Math. Z. 216, 147–155 (1994)

Mathematische Zeitschrift

Stabilization of solutions of a nonlinear parabolic equation with a gradient term

Keng Deng

Department of Mathematics, University of Southwestern Louisiana, Lafayette, LA 70504-1010, USA

Received 29 October 1992; in final form 19 February 1993

1 Introduction

In this paper, we investigate the large time behavior of nonnegative solutions of the initial-boundary value problem:

$$\text{(P)} \qquad \begin{aligned} u_t &= u_{xx} - u_x^2 + \lambda u^p & -1<x<1, \quad & t>0, \\ u(-1,t) &= u(1,t) = 0 & & t>0, \\ u(x,0) &= u_0(x) & -1 \leqq x \leqq 1, & \end{aligned}$$

here $\lambda > 0$, $p > 1$, and $u_0(x)$ is a nonnegative prescribed function. Problem (P) is an one-dimensional case for a model dealt with in [5], where Kawohl and Peletier established a relationship between (P) and a so-called dead core problem, and they proved that if $1 < p \leqq 2$, for each $\lambda > 0$, every solution is uniformly bounded; while for $p > 2$ with any given initial data, there exists a $\lambda^* > 0$ such that if $\lambda > \lambda^*$, u blows up in finite time.

Their work was motivated by a recent study of Chipot and Weissler [2], wherein the following problem was considered:

$$(\mathrm{P}_0) \qquad \begin{aligned} u_t = \Delta u - |\nabla u|^q + |u|^{p-1} u \quad & x \in \Omega, \quad & t>0, \\ u(x,t) = 0 \quad & x \in \partial\Omega, \quad & t>0, \\ u(x,0) = \phi(x) \quad & x \in \Omega, & \end{aligned}$$

with $1 < p < (n+2)/(n-2)$ and $1 < q \leqq 2p/(p+1)$. For sufficiently large $\phi(x)$, it was shown that a solution to (P_0) must blow up in finite time. Because the upper bound for q is arbitrarily close to 2 for large values of p, the consideration for $q = 2$ in [5] is a natural consequence, and not surprisingly, whether solutions exist globally relies on the balance between the power of the damping term and that of the source nonlinearity.

However, in contrast to the results in [5], our main objectives here are to obtain the bifurcation diagrams for the stationary states of (P), to determine the stability properties of these states, and to discuss the large time behavior of solutions of (P). It should be pointed out that the stability analysis for the case $1<p\leqq 2$ has already been made by Schaaf [7], but the argument is somewhat different from ours. It is also interesting to note that for $p>2$ with sufficiently large λ but small initial data the solution of (P) tends to zero instead of blowing up and with certain λ blow up in infinite time can occur.

By means of the transformation, $v=1-e^{-u}$, we have an equivalent problem to (P) as follows:

$$\text{(P')}\qquad \begin{aligned} v_t&=v_{xx}+\lambda(1-v)[-\ln(1-v)]^p && -1<x<1, \quad t>0,\\ v(-1,t)&=v(1,t)=0 && t>0,\\ v(x,0)&=v_0(x)=1-e^{-u_0(x)} && -1\leqq x\leqq 1. \end{aligned}$$

This time, without the presence of the gradient term, it is more convenient to conduct our discussions. Therefore, in the sequel, we shall mainly concentrate on (P′).

2 Stationary solutions

We begin with the study of classical stationary solutions. For simplicity, let $f(w)=(1-w)\,[-\ln(1-w)]^p$ for $0\leqq w<1$. A stationary solution $w(x)$ to P′ satisfies

$$\text{(S')}\qquad \begin{aligned} &w''(x)+\lambda f(w(x))=0 \quad -1<x<1,\\ &w(-1)=w(1)=0. \end{aligned}$$

Clearly $w(x)\equiv 0$ is always a solution of problem (S′). We are more concerned with positive solutions of (S′), that is, $0<w(x)<1$ on $(-1,1)$. From the equation in (S′), it follows that $w''(x)<0$ and thus $w(x)$ can attain one maximum at some $\xi\in(-1,1)$.

