4 Comparison with the Hilbert scheme of four points

The map $[\mathscr{F}] \to Z(\mathscr{F})$ for reflexive sheaves gives us a rational morphism from $M(0, 2, 4)$ to the Hilbert scheme $\mathrm{Hilb}^4(\mathbb{P}_3)$ of 0-dimensional subschemes of length 4 in $\mathbb{P}_3$. Using the normal forms of the last section we describe the fibres in more detail and show that the generic fibre is $\mathbb{P}_1$. Moreover, we study the subvarieties of reflexive sheaves whose singular locus of length 4 has a multiple structure.

4.1 Some facts about $H = \mathrm{Hilb}^4(\mathbb{P}_3)$

Let $H'_4 \subset H$ be the subvariety of 4-fold points p with structure sheaf $\mathcal{O}_p/\mathfrak{m}_p^2$. It is known, see [LB, I, F] that

(i) H is irreducible of dimension 12,

(ii) H'_4 is the singular locus of H.

We consider the following subvarieties of H. Let $H_{t2} \subset H_{t1}$ be the subvarieties of all $Z \in H$ such that there is a point $p \in Z$ with $\dim T_p Z \geqq 2$ (resp. $\geqq 1$), and let $H_{p\ell}$ be the subvariety of $Z \in H$ which are contained in a plane. Finally, let H_2^0, $H_{2,2}^0$, H_3^0, $H_4^0 \subset H \setminus H_{t2} \cup H_{p\ell}$ be the subvarieties with exactly one point of length 2 (resp. two points of length 2, resp. one point of length 3, resp. one point of length 4), and let H_2, $H_{2,2}$, H_3, H_4 be their closures in H.

4.1.1 Proposition. *$H'_4 \subset H_4 \subset H_3 \subset H_2$ and $H_4 \subset H_{2,2} \subset H_2$.*

Proof. The inclusions $H_3 \subset H_2$ and $H_{2,2} \subset H_2$ follow from the fact that 2- and 3-fold points can be split under deformation in their planes. $H'_4 \subset H_4$ and $H_4 \subset H_3$ is easy by using the normal forms of Remarks 3.8.2 and 3.9.2, or follow from $D'_4 \subset D_4 \subset D_3$ below.

We are left to prove $H_4 \subset H_{2,2}$. Let $Z \in H_4^0$. By Remark 3.9.2 we can assume that coordinates are chosen such that the ideal sheaf $\mathscr{I}_Z$ of Z is generated in degree 2 by the six forms as in Remark 3.9.2, and that Z is supported on p_0. It is easy to see that then also z_1^4 is a section of $\mathscr{I}_Z(4)$ and that, moreover, $\mathscr{I}_{Z,p_0}$ is generated by

$$x_2 - \lambda x_1^2 + \alpha x_1 x_2, \quad x_3 - \mu x_1 x_2, x_1^4,$$

where $x_i = z_i/z_0$ are the local coordinates, $\mu\lambda \neq 0$. Let $Z_t \subset \{z_0 \neq 0\} \subset \mathbb{P}_3$ be the family defined by the first two equations and $x_1^2(x_1 - t)^2$. This is a flat family of deformations of $Z = Z_0$ and defines a germ of a curve in H. For $t \neq 0$, Z_t consists of two double points. This proves that $H_4^0 \subset H_{2,2}$ and hence also $H_4 \subset H_{2,2}$.

4.2 The rational morphism $M \dashrightarrow H$

Let $M_r \subset M = M(0, 2, 4)$ the open subscheme of reflexive sheaves $\mathscr{F}$, and let $X \subset G_4(k^2 \otimes V^*)^{ss}$ be its inverse image. The dual of the universal homomorphism (UF) in 1.1 gives us a subscheme $\tilde{Z} \subset X \times \mathbb{P}_3$ by applying the functor Λ^2:

$$\Lambda^2(\Lambda) \boxtimes \mathcal{O}_{\mathbb{P}_3}(-2) \to \mathcal{O}_{X \times \mathbb{P}_3} \to \mathcal{O}_{\tilde{Z}} \to 0.$$

