappa(u,\theta)$ the following growth conditions: There are exponents $r\in[0,1]$, $q\geqq 1+r$ and positive constants v, p_1, p_2 and κ_0, and for any $\underline{u}>0$ there are positive constants $N(\underline{u})$ and $\kappa_1(\underline{u})$ such that for $u\geqq\underline{u}$ and $\theta\geqq 0$,

$$0\leqq e(u,0), \quad v(1+\theta^r)\leqq e_\theta(u,\theta)\leqq N(\underline{u})(1+\theta^r)\,, \tag{1.10}$$

$$-\frac{p_2(\ell+(1-\ell)\theta+\theta^{1+r})}{u^2}\leqq p_u(u,\theta)\leqq -\frac{p_1(\ell+(1-\ell)\theta+\theta^{1+r})}{u^2}, \quad \ell=0, \text{ or } \ell=1\,, \tag{1.11}$$

$$0\leqq p(u,\theta), \quad |p_\theta(u,\theta)|\leqq N(\underline{u})(1+\theta^r)\,, \tag{1.12}$$

$$\kappa_0(1+\theta^q)\leqq\kappa(u,\theta)\leqq\kappa_1(\underline{u})(1+\theta^q), \quad |\kappa_u(u,\theta)|+|\kappa_{uu}(u,\theta)|\leqq\kappa_1(\underline{u})(1+\theta^q)\,. \tag{1.13}$$

The above assumptions are motivated by the facts in [1, 20] where it is pointed out that e grows as θ^{1+r} with $r \approx 0.5$ and κ increases like θ^q with $4.5 \leqq q \leqq 5.5$. One can easily find power laws of similar types in some books on physical chemistry, e.g. in [19]. Note that for an ideal gas, r is zero.

For technical reasons we require that the viscosity $\mu(u, \theta)$ is independent of θ, uniformly positive and bounded

$$0 < \mu_0 \leqq \mu(u) \leqq \mu_1 \tag{1.14.b, c}$$

or, in the case of the boundary condition (1.5.a) even constant

$$\mu(u) \equiv \mu_0 > 0 \,. \tag{1.14.a}$$

where and in what follows the label a (or b, or c) indicates that a certain condition applies to the problem (1.1), (1.3), (1.4), (1.5.a) (or (1.1), (1.3), (1.4), (1.5.b), or (1.1), (1.3), (1.4), (1.5.c)) only. The assumptions (1.14) are not physically motivated because in general the viscosity of a gas varies with the temperature. Unfortunately, our techniques cannot handle the situation where the viscosity depends on the temperature. It should be noted that (1.14.b) can be replaced by the following: There is a constant $\mu_0 > 0$, and for any $\underline{u} > 0$ there is a constant $\mu_1(\underline{u}) > 0$ such that $\mu_0 \leqq \mu(u) \leqq \mu_1(\underline{u})$ for $u \geqq \underline{u}$. We also make the additional assumption that for $\theta \geqq 0$

$$p(u, \theta) \to 0 \quad \text{as } u \to \infty \,. \tag{1.15}$$

Under the assumptions (1.7)–(1.14) Kawohl [9] and the author [5] established the existence of global solutions to the initial boundary value problems (1.1), (1.3), (1.4) and (1.5). It is proved in [5, 9] that if

$$\begin{gathered} u_0, u_0', v_0, v_0', v_0'', \theta_0, \theta_0', \theta_0'' \in C^\alpha[0, 1] \text{ for some } \alpha \in (0, 1), \\ u_0(x),\ \theta_0(x) > 0 \text{ on } [0, 1] \,, \end{gathered} \tag{1.16}$$

then there exists a unique solution $\{u(x, t), v(x, t), \theta(x, t)\}$ with positive u and θ to (1.1), (1.3)–(1.4) and (1.5) on $[0, 1] \times [0, \infty)$ such that for every $T > 0$

$$u, u_x, u_t, u_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx} \in C^{\alpha, \alpha/2}(Q_T), \quad u_{tt}, v_{xt}, \theta_{xt} \in L^2(Q_T) \,. \tag{1.17}$$

Here $C^\alpha[0, 1]$ stands for the Banach space of functions on $[0, 1]$ which are uniformly Hölder continuous with exponent α and $C^{\alpha, \alpha/2}(Q_T)$ for the Banach space of functions on $Q_T := [0, 1] \times [0, T]$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t. The large-time behavior of solutions, however, is not discussed in [5, 9].

The aim of this paper is to study the asymptotic behavior of solutions to (1.1), (1.3), (1.4) and (1.5).

We also mention the works by Kanel [6], Kawashima [8], Okada and Kawashima [18], Zheng and Shen [21] who investigated the existence and large-time behavior of smooth solutions to the Cauchy problem for sufficiently small initial data. For a class of solidlike materials there are independent investigations by Dafermos [2], Dafermos and Hsiao [3], and the author [4].

From a physical point of view, it is possible that the gas is rarified under the conditions (1.5.b) and (1.5.c), so u or $\int_0^1 u(x, t)\,dx$ (the volume of the region occupied by the gas) may grow to infinity, and a confined gas (1.5.a) will probably not develop vacuous regions. We shall find this conjecture valid for smooth solutions of (1.1) and (1.3)–(1.5). Our main theorems in this paper are following.

Theorem 1.1 *Consider the initial boundary problem* (1.1), (1.3), (1.4) *and* (1.5.a) *under* (1.7)–(1.15). *Assume that* u_0, v_0 *and* θ_0 *satisfy* (1.16). *Also assume that* $u_0(x)$ *satisfies the normalization condition* $\int_0^1 u_0(x)\,dx=1$. *Let* $\{u(x,t), v(x,t), \theta(x,t)\}$ *be a solution in the function class indicated in* (1.17). *Then* $\{u, v, \theta\}$ *decays to the constant state* $\{1, 0, \theta^*\}$ *in* $H^1(0,1)$ *as* $t\to\infty$, *where the constant* $\theta^*>0$ *is determined by* $e(1,\theta^*)=\int_0^1[e(u_0,v_0)+v_0^2/2](x)\,dx$. *Moreover, there are constants* α, T_0, $C>0$, *independent of* t, *such that*

$$\|u(t)-1\|_{H^1}+\|v(t)\|_{H^1}+\|\theta(t)-\theta^*\|_{H^1}\leqq Ce^{-\alpha t}\quad \textit{for } t\geqq T_0\,. \tag{1.18}$$

Theorem 1.2 *Consider the problems* (1.1), (1.3), (1.4) *and* (1.5.b) *or* (1.5.c) *under* (1.7)–(1.15). *Let* (1.16) *be satisfied for* u_0, v_0 *and* θ_0 *and let* $\{u(x,t), v(x,t), \theta(x,t)\}$ *be a solution in the function class indicated in* (1.17). *Then for* (1.5.b) *we have*

$$1+\int_0^1 u(x,t)\,dx\geqq C_1 t\quad \textit{for } t\geqq 0\,, \tag{1.19}$$

and for (1.5.c) *we have* $\int_0^1 u(x,t)\,dx\leqq C_2(1+\sqrt{t})$ *for* $t\geqq 0$ *and*

$$\overline{\lim_{t\to\infty}}\frac{\int_0^1 u(x,t)\,dx}{\log t}=\infty\,, \tag{1.20}$$

where C_1, $C_2>0$ *are constants independent of* t.

We will prove Theorem 1.1 and 1.2 in Sect. 2 and Sect. 3, respectively.

Remark 1.1 The techniques in this paper work for the boundary conditions $v(0,t)=v(1,t)=0$ and $\theta(0,t)=\theta(1,t)=1$, and a convergence of solutions like (1.18) can be obtained.

Remark 1.2 The decay constant α in Theorem 1.1 may depend on the initial data, ν, p_1, p_2, κ_0, q and μ_0. From Theorem 1.2 we see that the boundary damping (1.5.c) slows down the growth of $\int_0^1 u(x,t)\,dx$ (the volume occupied by the gas) to infinity.