Set $F(w)=\int_0^w f(s)\,ds$. Then $w(x)$ also solves

$$\tfrac{1}{2}(w'(x))^2+\lambda F(w(x))=\lambda F(\mu), \tag{2.1}$$

where $\mu=w(\xi)$.

Integrating (2.1) leads to

$$\frac{1}{\sqrt{2}}\int_w^\mu (F(\mu)-F(\eta))^{-1/2}\,d\eta=\sqrt{\lambda}\,|\xi-x|. \tag{2.2}$$

Then substituting the homogeneous boundary conditions for $w(x)$ in (2.2) yields

$$\frac{1}{\sqrt{2}}\int_0^\mu (F(\mu)-F(\eta))^{-1/2}\,d\eta=\sqrt{\lambda}(1+\xi)=\sqrt{\lambda}(1-\xi). \tag{2.3}$$

Thus we find that $\xi=0$, which follows also from [4].

Let $G(\mu)=\frac{1}{\sqrt{2}}\int_0^{\mu}(F(\mu)-F(\eta))^{-1/2}\,d\eta$. Then (2.3) is equivalent to

$$G(\mu)=\sqrt{\lambda}. \tag{2.4}$$

From the above discussion, we conclude that $w(x)$ is a positive solution of (S′) if and only if for $-1<x<1$

$$\frac{1}{\sqrt{2}}\int_{w(x)}^{\mu}(F(\mu)-F(\eta))^{-1/2}\,d\eta=\sqrt{\lambda}|x| \tag{2.5}$$

with μ determined by (2.4).

Therefore in order to establish the characterization of the set of stationary solutions, we should only count the number of μ in (2.4).

First we note that although the integral in (2.4) is improper, $G(\mu)$ is continuous for $0<\mu<1$ since $F(\mu)-F(\eta)\geqq\delta(\mu-\eta)$ for some $\delta>0$ and η near μ.

Because $G(\mu)$ cannot be solved explicitly, we shall follow closely the argument in [8] by Smoller and Wasserman, where they studied the bifurcation of stationary solutions of a problem similar to (P′) but with the nonlinearity $f(v)=(v-a)(b-v)(v-c)(a<b<c)$. The same kind of idea has been applied to the work by Aronson et al. for a porous medium problem [1] and to that by the author on a singular plasma type equation [3]. It is worth mentioning that discussions in this paper will become more complicated, since for any $p>1$, our $F(v)$ can only be represented implicitly, whereas in [8] some properties were proved by elementary calculations using the explicit formula for F.

Lemma 2.1 *$G(\mu)$ is continuously differentiable on $(0,1)$ and there exist μ_1 and μ_2 with $0<\mu_1<\mu_2<1$ such that $G'(\mu)<0$ on $(0,\mu_1)$ and $G'(\mu)>0$ on $(\mu_2,1)$.*

Proof. By the change of variable $\eta=\mu\tau$, $G(\mu)$ is rewritten in

$$G(\mu)=\frac{\mu}{\sqrt{2}}\int_0^1(F(\mu)-F(\mu\tau))^{-1/2}\,d\tau. \tag{2.6}$$

Upon a formal differentiation on $G(\mu)$, we find that

$$\begin{aligned} G'(\mu)&=\frac{1}{\mu}G(\mu)-\frac{1}{2\sqrt{2}}\int_0^1\frac{\mu F'(\mu)-\mu\tau F'(\mu\tau)}{(F(\mu)-F(\mu\tau))^{\frac{3}{2}}}\,d\tau \\ &=\frac{1}{\mu}G(\mu)-\frac{1}{2\sqrt{2}\mu}\int_0^{\mu}\frac{\mu F'(\mu)-\eta F'(\eta)}{(F(\mu)-F(\eta))^{\frac{3}{2}}}\,d\eta \\ &=\frac{1}{2\sqrt{2}\mu}\int_0^{\mu}[(2F(\mu)-\mu f(\mu))-(2F(\eta)-\eta f(\eta))][F(\mu)-F(\eta)]^{-3/2}\,d\eta. \end{aligned} \tag{2.7}$$