For each $\Lambda \in X$ the fibre $\tilde{Z}(\Lambda) \subset \mathbb{P}_3$ is the singular locus of the reflexive sheaf $\mathscr{F}(\Lambda)$ and hence a point of H. Moreover, $\tilde{Z}$ is flat over X. This gives us a unique morphism $X \to H$ such that $\tilde{Z}$ becomes the pullback of the universal scheme over H. This morphism is SL(2)-equivariant since Λ^2 kills the SL(2)-action and, therefore, we obtain a morphism

$$M_r \xrightarrow{h} H,$$

whose underlying map is $[\mathscr{F}] \to Z(\mathscr{F}) = \operatorname{Supp} \mathscr{E}xt^1(\mathscr{F}, \mathcal{O})$.

We let $\tilde{M} \subset M \times H$ be the closure of the graph of h and thus get two morphisms

$$M \xleftarrow{\sigma} \tilde{M} \xrightarrow{\eta} H.$$

Clearly σ is birational and an isomorphism over M_r and h maps $M \setminus S_0 \cup S_1 \cup S_2 \cup D'_4$ into $H \setminus H_{t2} \cup H_{p\ell}$ by Proposition 3.3, Lemma 3.4. It follows from Proposition 4.4 that this map is also surjective. D'_4 is mapped onto H'_4.

4.2.1 Remark. For a point $([\mathscr{F}]), Z) \in \tilde{M}$ we necessarily have $Z \subset Z(\mathscr{F})$ if $\mathscr{F}$ is stable and $Z(\mathscr{F})$ is given by the Fitting ideal of the matrix A. If $\mathscr{F}$ is only semistable and $[\mathscr{F}] = [\mathscr{I}_\ell \oplus \mathscr{I}_{\ell'}]$ then $Z \subset \ell \cup \ell'$. Note that for a non-trivial extension $0 \to \mathscr{I}_\ell \to \mathscr{F} \to \mathscr{I}_{\ell'} \to 0$ the variety $Z(\mathscr{F})$ consists of ℓ and two points or a double point on ℓ', see Remark 1.8.

4.3 Notation. Let $D_2^0, D_{2,2}^0, D_3^0, D_4^0 \subset M \setminus S_0 \cup S_1 \cup S_2 \cup D'_4$ be the inverse images of H_2^0, $H_{2,2}^0$, H_3^0, H_4^0 under h respectively and let D_2, $D_{2,2}$, D_3, D_4 be their closures in M.

4.4 Proposition. *The restriction of h induces surjective morphisms*

$$D_2^0 \to H_2^0, \quad D_{2,2}^0 \to H_{2,2}^0, \quad D_3^0 \to H_3^0, \quad D_4^0 \to H_4^0.$$

Moreover, the first and second are fibrations with fibre k^* and the third and fourth are fibrations with fibre k.

Proof. The surjectivity in the first two cases is obvious by 3.6 and 3.7 since double structures on points are determined by the tangent lines. Then the normal forms give us sheaves over a given 0-dimensional scheme in H_2^0 or $H_{2,2}^0$. In the last two cases surjectivity follows from the Remarks 3.8.2 and 3.9.2: the ideal of any Z is obtained by a normal form. The structure of a fibration follows directly from the statements in Lemmas 3.6.1, 3.7.1, 3.8.1, 3.9.1. In the case of Lemma 3.8.1 the coefficient α does not occur in the Fitting ideal and in case 3.9.1 the same is true for the coefficient β. With some more effort one should be able to verify that the fibrations are in fact locally trivial.

4.5 Theorem II. (a) *The varieties D_2, $D_{2,2}$, D_3, D_4 are all irreducible and smooth along D_2^0, $D_{2,2}^0$, D_3^0, D_4^0 respectively.*
(b) *$D'_4 \subset D_4 \subset D_3 \subset D_2$ and $D_4 \subset D_{2,2} \subset D_2$.*
(c) $\dim D_4 = 10$, $\dim D_3 = \dim D_{2,2} = 11$, $\dim D_2 = 12$.