Now we explain the notations used in this paper. $\|\cdot\|_{H^1}$ denotes the norm in the usual Sobolev space $H^1(0,1)$ and $\|\cdot\|$ is the norm in $L^2(0,1)$. The same letter C (sometimes used as $C(a,b,\cdots)$ to emphasize that C depends on $a, b, \cdots$) will denote various positive constants which are in particular independent of t. In general and without danger of confusion we will use the same symbol to denote the state functions as well as their value along a thermodynamic process, e.g. $p(u,\theta)$, and $p(u(x,t),\theta(x,t))$ and $p(x,t)$.

2 Proof of Theorem 1.1

In this section we let the assumptions in Theorem 1.1 be satisfied. First we adapt and modify an idea of Kazhikhov [11] (also cf. the survey article [17]) for the polytropic ideal gas to give a representation of solutions of (1.1), (1.3)–(1.4) and (1.5.a).

Let $\phi(x,t):=\int_0^t\sigma(x,s)\,ds+\int_0^x v_0(y)\,dy$. Then by (1.7) and (1.14.a), $\phi_x=v$ and $\phi_t=\sigma=-p(u,\theta)+\mu_0 v_x/u$. Thus ϕ satisfies

$$\phi_t=\frac{\mu_0}{u}\phi_{xx}-p(u,\theta)\,. \tag{2.1}$$

Multiplying (2.1) by u and using $(1.1)_1$, we see that

$$(u\phi)_t-(v\phi)_x=\mu_0\phi_{xx}-up(u,\theta)-v^2\,. \tag{2.2}$$

Keeping in mind that $\phi_x=v$ vanishes on the boundary, we integrate (2.2) over $[0,1]\times[0,t]$ to obtain

$$\int_0^1(u\phi)(x,t)\,dx=\int_0^1(u\phi)(x,0)\,dx-\int_0^t\int_0^1(v^2+up(u,\theta))(x,s)\,dx\,ds=:\Phi(t)\,. \tag{2.3}$$

It follows from integration of $(1.1)_1$ over $[0,1]\times[0,t]$ and use of (1.5.a) that

$$\int_0^1 u(x,t)\,dx=\int_0^1 u_0(x)\,dx=1\,. \tag{2.4}$$

Note that $u>0$. If we apply the mean value theorem to (2.3) and use (2.4) we conclude that for each $t\geqq 0$ there is an $x_0(t)\in[0,1]$ such that

$$\phi(x_0(t),t)=\int_0^1\phi(x,t)u(x,t)\,dx=\Phi(t)\,. \tag{2.5}$$

Therefore by the definition of $\phi(x,t)$ and (2.3), we have

$$\begin{aligned}\int_0^t\sigma(x_0(t),s)\,ds&=\phi(x_0(t),t)-\int_0^{x_0(t)}v_0(y)\,dy=\Phi(t)-\int_0^{x_0(t)}v_0(y)\,dy\\&=-\int_0^t\int_0^1(v^2+up(u,\theta))\,dx\,ds+\int_0^1u_0(x)\int_0^x v_0(y)\,dy\,dx-\int_0^{x_0(t)}v_0(y)\,dy\end{aligned} \tag{2.6}$$

for $t\geqq 0$. Using (2.6), we now prove

Lemma 2.1 *For the problem* (1.1), (1.3)–(1.4) *and* (1.5.*a*) *we have the following representation*

$$u(x,t)=\frac{D(x,t)}{B(t)}\left\{1+\frac{1}{\mu_0}\int_0^t\frac{u(x,s)p(x,s)B(s)}{D(x,s)}\,ds\right\}, \tag{2.7}$$

where

$$D(x,t):=u_0(x)\exp\left\{\frac{1}{\mu_0}\left(\int_0^1u_0(x)\int_0^x v_0(y)\,dy\,dx-\int_0^{x_0(t)}v_0(y)\,dy+\int_{x_0(t)}^x(v-v_0)\,dy\right)\right\}, \tag{2.8}$$

$$B(t):=\exp\left(\frac{1}{\mu_0}\int_0^t\int_0^1(v^2+up(u,\theta))(y,\tau)\,dy\,d\tau\right) \tag{2.9}$$

and $x_0(t)\in[0,1]$ *is the same as in* (2.5).

Proof. In view of (1.7) and (1.14.a), we rewrite $(1.1)_2$ as follows

$$v_t+p(u,\theta)_x=(\mu_0\log u)_{xt}\quad(\Leftrightarrow v_t=\sigma_x)\,. \tag{2.10}$$

If we integrate (2.10) over $[0, t]$, then integrate over $[x_0(t), x]$ with respect to x and use (2.6), we obtain

$$\mu_0 \log u(x,t) - \int_0^t p(x,s)\,ds = \mu_0 \log u_0(x) - \int_0^t\int_0^1 (v^2 + up(u,\theta))(x,s)\,dx\,ds$$
$$+ \int_0^1 u_0(x)\int_0^x v_0(y)\,dy\,dx - \int_0^{x_0(t)} v_0\,dy + \int_{x_0(t)}^x (v - v_0)\,dy\,,$$

which, upon taking the exponential, turns into

$$u(x,t)\exp\left(-\frac{1}{\mu_0}\int_0^t p(x,s)\,ds\right) = \frac{D(x,t)}{B(t)}\,. \tag{2.11}$$

It follows from (2.11) and (2.8) that

$$\frac{p(x,t)}{\mu_0}\exp\left(\frac{1}{\mu_0}\int_0^t p(x,s)\,ds\right) = \frac{u(x,t)p(x,t)B(t)}{\mu_0 D(x,t)}\,.$$

Integrating the above identity over $(0, t)$, one has

$$\exp\left(\frac{1}{\mu_0}\int_0^t p(x,s)\,ds\right) = 1 + \frac{1}{\mu_0}\int_0^t \frac{u(x,s)p(x,s)B(s)}{D(x,s)}\,ds\,. \tag{2.12}$$

Inserting (2.12) into (2.11), we obtain (2.7). □

Next we exploit some relations associated with the second law of thermodynamics to derive estimates for solutions.

Lemma 2.2 *Let α and β be two (positive) roots of the equation*

$$\theta - \log\theta - 1 = E_0/v\,, \tag{2.13}$$

where $E_0 > 0$ is defined in (2.20) below. Then for each $t \geqq 0$ there is an $a(t) \in [0, 1]$ such that

$$0 < \alpha \leqq \theta(a(t), t) \leqq \beta\,. \tag{2.14}$$

Furthermore, the following estimates hold.

$$0 < \alpha \leqq \int_0^1 \theta(x,t)\,dx \leqq \beta \quad \forall t \geqq 0\,, \tag{2.15}$$

$$\int_0^1 (\theta + \theta^{1+r} + v^2)(x,t)\,dx + \int_0^t\int_0^1 \left(\frac{v_x^2}{u\theta} + \frac{(1+\theta^q)\theta_x^2}{u\theta^2}\right)dx\,ds \leqq C \quad \forall t \geqq 0\,. \tag{2.16}$$

Proof. Let $\psi(u,\theta) = e(u,\theta) - \theta\eta(u,\theta)$ denote the Helmholtz free energy function. Then

$$\psi_\theta(u,\theta) = -\eta(u,\theta), \quad \psi_u(u,\theta) = \sigma(u,\theta,0) \equiv -p(u,\theta), \quad \theta\psi_{\theta\theta}(u,\theta) = -e_\theta(u,\theta)\,. \tag{2.17}$$

We denote

$$E(u,\theta) := \psi(u,\theta) - \psi(1,1) - \psi_u(1,1)(u-1) - (\theta-1)\psi_\theta(u,\theta)\,. \tag{2.18}$$

Using (2.17), the Eqs. (1.1) (also cf. (2.60)), (1.7) – (1.9) and (1.14.a), we deduce after a straightforward calculation that

$$\partial_t\left(E(u,\theta)+\frac{v^2}{2}\right)+\frac{\mu_0 v_x^2}{u\theta^2}+\frac{\kappa(u,\theta)\theta_x^2}{u\theta^2}=(\sigma v)_x+p(1,1)v_x+\left(\frac{\theta-1}{\theta}\frac{\kappa(u,\theta)\theta_x}{u}\right)_x. \quad (2.19)$$