Set $H(\zeta)=2F(\zeta)-\zeta f(\zeta)$. Then we have

$$G'(\mu)=\frac{1}{2\sqrt{2}\mu}\int_0^{\mu}\frac{H(\mu)-H(\eta)}{(F(\mu)-F(\eta))^{\frac{3}{2}}}\,d\eta. \tag{2.8}$$

A straightforward computation shows

$$H'(\zeta)=f(\zeta)-\zeta f'(\zeta) \tag{2.9}$$
$$=(-\ln(1-\zeta))^{p-1}(-\ln(1-\zeta)-p\zeta).$$

For any η close to μ, by the Cauchy mean value theorem, there is a $c(\eta<c<\mu)$

$$\left|\frac{H(\mu)-H(\eta)}{F(\mu)-F(\eta)}\right|=\left|\frac{-\ln(1-c)-pc}{(1-c)(-\ln(1-c))}\right|<\infty.$$

Thus the integral in (2.8) is convergent, and $G'(\mu)\in C(0,1)$.

Moreover, from the fact that $H'(\zeta)$ has exactly one zero μ_1 on $(0,1)$, it follows that $H'(\zeta)<0$ on $(0,\mu_1)$ and $H'(\zeta)>0$ on $(\mu_1,1)$. Since $H(0)=0$ and $H(1)>0$, there exists a $\mu_2(>\mu_1)$ such that $H(\zeta)<0$ on $(0,\mu_2)$ and $H(\zeta)>0$ on $(\mu_2,1)$. The proof is complete.

Lemma 2.2 *$G'(\mu)$ has a unique root on* $(0,1)$.

Proof. Lemma 2.1 implies that $G'(\mu)$ has at least one zero $\bar{\mu}$ between μ_1 and μ_2. We will show that $G'(\mu)$ has at most one zero. For this purpose, we formally differentiate $G'(\mu)$ to obtain

$$G''(\mu)=\frac{1}{2\sqrt{2}\mu^2}\int_0^{\mu}\frac{\mu H'(\mu)-\eta H'(\eta)}{(F(\mu)-F(\eta))^{\frac{3}{2}}}d\eta \tag{2.10}$$
$$+\frac{3}{4\sqrt{2}\mu^2}\int_0^{\mu}\frac{(H(\mu)-H(\eta))(\eta f(\eta)-\mu f(\mu))}{(F(\mu)-F(\eta))^{\frac{5}{2}}}d\eta$$
$$=G_1(\mu)+G_2(\mu).$$

The feasibility of (2.10) can be verified as in Lemma 2.1, and hence we only focus on the sign of $G''(\mu)$ for $\mu>\mu_1$.

For $G_1(\mu)$, if $\zeta>\mu_1$, we calculate $\frac{d}{d\zeta}(\zeta H'(\zeta))$ to have

$$\frac{d}{d\zeta}(\zeta H'(\zeta))=(-\ln(1-\zeta))^{p-2}\left[-\frac{p\zeta\ln(1-\zeta)}{1-\zeta}-\frac{p(p-1)\zeta^2}{1-\zeta}\right. \tag{2.11}$$
$$\left.+(-\ln(1-\zeta))^2+2p\zeta\ln(1-\zeta)\right]$$
$$=(-\ln(1-\zeta))^{p-2}\left\{\left[\frac{p\zeta}{1-\zeta}-\ln(1-\zeta)\right](-\ln(1-\zeta)-p\zeta)\right.$$
$$\left.+p\zeta\left[\frac{\zeta}{1-\zeta}+\ln(1-\zeta)\right]\right\}$$
$$>0,$$

since $-\ln(1-\zeta)>p\zeta$ when $\zeta>\mu_1$ and $\frac{\zeta}{1-\zeta}+\ln(1-\zeta)>0$ if $0<\zeta<1$.