Proof. The subvariety H_2^0 consists of schemes Z which are determined by 3 points and a line through one of them, not contained in a plane with the other two. It follows that H_2^0 is an orbit under the action of PGL(4) and has dimension 11. By Proposition 4.4 the statements (a), (c) follow for D_2. A similar argument works in the case of $D_{2,2}$. Now by Remark 3.8.2 a scheme $Z \in H_3^0$ is in 1:1 correspondence with a tuple (p, ℓ, P, q, λ), where ℓ is the tangent line of p and P the plane containing the 3-fold point p, $\lambda \neq 0$. Again we conclude that H_3^0 is smooth, irreducible of dimension $3+3+2+1+1=10$. Then D_3 satisfies (a), (c). An analogous argument applies to H_4^0 by using 3.9.2. Here P becomes the osculating plane.

In order to derive the inclusions we consider 1-parameter deformations of the normal forms: $D_4' \subset D_4$ and $D_4 \subset D_3$ follow from the families

$$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ tz_0 & z_1 & z_2 & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 & z_2 & z_3 & tz_3 \\ z_0+\alpha z_1+\beta z_2 & \lambda z_1 & \mu z_2 & z_3 \end{pmatrix}$$

which give points in D_4^0 resp. D_3^0 for $t \neq 0$.

The inclusions $D_4 \subset D_{2,2}$ and D_3, $D_{2,2} \subset D_2$ follow from the corresponding inclusions in H and the dimension of the fibres of $M_r \xrightarrow{h} H$. Let us prove this in the less obvious case $D_4 \subset D_{2,2}$. By Proposition 4.4 the fibres of h over H_4^0 and $H_{2,2}^0$ are isomorphic to k^*. Since $D_{2,2} = \overline{h^{-1}(H_{2,2}^0)}$, we consider the restriction $\alpha = h \mid D_{2,2}$. As for any dominant morphism we obtain for $D_{2,2} \xrightarrow{\alpha} H_{2,2}$: any component Y of $\alpha^{-1}(H_4^0)$ has

$$\dim Y \geqq \dim H_4^0 + \dim D_{2,2} - \dim H_{2,2} = \dim H_4^0 + 1.$$

Now $\alpha^{-1}(H_4^0) \subset D_{2,2} \cap h^{-1}(H_4^0) \subset D_{2,2} \cap D_4 \subset D_4$, and therefore we obtain

$$\dim H_4^0 + 1 \leqq \dim Y \leqq \dim D_4 = \dim H_4^0 + 1.$$

Since D_4 is irreducible, $Y = D_{2,2} \cap D_4 = D_4$, hence $D_4 \subset D_{2,2}$.

4.6 Fibres of $\tilde{M} \xrightarrow{\eta} H$

As a last part we discuss the fibres of the projection η in the different cases. It turns out that all the fibres over $H \setminus H_{t2} \cup H_{p\ell}$ are isomorphic to $\mathbb{P}_1$.

4.6.1 Case of 4 simple points. Let $Z = \{p_0, \ldots, p_3\} \in H$. Then by 3.5.1 the reflexive sheaves over Z are parametrized by $\lambda \in k \setminus \{0, 1\}$ by the normal forms $\begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & \lambda z_1 & z_2 & z_3 \end{pmatrix}$.

We consider now the morphism $\mathbb{P}_1 \to \eta^{-1}(Z) \subset \tilde{M}$ defined by $\langle \lambda, \mu \rangle \to ([\mathscr{F}_{\lambda:\mu}], Z)$ where $\mathscr{F}_{\lambda:\mu}$ is presented by

$$\begin{pmatrix} z_0 & \mu z_1 & z_2 & 0 \\ 0 & \lambda z_1 & z_2 & z_3 \end{pmatrix}$$

and Z is the given scheme. For $\langle \lambda, \mu \rangle \neq \langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 1, 1 \rangle$ clearly $\varphi \langle \lambda, \mu \rangle \in \eta^{-1}(Z) \cap M_r$ and φ is an isomorphism away from the three points.