Integrating (2.19) over $Q_t:=[0,1]\times[0,t]$, remembering that v and θ_x vanish at $x=0$ and $x=1$ and taking (1.13) into account, we arrive at

$$\int_0^1\left(E(u,\theta)+\frac{v^2}{2}\right)(x,t)\,dx+\int_0^t\int_0^1\left(\frac{\mu_0 v_x^2}{u\theta}+\frac{\kappa_0(1+\theta^q)\theta_x^2}{u\theta^2}\right)dx\,ds$$
$$\leq\int_0^1\left(E(u_0,\theta_0)+\frac{v_0^2}{2}\right)(x)\,dx=:E_0 \quad \text{for } t\geq 0. \quad (2.20)$$

It follows from Taylor's theorem, (2.17) and (1.10)–(1.11) that

$$E(u,\theta)-\psi(u,\theta)+\psi(u,1)+(\theta-1)\psi_\theta(u,\theta)=(u-1)^2\int_0^1(1-\tau)$$
$$\times\psi_{uu}(1+\tau(u-1),1)\,d\tau\geq 0\,, \quad (2.21)$$

$$\psi(u,\theta)-\psi(u,1)-(\theta-1)\psi_\theta(u,\theta)\geq\nu(1-\theta^2)\int_0^1\frac{(1-\tau)[1+(\theta+\tau(1-\theta))^r]}{\theta+\tau(1-\theta)}d\tau \quad (2.22)$$

$$=\nu(\theta-\log\theta-1)+\begin{cases}\nu(\theta-\log\theta-1)\,, & r=0\\ \theta^{1+r}/(1+r)+[1/(1+r)-\theta^r]/r\,, & r>0\end{cases}\geq 0\,.$$

Adding (2.22) to (2.21), one has

$$E(u,\theta)\geq\nu(\theta-\log\theta-1),\quad\text{or } E(u,\theta)\geq\frac{\nu}{2}\theta+\frac{\theta^{1+r}}{2(1+r)}-C\,, \quad (2.23)$$

where we have used

$$ab\leq a^{\lambda_1}/\lambda_1+b^{\lambda_2}/\lambda_2 \quad \forall a,b\geq 0,\ \lambda_1,\lambda_2>1,\ 1/\lambda_1+1/\lambda_2=1\,, \quad (2.24)$$

which will be frequently used throughout the paper. From (2.20) and (2.23), (2.16) follows immediately. To show (2.14)–(2.15) we substitute (2.23) into (2.20) to see that

$$\nu\int_0^1(\theta(x,t)-\log\theta(x,t)-1)\,dx\leq E_0,\quad t\geq 0\,. \quad (2.25)$$

So, using the mean value theorem, for each $t\geq 0$ there is an $a(t)\in[0,1]$ such that $\theta(a(t),t)-\log\theta(a(t),t)-1\leq E_0/\nu$, from which (2.14) follows. If we use (2.25) and apply Jensen's inequality to the convex function $y-\log y-1$, we obtain: $\int_0^1\theta(x,t)\,dx-\log\int_0^1\theta(x,t)\,dx-1\leq E_0/\nu$, which yields (2.15). This proves the lemma. □

Recalling the definition (2.8) of $D(x,t)$, as a result of (2.16) and Schwarz's inequality we have

$$0<C^{-1}\leq D(x,t)\leq C \quad \forall x\in[0,1],\ t\geq 0\,. \quad (2.26)$$

By virtue of Schwarz's inequality, (2.4) and (2.24), recalling that $q \geqq 1+r$, we find

$$|\theta^{(r+1)/2}(x,t)-\theta^{(r+1)/2}(a(t),t)| \leqq C\int_0^1 |\theta_x|\,\theta^{(r-1)/2}\,dx$$

$$\leqq C\left(\int_0^1 \frac{\theta_x^2\theta^{r-1}\,dx}{u}\right)^{1/2} \leqq CV^{1/2}(t)\,, \tag{2.27}$$

where

$$V(t) := \int_0^1 \left(\frac{(1+\theta^q)\theta_x^2}{u\theta^2}\right)(x,t)\,dx\,. \tag{2.28}$$

(2.27) together with (2.14) yields

$$\alpha^{r+1}/2 - CV(t) \leqq \theta^{r+1}(x,t) \leqq 2\theta^{r+1}(a(t),t)+CV(t) \leqq 2\beta^{r+1}+CV(t) \tag{2.29}$$

for $x \in [0,1]$ and $t \geqq 0$. Now integration of (1.11) with respect to u over (u, ∞) and use of (1.15) imply

$$p_1[\ell+(1-\ell)\theta+\theta^{r+1}] \leqq up(u,\theta) \leqq p_2[\ell+(1-\ell)\theta+\theta^{r+1}],$$

$$\ell=0, \text{ or } \ell=1\,, \tag{2.30}$$

which combined with (2.15) implies

$$\int_0^1 up(u,\theta)\,dx \geqq p_1\int_0^1 (\ell+(1-\ell)\theta(x,t))\,dx \geqq C_0>0\,. \tag{2.31}$$

Here C_0 is a constant independent of t. Recalling the definitions (2.9) and (2.28), if we make use of (2.7), (2.26), (2.30), and (2.24), (2.31), (2.29) and (2.16), we deduce

$$u(x,t) \leqq C\int_0^t up(x,s)\exp\left(-\frac{1}{\mu_0}\int_s^t\int_0^1 (v^2+up)\,dy\,d\tau\right)ds$$

$$\leqq C\int_0^t (1+\theta^{1+r}(x,s))\,e^{-C_1(t-s)}\,ds$$

$$\leqq C\int_0^t (1+V(s))\,e^{-C_1(t-s)}\,ds \leqq C\left(1+\int_0^t V(s)\,ds\right) \leqq C \tag{2.32}$$

for all $x \in [0,1]$ and $t \geqq 0$. In the same manner we have

$$u(x,t) \geqq C\int_0^t up(x,s)\exp\left(-\frac{1}{\mu_0}\int_s^t\int_0^1 (v^2+up)\,dy\,d\tau\right)ds$$

$$\geqq C^{-1}\int_0^t \theta^{1+r}(x,s)\exp\left(-C\int_0^t\int_0^1 (v^2+1+\theta^{r+1}+\theta)\,dy\,d\tau\right)ds$$

$$\geqq C^{-1}\int_0^t \left(\frac{1}{2}\alpha^{r+1}-CV(s)\right)e^{-C_2(t-s)}\,ds$$

$$\geqq C^{-1}(1-e^{-C_2 t}) - C\int_0^{t/2} V(s)\,ds\,e^{-C_2 t/2} - C\int_{t/2}^t V(s)\,ds \geqq (2C)^{-1}>0 \tag{2.33}$$

for any $t \geqq T_0$ and some $T_0 > 0$. Here C_1 and C_2 in (2.32)–(2.33) are positive constants. It follows from (2.7), (2.26), (2.30) and (2.16) that

$$u(x,t) \geqq \frac{D(x,t)}{B(t)} \geqq \frac{1}{C} \exp\left(-C \int_0^t \int_0^1 (v^2 + 1 + \theta + \theta^{1+r})\, dy\, d\tau \right) \geqq \frac{e^{-Ct}}{C} \tag{2.34}$$

for $x \in [0,1]$ and $t \geqq 0$. Putting (2.32)–(2.34) together, we have proved

Lemma 2.3 *There are positive constants $\underline{u}$ and $\bar{u}$, independent of t, such that*

$$\underline{u} \leqq u(x,t) \leqq \bar{u} \quad \textit{for any } t \geqq 0 \textit{ and } x \in [0,1] . \tag{2.35}$$

In the sequel we derive Sobolev-norm estimates of derivatives for u, v, θ. We first observe that by (2.16) and (2.35)

$$\int_0^t \max_{[0,1]} v^2(\cdot, s)\, ds \leqq \int_0^t \left\{ \int_0^1 |v_x|\, dx \right\}^2 ds \leqq \int_0^t \int_0^1 \frac{v_x^2}{\theta}\, dx\, ds \leqq C . \tag{2.36}$$