Thus for $\mu_1 < \mu < 1$,

$$G_1(\mu) = \frac{1}{2\sqrt{2}\mu^2}\left\{\int_0^{\mu_1} + \int_{\mu_1}^{\mu}\right\} \frac{\mu H'(\mu) - \eta H'(\eta)}{(F(\mu) - F(\eta))^{\frac{3}{2}}} d\eta > 0.$$

Then for $G_2(\mu)$, after rewriting it, we find

$$G_2(\mu) = \frac{3}{4\sqrt{2}\mu^2} \int_0^{\mu} \frac{(H(\mu) - H(\eta))^2}{(F(\mu) - F(\eta))^{\frac{5}{2}}} d\eta - \frac{3}{\mu} G'(\mu).$$

Hence at any critical point of $G(\mu)$, $G_2(\mu) > 0$.

From the above, we rule out the possibility that there are more than one zero of $G'(\mu)$.

Lemmas 2.1 and 2.2 show that $G(\mu)$ first decreases and then increases. To study the behavior of $G(\mu)$ near the two endpoints, we notice that for small μ, $(F(\mu) - F(\eta))^{-1/2} = (f(\sigma))^{-1/2}(\mu - \eta)^{-1/2}$ with $\eta < \sigma < \mu$. Thus $G(\mu) \to +\infty$ as $\mu \to 0^+$. On the other hand, since

$$\lim_{\eta \to 1^-} \frac{(1-\eta)^2(-\ln(1-\eta))^p}{F(1) - F(\eta)} = 2, \tag{2.12}$$

$$(F(1) - F(\eta))^{-1/2} = O((1-\eta)^{-1}(-\ln(1-\eta))^{-p/2}) \quad \text{as} \quad \eta \to 1^-. \tag{2.13}$$

Hence $\lim\limits_{\eta \to 1^-} G(\mu) = +\infty$ if $1 < p \leqq 2$ and $\lim\limits_{\eta \to 1^-} G(\mu) < \infty$ for $p > 2$.

To sum up, we state the following:

Theorem 2.3 *For $1 < p \leqq 2$, there exists a critical number $\lambda(p)$ such that*
(i) *If $0 < \lambda < \lambda(p)$, there are none positive stationary solutions of* (P);
(ii) *If $\lambda = \lambda(p)$, there is a unique positive stationary solution*;
(iii) *If $\lambda > \lambda(p)$, there are two positive stationary solutions.*

Theorem 2.4 *For the case $p > 2$, there are two positive numbers $\lambda_1(p)$ and $\lambda_2(p)$ $(\lambda_1 < \lambda_2)$ such that*
(i) *If $\lambda < \lambda_1(p)$, there are no positive stationary solutions of* (P);
(ii) *If $\lambda = \lambda_1(p)$ or $\lambda \geqq \lambda_2(p)$, there is only one positive stationary solution*;
(iii) *If $\lambda_1(p) < \lambda < \lambda_2(p)$, there are two positive stationary solutions.*

Let $s(x)$, or sometimes $s(x, \lambda)$, to denote the stationary solution of (P). For some λ if there are two positive solutions, we note that they are ordered, and $s_+(x)$ will be written for the larger solution while $s_-(x)$ for the smaller one. Since $s(x, \lambda)$ depends on $s(0, \lambda)$ continuously and $s(0, \lambda)$ is a continuous function of λ, $s(x, \lambda)$ is a continuous function of λ.

Finally we turn our attention to the existence of singular stationary solutions s_s of (P), i.e. solutions in $C^2((-1, 1)\backslash\{0\})$ with $\lim\limits_{x \to 0} s_s(x) = \infty$. The corresponding solutions of (S′) should satisfy

$$\frac{1}{\sqrt{2}} \int_{w_s(x)}^{1} (F(1) - F(\eta))^{-1/2} d\eta = \sqrt{\lambda}|x| \tag{2.14}$$

with

$$\frac{1}{\sqrt{2}}\int_0^1 (F(1)-F(\eta))^{-1/2}\,d\eta=\sqrt{\lambda}. \tag{2.15}$$

Taking (2.13) into account we conclude

Theorem 2.5 *If* $1<p\leqq 2$ (P) *has no singular stationary solutions. For* $p>2$ *there is exactly one when* $\lambda=\lambda_2(p)$.