It follows that φ is an isomorphism at all. The three extra points in $\eta^{-1}(Z) \subset M \times H$ consist of the three possible pairs of lines through the four points, namely

$$((\ell_{01}, \ell_{23}), Z), ((\ell_{02}, \ell_{13}), Z), ((\ell_{12}, \ell_{03}), Z)$$

according to the classes of matrices

$$\begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & 0 & z_2 & z_3 \end{pmatrix}, \quad \begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}, \quad \begin{pmatrix} z_0 & 0 & z_2 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}.$$

4.6.2 Case of a double and 2 simple points. Let $\operatorname{Supp} Z = \{p_0, p_2, p_3\}$ with double structure in p_0 defined by the line ℓ_{01}. In this case the isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ is defined by the normal form, see Lemma 3.6.1,

$$\begin{pmatrix} z_1 & z_2 & \mu z_3 & 0 \\ z_0 & 0 & \lambda z_3 & z_1 \end{pmatrix}.$$

For the exceptional points corresponding to $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$ we obtain

$$((\ell_{01}, \ell_{23}), Z) \quad \text{and} \quad ((\ell_{02}, \ell_{03}), Z).$$

4.6.3 Case of two double points. Let $\operatorname{Supp} Z = \{p_0, p_3\}$ with double structures in p_0 and p_3 defined by ℓ_{01} and ℓ_{23}. Again we get an isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ by the normal form, see Lemma 3.7.1,

$$\begin{pmatrix} z_1 & z_2 & \mu z_3 & 0 \\ z_0 & 0 & \lambda z_2 & z_1 \end{pmatrix}.$$

The exceptional points are now

$$((\ell_{01}, \ell_{23}), Z) \quad \text{and} \quad ((\ell_{03}, \ell_{03}), Z).$$

4.6.4 Case of a 3-fold and a simple point. Starting with the normal form Lemma 3.8.1 for $\operatorname{Supp} Z = \{p_0, p_3\}$ with triple structure in p_0, then $Z = Z(\mathscr{F}(A_{\lambda,\alpha}))$ has the ideal $\mathscr{I}_\lambda$ generated by

$$z_0 z_3, z_1 z_2, z_1 z_3, z_2 z_3, z_2^2, \lambda z_1^2 - z_0 z_2$$

which depends on λ but not on α. It follows that $\mathscr{I}_\lambda = \mathscr{I}_{\lambda'}$ if and only if $\lambda = \lambda'$. Therefore, if we fix λ we also get an isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ by using the parameter α. For $\alpha = \infty$ we get the only exceptional point $((\ell_{01}, \ell_{03}), Z)$ representing the tangent line of Z at p_0 and the line connecting p_0, p_3.

4.6.5 Case of a 4-fold point. Let $Z = Z(\mathscr{F})$ be a 4-fold point in p_0 where $\mathscr{F}$ is given by the normal form in Lemma 3.9.1. Then the ideal $\mathscr{I}_{\lambda,\mu,\alpha}$ is generated by

$$z_1 z_3, z_2 z_3, z_3^2, z_2^2, z_0 z_2 - \lambda z_1^2 + \alpha z_1 z_2, z_0 z_3 - \mu z_1 z_2$$

and does not depend on β. Again λ, μ, α is uniquely determined by $\mathscr{I}_{\lambda,\mu,\alpha}$ as it is easily verified (note that Z is not contained in a plane). Therefore, if we fix (λ, μ, α) we get an isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ by using the parameter β. For $\beta = \infty$ we get the only exceptional point $((\ell_{01}, \ell_{01}), Z)$ representing the tangent line of Z at p_0.

4.6.6 If $\mathscr{F}\in D'_4$ and reflexive, then the fibre of a singular point $Z\in H'_4$ is isomorphic to the $\mathbb{P}_5$ of all conics in the plane dual to Supp Z, see 2.4.