Using (1.7) and (1.14.a), we write $(1.1)_2$ as follows

$$v_t + p(u,\theta)_x = \left[\frac{\mu_0}{u} v_x \right]_x \quad \left(\equiv \left[\frac{\mu_0}{u} u_x \right]_t \right) . \tag{2.37}$$

Multiply (2.37) by v and integrate over $[0,1] \times [0,t]$. Integrating by parts with respect to x, and utilising Lemma 2.3, (1.11)–(1.12), (2.16), (2.24) (recalling $q \geqq 1 + r$), and (2.36), we see that

$$\begin{aligned}
\|v(t)\|^2 + \int_0^t \int_0^1 \frac{\mu_0 v_x^2}{u}\, dx\, ds &\leqq C + \int_0^t \int_0^1 |p(u,\theta)_x v|\, dx\, ds \\
&\leqq C + C \int_0^t \int_0^1 \{ [\ell + (1-\ell)\theta + \theta^{1+r}] |u_x| \\
&\quad + (1 + \theta^r) |\theta_x| \} |v|\, dx\, ds \\
&\leqq C(\varepsilon) + \varepsilon \int_0^t \int_0^1 [\ell + (1-\ell)\theta + \theta^{1+r}] u_x^2\, dx\, ds \\
&\quad + C(\varepsilon) \int_0^t \max_{[0,1]} v^2 \int_0^1 (1 + \theta^{1+r})\, dx\, ds \\
&\quad + C \int_0^t \int_0^1 \frac{(1+\theta^r)\theta_x^2}{\theta}\, dx\, ds \\
&\leqq C(\varepsilon) + \varepsilon \int_0^t \int_0^1 [\ell + (1-\ell)\theta + \theta^{1+r}] u_x^2\, dx\, ds .
\end{aligned} \tag{2.38}$$

To bound u_x we multiply (2.37) by u_x/u and integrate. We apply the assumption (1.11), (2.35), (2.24) and (2.16) to obtain

$$\begin{aligned}
\mu_0 \int_0^1 \frac{u_x^2}{u^2}\, dx &\leqq C + \int_0^t \int_0^1 v_t \frac{u_x}{u}\, dx\, ds - p_1 \int_0^t \int_0^1 \frac{[\ell + (1-\ell)\theta + \theta^{1+r}] u_x^2}{u^3}\, dx\, ds \\
&\quad + C \int_0^t \int_0^1 (1 + \theta^r) |\theta_x| |u_x|\, dx\, ds \\
&\leqq C + \int_0^t \int_0^1 v_t \frac{u_x}{u}\, dx\, ds - \frac{p_1}{2} \int_0^t \int_0^1 \frac{[\ell + (1-\ell)\theta + \theta^{1+r}] u_x^2}{u^3}\, dx\, ds .
\end{aligned} \tag{2.39}$$

Keeping in mind that $(u_x/u)_t=(u_t/u)_x$, we employ integration by parts, and use (2.35) and (2.16) to arrive at

$$\int_0^t\int_0^1 v_t \frac{u_x}{u}\,dx\,ds=\int_0^1 v\frac{u_x}{u}\Big|_0^t dx-\int_0^t\int_0^1 v\left[\frac{u_t}{u}\right]_x dx\,ds$$

$$\leqq C(\varepsilon)+\varepsilon\int_0^1 u_x^2(x,t)\,dx+\frac{1}{\underline{u}}\int_0^t\int_0^1 v_x^2\,dx\,ds\,.$$

Inserting the above inequality into (2.39) and letting ε appropriately small, we get in view of (2.35) that

$$\int_0^1 u_x^2(x,t)\,dx+\int_0^t\int_0^1 [\ell+(1-\ell)\theta+\theta^{1+r}]u_x^2\,dx\,ds\leqq C+C\int_0^t\int_0^1 v_x^2\,dx\,ds\,. \tag{2.40}$$

Letting ε appropriately small, then $(2.38)+(2.40)\times\sqrt{\varepsilon}$ implies

$$\|v(t)\|^2+\|u_x(t)\|^2+\int_0^t\int_0^1\{v_x^2+[\ell+(1-\ell)\theta+\theta^{1+r}]u_x^2\}(x,s)\,dx\,ds\leqq C\quad \forall t\geqq 0\,. \tag{2.41}$$

We proceed to get estimates concerning derivatives of solutions. Let $\bar\theta(t):=\theta(a(t),t)$, where $a(t)\in[0,1]$ for $t\geqq 0$ is defined in (2.14), then $\alpha\leqq\bar\theta(t)\leqq\beta$ in view of (2.14). It thus follows from (2.28) and Schwarz's inequality that

$$|\theta^{1+(q+r)/2}(x,t)-\bar\theta^{1+(q+r)/2}(t)|^2\leqq\left(\int_0^1\theta^{(q+r)/2}|\theta_x|\,dx\right)^2$$

$$\leqq\int_0^1\theta^{q-2}\theta_x^2(x,t)\,dx\int_0^1\theta^{r+2}(x,t)\,dx$$

$$\leqq V(t)\int_0^1\theta^{r+2}(x,t)\,dx \tag{2.42}$$

for $0\leqq x\leqq 1$ and $t\geqq 0$. Similarly, by (2.16),

$$|\theta^{(r+1)/2}(x,t)-\bar\theta^{(r+1)/2}(t)|^2,\ |\theta^{r+1}(x,t)-\bar\theta^{r+1}(t)|^2,$$

$$|\theta(x,t)-\bar\theta(t)|^2\leqq CV(t)\quad \forall x\in[0,1],\ t\geqq 0\,. \tag{2.43}$$

Using (1.9), (2.30), Lemma 2.3, (1.12)–(1.13), and (2.24), (2.16), (2.41)–(2.43), we get

$$\int_0^t\int_0^1\frac{\kappa(u,\theta)}{u}|\theta_x||e_u(u,\theta)||u_x|\,dx\,ds\leqq C\int_0^t\int_0^1(1+\theta^{q+r+1})|\theta_x||u_x|\,dx\,ds$$

$$\leqq\int_0^t\int_0^1(\theta^{-1-r}\theta_x^2+\theta^{1+r}u_x^2)\,dx\,ds$$

$$+\frac{\nu\kappa_0}{2\bar u}\int_0^t\int_0^1\theta^{q+r}\theta_x^2\,dx\,ds+C\int_0^t\int_0^1\theta^{q+r+2}u_x^2\,dx\,ds$$

$$\leqq C+\frac{\nu\kappa_0}{2\bar{u}}\int_0^t\int_0^1 \theta^{q+r}\theta_x^2\,dx\,ds$$

$$+C\int_0^t\int_0^1\{(\theta^{1+(q+r)/2}-\bar{\theta}^{1+(q+r)/2})^2$$

$$+\bar{\theta}^{q+1}(\theta^{(1+r)/2}-\bar{\theta}^{(1+r)/2})^2+\bar{\theta}^{q+1}\theta^{r+1}\}u_x^2$$

$$\leqq C+\frac{\nu\kappa_0}{2\bar{u}}\int_0^t\int_0^1 \theta^{q+r}\theta_x^2\,dx\,ds$$

$$+C\int_0^t V(s)\int_0^1 \theta^{r+2}(x,s)\,dx\,ds\,. \tag{2.44}$$

In the same manner we can show

$$\int_0^t\int_0^1 u_x^2(x,s)\,dx\,ds\leqq C\int_0^t\int_0^1 \bar{\theta}^{r+1}(s)u_x^2(x,s)\,dx\,ds$$

$$\leqq C\int_0^t\int_0^1(\theta^{(1+r)/2}-\bar{\theta}^{(1+r)/2})^2u_x^2\,dx\,ds+C\int_0^t\int_0^1\theta^{1+r}u_x^2\,dx\,ds\leqq C\,. \tag{2.45}$$