Remark. In [2] the authors mentioned without proof that if $q>2p/(p+1)$, there can exist singular solutions. Here we show that such claim need not be valid for all $p>1$. Furthermore, our result contrasts sharply with that in [6] where Levine proved that for (P) with the nonlinearity u^p replaced by $e^{\alpha u}$ there is one singular solution if $1<\alpha<2$ while none for $\alpha\geqq 2$.

3 Stability and large time behavior

At the beginning of this section, we recall the result in [9], a solution of problem (P) remaining uniformly bounded must converge to its steady state. We first establish stability-instability results. Without making any confusion, sometimes we shall write the solution of (P) as $u(x,t,\lambda)$ or $u(x,t;u_0)$ with u_0 being the initial datum.

We begin by formulating the precise notion of stability.

Definition. A stationary solution, $s(x)$, of (P) is stable from above if for any given $\varepsilon>0$ there exists a positive function $z(x)\in C_0^2(-1,1)$ such that if $u(x,t)$ is a nonnegative solution of (P) with $s(x)\leqq u(x,0)\leqq z(x)$ on $(-1,1)$, then $\|u(\cdot,t)-s(\cdot)\|_\infty<\varepsilon$ for all $t\geqq 0$, and $\lim_{t\to\infty} u(x,t)=s(x)$ for each $x\in[-1,1]$.

The stability from below can be defined analogously, and we say that a stationary solution $s(x)$ is stable if it is both stable from above and below.

Theorem 3.1 *Suppose that* $1<p\leqq 2$ *and* $\lambda>\lambda(p)$ *or* $p>2$ *and* $\lambda_1(p)<\lambda<\lambda_2(p)$. *Then* $s_+(x)$ *is stable whereas* $s_-(x)$ *is unstable. If* $p>2$ *and* $\lambda\geqq\lambda_2(p)$, *the unique positive stationary solution is unstable.*

Proof. We shall prove it only for $1<p\leqq 2$, the other case can be argued in a similar manner.

We first show that $s_+(x,\lambda)$ is an increasing function of λ. To this end, let $\lambda(p)<\lambda'<\lambda''$ and $w_+(x,\lambda)$ be the corresponding solution of (S′). Combining (2.3) and (2.5) produces

$$\frac{1}{\sqrt{2}}\int_0^{w(x)} (F(\mu)-F(\eta))^{-1/2}\,d\eta=\sqrt{\lambda}(1-|x|).$$

Thus

$$\frac{1}{\sqrt{2}}\int_0^{w_+(x,\lambda'')} (F(\mu(\lambda''))-F(\eta))^{-1/2}\,d\eta=\sqrt{\lambda''}(1-|x|)$$

$$>\sqrt{\lambda'}(1-|x|)=\frac{1}{\sqrt{2}}\int_0^{w_+(x,\lambda')} (F(\mu(\lambda'))-F(\eta))^{-1/2}\,d\eta,$$

since $F(\mu(\lambda''))>F(\mu(\lambda'))$,

$$\int_{w_+(x,\lambda')}^{w_+(x,\lambda'')} (F(\mu(\lambda'')-F(\mu(\eta))^{-1/2}\,d\eta>0,$$

which implies that $w_+(x,\lambda'')>w_+(x,\lambda')$, and consequently, $s_+(x,\lambda'')>s_+(x,\lambda')$ on $(-1,1)$.