4.6.7 Fibres over schemes $Z\in H_{p\ell}$. Let $\tilde{S}_2$ be the "proper transform" of $S_2\subset M$, i.e. $\tilde{S}_2$ is the closure of $\tilde{S}_2^0=\sigma^{-1}(S_2\backslash S_0)$ in $\tilde{M}$. The points of $\tilde{S}_2^0$ consist of pairs $([\mathscr{F}], Z)$ where $\mathscr{F}$ has a representation $0\to\mathscr{F}\to 2\mathcal{O}\to\mathcal{O}_C(1)\to 0$ with smooth conic $C=Z(\mathscr{F})$ and $Z\subset C$. Since here $[\mathscr{F}]$ is determined by C, the fibres of the morphism $\tilde{S}_2\to H_{p\ell}$ consist of conics through fixed 4 points in a plane, and thus are again isomorphic to $\mathbb{P}_1$.

4.6.8 Remark. The fibres over points $Z\in H_{t2}$ arise from limit points in $\tilde{M}$. We omit the details for this.

In addition to the results on the subvarieties S_i in Proposition 2.3 and D_i in Theorem 4.5 we have the following proposition on the relation between them.

4.7 Proposition. $S_0\subset D_{2,2}$; $S_1, S_2\subset D_4$; $S_0\cap D_3\subset S'_0$.

It follows from the last statement and from $S'_0\subset D'_4$ that $S_0\cap D_4=S_0\cap D_3=S'_0$.

Proof. (1) For $t\neq 0$ it is easy to see that

$$\begin{pmatrix} z_1 & z_2 & z_3 & 0\\ z_0 & 0 & t^2 z_2 & z_1\end{pmatrix}\sim\begin{pmatrix} tz_1 & z_2 & z_3 & 0\\ z_0 & 0 & tz_2 & z_1\end{pmatrix}$$

and that the matrices represent a sheaf $\mathscr{F}_t$ in $D^0_{2,2}$. For $t=0$ we obtain a given sheaf $\mathscr{F}_0=\mathscr{I}_{\ell_{01}}\oplus\mathscr{I}_{\ell_{23}}$ in $S_0\backslash S'_0$. This proves $S_0\subset D_{2,2}$.
(2) A sheaf $\mathscr{F}_0$ in $S_1\backslash S_0$ resp. $S_2\backslash S_0$ can be represented in normal form by

$$\begin{pmatrix} z_0 & z_2 & z_3 & 0\\ z_1 & 0 & z_2 & z_3\end{pmatrix}\ \text{resp.}\ \begin{pmatrix} z_0 & z_1 & z_3 & 0\\ z_1 & z_2 & 0 & z_3\end{pmatrix}.$$

The 1-parameter families

$$\begin{pmatrix} z_0 & z_2 & z_3 & tz_1\\ z_1 & 0 & z_2 & z_3\end{pmatrix}\ \text{resp.}\ \begin{pmatrix} z_0 & z_1 & z_3 & 0\\ z_1 & z_2 & tz_2 & z_3\end{pmatrix}$$

show that in each case $\mathscr{F}_0$ can be deformed into a reflexive sheaf $\mathscr{F}_t$ with a 4-fold point. This proves the second statement.
(3) If $m=[\mathscr{I}_\ell\oplus\mathscr{I}_{\ell'}]\in S_0\cap D_3$ we can find a 1-parameter family A_t of matrices such that A_0 represents m and $\mathscr{F}(A_t)\in D^0_3$ for $t\neq 0$, i.e. $Z(\mathscr{F}(A_t))$ has a curvi-linear 3-fold structure in one of its points. Let $Z_0\in H_3$ be the limit in H, so that we get a point $(m, Z_0)\in\tilde{M}$, in particular $Z_0\subset\ell\cup\ell'$, see Remark 4.2.1. We distinguish the following cases. If $Z_0\notin H_{t2}\cap H_{p\ell}$, then $Z_0\in H^0_3$ or H^0_4. In both cases the lines ℓ and ℓ' of m meet by 4.6.4 and 4.6.5. If $Z_0\in H_{t2}$ then Z_0 contains a point p with $\dim T_pZ_0\geqq 2$, and since $Z_0\subset\ell\cup\ell'$ the two lines must meet in p. If, finally, $Z_0\in H_{p\ell}$ then $m\in S_2$ by 4.6.7, and again the lines meet. This proves the last statement of the proposition.