Now note that

$$p(u,\theta)v_xe(u,\theta)=[p(u,\theta)-p(u,\bar{\theta})]\,v_xe(u,\theta)+[p(u,\bar{\theta})-p(1,\bar{\theta})]\,v_xe(u,\theta)$$

$$+p(1,\bar{\theta})v_x[e(u,\theta)-e(u,\bar{\theta})]$$

$$+p(1,\bar{\theta})v_x[e(u,\bar{\theta})-e(1,\bar{\theta})]$$

$$+p(1,\bar{\theta})v_xe(1,\bar{\theta})=:\sum_{j=1}^5 I_j(x,t)\,. \tag{2.46}$$

We want to bound every term in (2.46). By virtue of (1.10)–(1.12), (2.43), and (2.16), (2.4), Poincaré's inequality and (2.45), I_1 and I_2 can be estimated as follows

$$\left|\int_0^t\int_0^1 I_1(x,s)\,dx\,ds\right|\leqq C\int_0^t\int_0^1|\theta-\bar{\theta}|(1+\theta^{2r+1})|v_x|\,dx\,ds$$

$$\leqq C\int_0^t\int_0^1(\theta-\bar{\theta})^2(1+\theta^{2r+2})\,dx\,ds$$

$$+\int_0^t\int_0^1 v_x^2(1+\theta^{2r})\,dx\,ds$$

$$\leqq C\left(1+\int_0^t V(s)\int_0^1\theta^{2+2r}\,dx\,ds+\int_0^t\max_{[0,1]}v_x^2\,ds\right), \tag{2.47}$$

$$\left|\int_0^t\int_0^1 I_2(x,s)\,dx\,ds\right|\leqq C\int_0^t\int_0^1|u-1|\,|v_x|(1+\theta^{1+r})\,dx\,ds$$

$$\leqq C\int_0^t\max_{[0,1]}|u-1|\max_{[0,1]}|v_x|\,ds$$

$$\leqq C\int_0^t\int_0^1 u_x^2\,dx\,ds+C\int_0^t \max_{[0,1]} v_x^2\,ds$$

$$\leqq C\left(1+\int_0^t \max_{[0,1]} v_x^2\,ds\right). \tag{2.48}$$

Similarly,

$$\left|\int_0^t\int_0^1 I_3(x,s)\,dx\,ds\right|+\left|\int_0^t\int_0^1 I_4(x,s)\,dx\,ds\right|\leqq C\left(1+\int_0^t \max_{[0,1]} v_x^2\,ds\right). \tag{2.49}$$

Finally, since $v(0,t)=v(1,t)=0$, it is easy to see that

$$\int_0^t\int_0^1 I_5(x,s)\,dx\,ds=0. \tag{2.50}$$

If we integrate (2.46) over $[0,1]\times[0,t]$ and combine (2.47)–(2.50), we find

$$\left|\int_0^t\int_0^1 p(u,\theta)\,v_x e(u,\theta)\,dx\,ds\right|\leqq C\left(1+\int_0^t \max_{[0,1]} v_x^2\,ds\right)$$
$$+C\int_0^t V(s)\int_0^1 \theta^{2+2r}(x,s)\,dx\,ds. \tag{2.51}$$

We rewrite $(1.1)_3$, using $(1.1)_2$ and (1.8), as

$$e_t-\sigma v_x-\left[\frac{\kappa(u,\theta)}{u}\theta_x\right]_x=0. \tag{2.52}$$

Multiply (2.52) by e, integrate over $[0,1]\times[0,t]$ and employ partial integration. In view of (1.7) we make use of (1.10), (1.13), and (2.35), (2.44), (2.51) and (2.16) to deduce that

$$\frac{v^2}{8}\int_0^1(\theta^2+\theta^{2+2r})(x,t)\,dx+\frac{\nu\kappa_0}{2\bar u}\int_0^t\int_0^1(1+\theta^{q+r})\,\theta_x^2\,dx\,ds\leqq C\left(1+\int_0^t \max_{[0,1]} v_x^2\,ds\right)$$
$$+C\int_0^t V(s)\int_0^1 \theta^{1+r}(x,s)\,dx\,ds+\mu_0\int_0^t \max_{[0,1]} v_x^2\int_0^1\frac{e(u,\theta)}{u}\,dx\,ds$$
$$\leqq C(\varepsilon)+C\varepsilon\int_0^t\int_0^1 v_{xx}^2\,dx\,ds+C\int_0^t V(s)\int_0^1 \theta^{2+2r}\,dx\,ds,\quad \varepsilon\in(0,1), \tag{2.53}$$

where we have also used (2.41) and the inequality (Sobolev's imbedding theorem $W^{1,1}(0,1)\hookrightarrow L^\infty(0,1)$)

$$\max_{[0,1]} v_x^2(\cdot,t)\leqq C\int_0^1(v_x^2+|v_x|\,|v_{xx}|)\,dx\leqq C(\varepsilon)+\varepsilon\int_0^1 v_{xx}^2(x,t)\,dx. \tag{2.54}$$

In order to bound $\int_0^t\int_0^1 v_{xx}^2\,dx\,ds$ we multiply (2.37) by $-v_{xx}$ and integrate. By virtue of (1.11)–(1.12), (2.24), (2.35), and (2.41), (2.43), (2.45) and (2.54) with appropriately small ε, one has

$$\|v_x(t)\|^2+\frac{\mu_0}{2\bar{u}}\int_0^t\int_0^1 v_{xx}^2\,dx\,ds \leqq C+C\int_0^t\int_0^1\{u_x^2v_x^2+u_x^2(1+\theta^{2+2r})+\theta_x^2(1+\theta^{2r})\}\,dx\,ds$$

$$\leqq C+\frac{\mu_0}{4\bar{u}}\int_0^t\int_0^1 v_{xx}^2\,dx\,ds+C\int_0^t\int_0^1 u_x^2(\theta^{1+r}-\bar{\theta}^{1+r})^2\,dx\,ds$$

$$+C\int_0^t\int_0^1(1+\theta^{q+r})\,\theta_x^2\,dx\,ds$$

$$\leqq C+\frac{\mu_0}{4\bar{u}}\int_0^t\int_0^1 v_{xx}^2\,dx\,ds+C\int_0^t\int_0^1(1+\theta^{q+r})\,\theta_x^2\,dx\,ds\,. \tag{2.55}$$

Combining (2.53) with (2.55) and letting ε appropriately small, we obtain

$$\int_0^1(\theta^2+\theta^{2+2r})(x,t)\,dx+\int_0^t\int_0^1(1+\theta^{q+r})\,\theta_x^2\,dx\,ds \leqq C+C\int_0^t V(s)\int_0^1\theta^{2+2r}(x,s)\,dx\,ds\,,$$

from which, (2.16) and Gronwall's inequality, it follows that

$$\int_0^1(\theta^2+\theta^{2+2r})(x,t)\,dx+\int_0^t\int_0^1(1+\theta^{q+r})\,\theta_x^2(x,s)\,dx\,ds\leqq C \quad \forall t\geqq 0\,. \tag{2.56}$$

Putting (2.55), (2.56) and (2.54) together, we conclude that

$$\|v_x(t)\|^2+\int_0^t\max_{[0,1]}v_x^2(\cdot,s)\,ds+\int_0^t\int_0^1 v_{xx}^2\,dx\,ds\leqq C \quad \forall t\geqq 0\,. \tag{2.57}$$

Next we employ the bounds obtained thus far to estimate $\|\theta_x(t)\|^2$. To this end let

$$Y(t):=\max_{0\leqq s\leqq t}\int_0^1(1+\theta^{2q})\,\theta_x^2(x,s)dx, \quad X(t):=\int_0^t\int_0^1(1+\theta^{q+r})\,\theta_t^2\,dx\,ds\,. \tag{2.58}$$

Thus, (2.56) implies

$$\theta^{q+2+r}(x,t)-\bar{\theta}^{q+2+r}(t)\leqq\int_0^1\theta^{q+r+1}\,|\theta_x|\,dx$$

$$\leqq\left(\int_0^1\theta^{2q}\,\theta_x^2\,dx\right)^{1/2}\left(\int_0^1\theta^{2r+2}\,dx\right)^{1/2}\leqq CY^{1/2}(t)\,,$$

whence (also by (2.14))

$$\max_{Q_t}\theta\leqq C+CY^{1/(2q+4+2r)}\,, \tag{2.59}$$

where $Q_t=[0,1]\times[0,t]$. Using $(1.1)_1$–$(1.1)_2$, (1.7)–(1.9) and (1.14.a), we rewrite $(1.1)_3$ as follows

$$e_\theta(u,\theta)\theta_t+\theta p_\theta(u,\theta)v_x-\frac{\mu_0}{u}v_x^2=\left[\frac{\kappa(u,\theta)}{u}\theta_x\right]_x\,. \tag{2.60}$$

Define $K(u,\theta):=\int_0^\theta(\kappa(u,\xi)/u)\,d\xi$, and consider K to be a function of x and t. Multiplying (2.60) by K_t and integrating over Q_t, we perform partial integration to conclude

$$\int_0^t\int_0^1 (e_\theta\theta_t+\theta p_\theta v_x-\frac{\mu_0}{u}v_x^2)K_t\,dx\,ds+\int_0^t\int_0^1\frac{\kappa}{u}\theta_x K_{tx}\,dx\,ds=0\,. \tag{2.61}$$

We have to estimate every term in (2.61). Note that

$$K_t=K_u v_x+\frac{\kappa}{u}\theta_t,\quad K_{xt}=\left[\frac{\kappa}{u}\theta_x\right]_t+K_u v_{xx}+K_{uu}v_x u_x+\left[\frac{\kappa}{u}\right]_u u_x\theta_t\,. \tag{2.62}$$

It follows from (1.13) and (2.35) that $|K_u|$, $|K_{uu}|\leqq C(1+\theta^{q+1})$. Therefore using (1.10), (1.12)–(1.13), and (2.35), (2.24), (2.56)–(2.57) and (2.41), we infer

$$\begin{aligned}\int_0^t\int_0^1 e_\theta\theta_t K_t\,dx\,ds&\geqq\frac{\kappa_0\nu}{\bar u}X-C\int_0^t\int_0^1(1+\theta^{q+r+1})|\theta_t||v_x|\,dx\,ds\\&\geqq\frac{\kappa_0\nu}{2\bar u}X-C\int_0^t\int_0^1(1+\theta^{q+r+2})v_x^2\,dx\,ds\\&\geqq\frac{\kappa_0\nu}{2\bar u}X-C(1+\max_{Q_t}\theta^{q+r+2})\,,\end{aligned} \tag{2.63}$$

$$\begin{aligned}\left|\int_0^t\int_0^1(\theta p_\theta v_x-\frac{\mu_0}{u}v_x^2)K_t\,dx\,ds\right|&\leqq C\int_0^t\int_0^1\{(1+\theta^{q+r+2})v_x^2+(1+\theta^{q+1})|v_x|^3\,dx\,ds\\&\quad+(1+\theta^{q+r+1})|v_x||\theta_t|+(1+\theta^q)v_x^2|\theta_t|\}\,dx\,ds \qquad (2.64)\\&\leqq C(1+\max_{Q_t}\theta^{q+2+r})+C(1+\max_{Q_t}\theta^{q+1})\\&\quad\times\int_0^t\max_{[0,1]}v_x^2\int_0^1|v_x|\,dx\,ds+\frac{\nu\kappa_0}{4\bar u}X\\&\quad+C(1+\max_{Q_t}\theta^{q-r})\int_0^t\max_{[0,1]}v_x^2\int_0^1 v_x^2\,dx\,ds\\&\leqq C(1+\max_{Q_t}\theta^{q+2+r})+\frac{\nu\kappa_0}{4\bar u}X\,,\end{aligned}$$

$$\int_0^t\int_0^1\frac{\kappa}{u}\theta_x\left[\frac{\kappa}{u}\theta_x\right]_t dx\,ds\geqq\frac{\kappa_0^2}{2\bar u^2}\int_0^1[(1+\theta^{2q})\theta_x^2](x,t)\,dx-C\,, \tag{2.65}$$

$$\begin{aligned}\left|\int_0^t\int_0^1\frac{\kappa}{u}\theta_x(K_u v_{xx}+K_{uu}v_x u_x)\,dx\,ds\right|&\leqq C\int_0^t\int_0^1(1+\theta^{2q+1})|\theta_x|(|v_{xx}|+|v_x|\,|u_x|)\,dx\,ds\\&\leqq C\left(\int_0^t\int_0^1(1+\theta^{4q+2})\theta_x^2\right)^{1/2}\\&\leqq C(1+\max_{Q_t}\theta^{1+3q/2})\,,\end{aligned} \tag{2.66}$$

$$\left|\int_0^t\int_0^1 \frac{\kappa}{u}\theta_x\left[\frac{\kappa}{u}\right]_u u_x\theta_t\,dx\,ds\right| \leqq \frac{\kappa_0 \nu}{16\bar{u}}X + C\int_0^t\int_0^1\left[\frac{\kappa}{u}\theta_x\right]^2(1+\theta^{q-r})u_x^2\,dx\,ds$$

$$\leqq \frac{\kappa_0\nu}{16\bar{u}}X + C(1+\max_{Q_t}\theta^{q-r})\int_0^t \max_{[0,1]}\left[\frac{\kappa}{u}\theta_x\right]^2(\cdot, s)\,ds$$

$$\leqq \frac{\kappa_0\nu}{16\bar{u}}X + C(1+\max_{Q_t}\theta^{q-r})\int_0^t\int_0^1\left\{\left[\frac{\kappa}{u}\theta_x\right]^2 + \left|\frac{\kappa}{u}\theta_x\right|\left|\left[\frac{\kappa}{u}\theta_x\right]_x\right|\right\}dx\,ds$$

$$\leqq \frac{\kappa_0\nu}{16\bar{u}}X + C + C\max_{Q_t}\theta^{2q-2r}) + C(1+\max_{Q_t}\theta^{q-r}) \times\int_0^t\int_0^1\left|\frac{\kappa}{u}\theta_x\right|\left|\left[\frac{\kappa}{u}\theta_x\right]_x\right|dx\,ds\,. \tag{2.67}$$

The last integral in the above inequality can be estimated as follows, using Schwarz's inequality, (2.60), (1.10), (1.12)–(1.13), (2.56)–(2.57) and (2.24) (recalling $q \geqq 1+r$),

$$\int_0^t\int_0^1\left|\frac{\kappa}{u}\theta_x\right|\left|\left[\frac{\kappa}{u}\theta_x\right]_x\right|dx\,ds \leqq C\left\{\int_0^t\int_0^1(1+\theta^{q-r})\left[\frac{\kappa}{u}\theta_x\right]_x^2 dx\,ds\right\}^{1/2}$$

$$\leqq C\left\{\int_0^t\int_0^1[(1+\theta^{q+r})\theta_t^2 + (1+\theta^{q+2+r})v_x^2 + (1+\theta^{q-r})v_x^4]\,dx\,ds\right\}^{1/2}$$

$$\leqq CX^{1/2} + C + C\max_{Q_t}\theta^{(q+2+r)/2} + C\max_{Q_t}\theta^{(q-r)/2}$$

$$\leqq CX^{1/2} + C + C\max_{Q_t}\theta^{(q+2+r)/2}\,,$$

which together with (2.67) and (2.24) gives

$$\left|\int_0^t\int_0^1\frac{\kappa}{u}\theta_x\left[\frac{\kappa}{u}\right]_u u_x\theta_t\,dx\,ds\right| \leqq \frac{\kappa_0\nu}{8\bar{u}}X + C + C\max_{Q_t}\theta^{2q+1}\,. \tag{2.68}$$

Inserting (2.63)–(2.66) and (2.68) into (2.61), using (2.59) and (2.24), we obtain

$$\frac{\kappa_0\nu}{16\bar{u}}X + \frac{\kappa_0^2}{2\bar{u}^2}Y \leqq C + C\max_{Q_t}\theta^{2q+1} \leqq C + CY^{(2q+1)/(2q+4+2r)}\,,$$

whence $X(t)+Y(t) \leqq C$ for any $t \geqq 0$. Thus we have proved the following estimate.

$$\int_0^1(1+\theta^{2q})\theta_x^2(x,t)\,dx + \int_0^t\int_0^1(1+\theta^{q+r})\theta_t^2\,dx\,ds + \max_{Q_t}\theta \leqq C \quad \forall t \geqq 0\,. \tag{2.69}$$

It follows from (2.60), (2.35), (1.10), and (1.12)–(1.13), (2.41), (2.57), (2.69) and (2.56) that

$$\int_0^t\int_0^1 \theta_{xx}^2(x,s)\,dx\,ds \leqq C + C\int_0^t \max_{[0,1]}\theta_x^2\,ds \leqq C + C\int_0^t\int_0^1 |\theta_x||\theta_{xx}|\,dx\,ds \leqq C + \frac{1}{2}\int_0^t\int_0^1 \theta_{xx}^2\,dx\,ds\,,$$

which implies

$$\int_0^t\int_0^1 \theta_{xx}^2(x,s)\,dx\,ds \leqq C \quad \text{for } t \geqq 0\,. \tag{2.70}$$

Similarly, by $(1.1)_2$, (2.45), (2.56)–(2.57) and (2.41),

$$\int_0^t\int_0^1 v_t^2(x,s)\,dx\,ds \leqq C \quad \text{for } t \geqq 0\,. \tag{2.71}$$

Now we are able to show the convergence of $\{u, v, \theta\}$ to the constant state $\{1, 0, \theta^*\}$ in $H^1(0, 1)$ as t goes to infinity. To this end we observe that by (2.45), $(1.1)_1$, (2.57), (2.69)–(2.71), and the equalities $\int_0^1 v_x v_{xt}\,dx = -\int_0^1 v_{xx} v_t\,dx$ and $\int_0^1 \theta_x\theta_{xt}\,dx = -\int_0^1 \theta_{xx}\theta_t\,dx$, we have

$$\int_0^\infty \left\{\left|\frac{d}{dt}\|u_x(t)\|^2\right| + \left|\frac{d}{dt}\|v_x(t)\|^2\right| + \left|\frac{d}{dt}\|\theta_x(t)\|^2\right|\right\}dt \leqq C\,,$$

which combined with (2.41), (2.45) and (2.56) yields

$$\|u_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_x(t)\|^2 \to 0 \quad \text{as } t \to \infty\,. \tag{2.72}$$

By virtue of (2.4) and Poincaré's inequality, $\|u(t)-1\|_{H^1} + \|v(t)\|_{H^1} \to 0$ as $t \to \infty$. So, in view of (2.72), in order to complete the proof of the convergence of solutions it remains to show that $\|\theta - \theta^*\| \to 0$ as $t \to \infty$. Recalling the definition of θ^*, we integrate $(1.1)_3$ over $[0, 1] \times [0, t]$ to get

$$\int_0^1 \left\{\left(e(u,\theta) + \frac{v^2}{2}\right)(x,t) - e(1,\theta^*)\right\}dx = 0\,, \tag{2.73}$$

which together with Poincaré's inequality, (2.35), (2.57) and (2.69) implies

$$\begin{aligned}\|(e(u,\theta) - e(1,\theta^*) + v^2/2)(t)\|^2 &\leqq C\|(e_u u_x + e_\theta\theta_x + v v_x)(t)\|^2\\ &\leqq C(\|u_x\|^2 + \|\theta_x\|^2 + \max_{[0,1]} v^2\,\|v_x\|^2)\\ &\leqq C(\|u_x(t)\|^2 + \|\theta_x(t)\|^2 + \|v_x(t)\|^2)\,. \end{aligned}\tag{2.74}$$

It follows from the mean value theorem, (1.10), (2.35), (2.69), (2.74) and (2.72) that

$$\begin{aligned}\|\theta(t) - \theta^*\|^2 &\leqq C\|e(1,\theta) - e(1,\theta^*)\|^2 \leqq C\|e(u,\theta) - e(1,\theta^*) + v^2/2\|^2\\ &\quad + C(\|e(u,\theta) - e(1,\theta)\|^2 + \|v^2\|^2)\\ &\leqq C(\|u_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_x(t)\|^2) \to 0 \end{aligned}\tag{2.75}$$

as $t \to \infty$. Thus we have proved that $\|u(t)-1\|_{H^1} + \|v(t)\|_{H^1} + \|\theta(t) - \theta^*\|_{H^1} \to 0$ as $t \to \infty$.

We have known that $u-1$, v and $\theta - \theta^*$ become small in H^1-norm for large t, thus we can apply arguments similar to those used in [18, Theorem 2.2] to obtain (1.18) in Theorem 1.1 (the exponential convergence of $\{u, v, \theta\}$ to the constant state as $t \to \infty$). This completes the proof of Theorem 1.1.

3 Proof of Theorem 1.2

We start with the following identity which follows from integration of $(1.1)_3$ over $[0,1]\times[0,t]$, and use of (1.5.b) and (1.5.c).

$$\int_0^1\left(e(u,\theta)+\frac{v^2}{2}\right)(x,t)\,dx+\lambda\int_0^t[v^2(0,s)+v^2(1,s)]\,ds=\int_0^1\left(e(u_0,\theta_0)+\frac{v_0^2}{2}(x)\right)dx=:e_0\,, \tag{3.1}$$

where $\lambda=0$ for the condition (1.5.b) and $\lambda=1$ for (1.5.c).

Lemma 3.1 *We have*

$$\int_0^t\int_0^1[v^2+up(u,\theta)](x,s)\,dx\,ds\le C\left\{\left(\int_0^1 u(x,t)\,dx\right)^n+1\right\},\quad t\ge 0\,, \tag{3.2}$$

where $n=1$ *for* (1.5.b) *and* $n=2$ *for* (1.5.c).

Proof. Integrating $(1.1)_2$ over $[0,x]$ $(x\in[0,1])$, using (1.5.b)–(1.5.c) and (1.7), and then multiplying by u, we obtain

$$\begin{aligned}u(x,t)\left[\int_0^x v(y,t)\,dy\right]_t&+[up(u,\theta)](x,t)+\lambda v(0,t)u(x,t)\\&=\mu(u)u_t(x,t)\equiv[Au(x,t)]_t\,,\end{aligned} \tag{3.3}$$

where $\lambda=0$ for (1.5.b) and $\lambda=1$ for (1.5.c), and $Au:=\int_{\min u_0(\cdot)}^u\mu(\xi)\,d\xi$. We distinguish two cases.

(i) *In the case of* (1.5.c). Recalling (1.5.c), we integrate by parts to get

$$\begin{aligned}\int_0^t\int_0^1 u\left(\int_0^x v\,dy\right)_t dx\,ds&=-\int_0^t\int_0^1 v_x\int_0^x v\,dy\,dx\,ds+\int_0^1 u\int_0^x v\,dy\,dx\Big|_0^t\\&=\int_0^t\int_0^1 v^2\,dx\,ds-\int_0^t v(1,s)\int_0^1 v\,dy\,ds+\int_0^1 u\int_0^x v\,dy\,dx\Big|_0^t\,.\end{aligned} \tag{3.4}$$

Keeping in mind that $Au\le\mu_1 u$ for $u>0$ by virtue of (1.14.b,c), we integrate (3.3) with $\lambda=1$ over $[0,1]\times[0,t]$, and use (3.4), (3.1) and Schwarz's inequality to arrive at (note $u>0$)

$$\begin{aligned}&\int_0^t\int_0^1(v^2+up(u,\theta))\,dx\,ds-\int_0^t v(1,s)\int_0^1 v\,dx\,ds+\int_0^t v(0,s)\int_0^1 u\,dx\,ds\\&\quad=\int_0^1 Au\,dx-\int_0^1 Au_0\,dx-\int_0^1 u\int_0^x v\,dy\,dx\Big|_0^t\le C\left(\int_0^1 u\,dx+1\right).\end{aligned} \tag{3.5}$$

To estimate the second term on the left hand side of (3.5), we observe by integrating $(1.1)_1$–$(1.1)_2$ and using (1.5.c) that

$$\left(\int_0^1 u(x,t)\,dx\right)_t=v(1,t)-v(0,t),\quad\left(\int_0^1 v(x,t)\,dx\right)_t=-(v(1,t)+v(0,t))\,. \tag{3.6}$$

It follows from (3.6) and Schwarz's inequality that

$$\int_0^t\int_0^1(-v(1,s)v(x,s)+v(0,s)u(x,s))\,dx\,ds=-\int_0^t\int_0^1(v(1,s)+v(0,s))v(x,s)\,dx\,ds$$
$$+\int_0^t v(0,s)\int_0^1(u+v)\,dx\,ds$$
$$=\frac{1}{2}\int_0^t\frac{d}{dt}\left\{\left(\int_0^1 v\,dx\right)^2-\frac{1}{2}\left(\int_0^1(u+v)\,dx\right)^2\right\}ds$$
$$\geqq -C-\frac{1}{2}\left(\int_0^1(u+v)\,dx\right)^2$$
$$\geqq -C-\int_0^1 v^2\,dx-\left(\int_0^1 u\,dx\right)^2. \tag{3.7}$$

Inserting (3.7) into (3.5), and taking (3.1) into account, we obtain the estimate (3.2) for (1.5.c).

(ii) *In the case of* (*1.5.b*). We note that by integrating $(1.1)_2$, $\int_0^1 v(x,t)\,dx=\int_0^1 v_0(x)\,dx$. Therefore (if necessary we take $\tilde{v}=v-\int_0^1 v_0\,dx$ as an unknown function instead of v), we may assume $\int_0^1 v(x,t)\,dx\equiv 0$. Hence, analogous to (3.4), one has

$$\int_0^t\int_0^1 u\left(\int_0^x v\,dy\right)_t dx\,ds=-\int_0^t\int_0^1 v_x\int_0^x v\,dy\,dx\,ds+\int_0^1 u\int_0^x v\,dy\,dx\Big|_0^t$$
$$\geqq\int_0^t\int_0^1 v^2\,dx\,ds-C\left(\int_0^1 u\,dx+1\right), \tag{3.8}$$

where Schwarz's inequality and (3.1) have been used. Recalling that $Au\leqq\mu_1 u$ for $u>0$, we integrate (3.3) with $\lambda=0$ and make use of (3.8) to obtain (3.2) for (1.5.b). □

To show Theorem 1.2 we first note that for $\ell=1$ in the assumption (1.11) we have by virtue of Lemma 3.1 and (2.30) that

$$C\left\{\left(\int_0^1 u(x,t)\,dx\right)^n+1\right\}\geqq\int_0^t\int_0^1 up(u,\theta)\,dx\,ds\geqq p_1 t,\quad t\geqq 0$$

with the same n as in Lemma 3.1, which implies (1.19) and (1.20) in Theorem 1.2. Moreover, for (1.5.c) we integrate $(1.1)_1$ over $[0,1]\times[0,t]$, use Schwarz's inequality and (3.1) with $\lambda=1$ to infer that $\int_0^1 u(x,t)\,dx\leqq C(1+\sqrt{t})$. Therefore, to prove Theorem 1.2 it suffices to show that (1.19) and (1.20) hold for $\ell=0$ in (1.11). We divide the proof in two steps.

(i) *In the case of* (*1.5.b*). Using (1.7), we rewrite $(1.1)_2$ as follows

$$v_t+p(u,\theta)_x=(Mu)_{xt}\,. \tag{3.9}$$

Here $Mu:=\int_{\min u_0(\cdot)}^u\mu(\xi)/\xi\,d\xi$. By virtue of (1.14.b), Mu is a strictly increasing function which maps $(0,\infty)$ onto $(-\infty,\infty)$. If we integrate (3.9) over $[0,x]\times[0,t]$, and utilise (1.5.b), the fact that $p\geqq 0$, Schwarz's inequality and (3.1) with $\lambda=0$, we infer that

$$Mu(x,t)=Mu(x,0)+\int_0^t p(x,s)\,ds+\int_0^x(v(y,t)-v_0(y))\,dy\geqq -2\sqrt{e_0}\,.$$

Hence, u is bounded from below

$$u(x,t)\geqq \underline{u}>0 \quad \text{for any } x\in[0,1] \text{ and } t\geqq 0\,, \tag{3.10}$$

where $\underline{u}\equiv\underline{u}(\min u_0, E_0, \mu_0)=M^{-1}(-2\sqrt{e_0})$ is independent of t. In view of (1.9) and (2.30) with $\ell=0$ we find that $\partial_t e(u(x,t),0)=e_u(u,0)u_t=-p(u,0)u_t=0$. So $\int_0^1 e(u(x,t),0)\,dx=\int_0^1 e(u_0(x),0)\,dx$. Denote $c_0:=\int_0^1(e(u_0,\theta_0)-e(u_0,0)+v_0^2/2)(x)\,dx>0$. Then it follows from (3.1), (3.10), the mean value theorem, (1.10) and (2.30) with $\ell=0$ that

$$\begin{aligned}c_0&=\int_0^1(e(u,\theta)+v^2/2)(x,t)\,dx-\int_0^1 e(u_0(x),0)\,dx=\int_0^1(e(u,\theta)-e(u,0)+v^2/2)(x,t)\,dx\\&\leqq C\int_0^1(\theta+\theta^{1+r}+v^2)(x,t)\,dx\leqq C\int_0^1[v^2+up(u,\theta)](x,t)\,dx \quad \text{for } t\geqq 0,\end{aligned}$$

which together with Lemma 3.1 proves (1.19).

(ii) *In the case of (1.5.c).* Similar to (2.18) we define

$$E_m(u,\theta):=\psi(u,\theta)-\psi(m,1)-\psi_u(m,1)(u-m)-(\theta-1)\psi_\theta(u,\theta)\,, \tag{3.11}$$

where $m>0$ is a constant determined later. The same procedure as used for (2.23) yields

$$E_m(u,\theta)\geqq v(\theta-\log\theta-1)\,. \tag{3.12}$$

By the same calculations as in derivation of (2.19) we obtain

$$\partial_t\left(E_m(u,\theta)+\frac{v^2}{2}\right)_t+\frac{\mu(u)v_x^2}{u\theta}+\frac{\kappa(u,\theta)\theta_x^2}{u\theta^2}=(\sigma v)_x+p(m,1)u_t+\left[\frac{\theta-1}{\theta}\frac{\kappa(u,\theta)\theta_x}{u}\right]_x. \tag{3.13}$$

Integrating (3.13) over $[0,1]\times[0,t]$, utilising (1.5.c) and (3.12), and applying Jensen's inequality to the convex function $y-\log y-1$, we deduce that

$$\int_0^1\theta(x,t)\,dx-\log\int_0^1\theta(x,t)\,dx-1\leqq\frac{p(m,1)}{v}\int_0^1 u(x,t)\,dx+C\,. \tag{3.14}$$

We prove (1.20) by contradiction. If (1.20) does not hold, then there are constants T_0, $\Lambda>0$ such that

$$\int_0^1 u(x,t)\,dx\leqq\Lambda\log t \quad \text{for any } t\geqq T_0\,. \tag{3.15}$$

Since $p(u,1)\to 0$ as $u\to\infty$, we choose m large enough such that $p(m,1)\Lambda/v\leqq 1/2$. Thus, inserting (3.15) into (3.14), one gets

$$\int_0^1\theta(x,t)\,dx-\log\int_0^1\theta(x,t)\,dx-1-\frac{1}{2}\log t-C\leqq 0, \quad \forall t\geqq T_0\,. \tag{3.16}$$

Since $x-\log x-\log t/2-1-C>0$ for $0<x<t^{-1/2}e^{-(1+C)}$ and $t\geqq T_0$, we conclude that $\int_0^1\theta(x,t)\,dx\geqq t^{-1/2}e^{-(1+C)}$ for any $t\geqq T_0$. Using this and (2.30) with $\ell=0$,

inserting (3.15) into (3.2), we see that

$$C[(\Lambda \log t)^2+1] \geqq p_1 \int_0^t \int_0^1 \theta(x,s)\,dx\,ds$$

$$\geqq p_1 e^{-(1+C)} \int_{T_0}^t s^{-1/2}\,ds = \frac{2p_1(\sqrt{t}-\sqrt{T_0})}{e^{(1+C)}} \quad \forall t \geqq T_0\,,$$

which is not true. Therefore, (1.20) holds. This completes the proof of Theorem 1.2.

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