We now prove the stability of $s_+(x)$. Let $u(x,t,\lambda')$ be a solution of (P) with $u_0(x)=s_+(x,\lambda'')$. By the definition for stability, it suffices to show that $u(x,t,\lambda')$ converges to $s_+(x,\lambda')$. Turning our attention to problem (P′) we then have that on $(-1,1)$

$$\begin{aligned} v_t(x,0,\lambda') &= v_0''+\lambda' f(v_0)\\ &= w_{+xx}(x,\lambda'')+\lambda' f(w_+(x,\lambda''))\\ &< w_{+xx}(x,\lambda'')+\lambda'' f(w_+(x,\lambda''))\\ &=0. \end{aligned}$$

Since v_t satisfies a linear problem with homogeneous boundary conditions, $v_t\leqq 0$ on $(-1,1)\times\{t>0\}$, and so is u_t. Moreover, by the comparison principle

$$s_+(x,\lambda')\leqq u(x,t,\lambda')\leqq s_+(x,\lambda'').$$

Noticing the opening remark of this section, we can see that $\phi(x,\lambda') = \lim_{t\to\infty} u(x,t,\lambda')$ is a stationary solution of (P) bounded by $s_+(x,\lambda')$ and $s_+(x,\lambda'')$, and in particular that $s_+(0,\lambda')\leqq\phi(0,\lambda')\leqq s_+(0,\lambda'')$. But ϕ is a steady state of (P) with λ', and so $\phi(x,\lambda')$ is either $s_+(x,\lambda')$ or else the other solution $s_-(x,\lambda')$. From the graph of $G(\mu)=\sqrt{\lambda}$, it follows that $s_-(0,\lambda')<s_+(0,\lambda')$, which excludes the possibility $\phi(x,\lambda')=s_-(x,\lambda')$. We thus show that $s_+(x,\lambda')$ is stable from above. From $\lambda'>\lambda''$, using a similar argument we can also prove that $s_+(x,\lambda')$ is stable from below.

Then for $s_-(x,\lambda)$, we know that $s_-(x,\lambda'')<s_-(x,\lambda')$ in $[-\delta,\delta]$ for $0<\lambda'<\lambda''<\lambda(p)$. Let $u(x,t,\lambda'')$ be a solution of (P) with $u_0(x)=s_-(x,\lambda')$. Then by similar reasoning, we have $u_t\geqq 0$ on $(-1,1)\times\{t>0\}$. Therefore $u(x,t,\lambda'')$ is increasing in t, which indicates that $s_-(x,\lambda'')$ is unstable from above. Similarly, it can be shown that $s_-(x,\lambda'')$ is unstable from below. □

Next we study the large time behaviour of solutions of (P).

Theorem 3.2 *Let* $1<p\leqq 2$.

(i) *For* $0<\lambda<\lambda(p)$, *every solution tends to zero as* $t\to\infty$.

(ii) *For* $\lambda=\lambda(p)$, *if* $0\leqq u_0(x)<s(x,\lambda(p))$ *on* $(-1,1)$, *then* $\lim_{t\to\infty} u(x,t)=0$; *while* $\lim_{t\to\infty} u(x,t)=s(x,\lambda(p))$ *if* $u_0\geqq s(x,\lambda(p))$.

(iii) *For* $\lambda>\lambda(p)$, *if* $0\leqq u_0(x)<s_-(x,\lambda)$, *then u converges to zero, but u approaches* $s_+(x,\lambda)$ *if* $u_0(x)>s_-(x,\lambda)$.

Proof. (i) Recalling Theorem 2 of [5], $\sup\{u(x,t)|-1\leqq x\leqq 1, t\geqq 0\}<\infty$, since there are no positive stationary solutions of (P), the assertion holds.

(ii) For $u_0(x)<s(x,\lambda(p))$, we choose a $\gamma(\gamma>\lambda(p))$ so close to $\lambda(p)$ that $u_0(x)\leqq\bar{u}_0(x)=s_-(x,\gamma)$, then by comparison, $u(x,t;u_0)\leqq u(x,t;\bar{u}_0)$. Since $u_t(x,t;\bar{u}_0)$

$\leqq 0$, $\lim_{t\to\infty} u(x,t;\bar{u}_0)=0$, and it follows that $\lim_{t\to\infty} u(x,t;u_0)=0$. If $s(x,\lambda(p))\leqq u_0(x)$, $s(x,\lambda(p))\leqq u(x,t)<\infty$, hence $\lim_{t\to\infty} u(x,t)=s(x,\lambda(p))$.

(iii) Proof for the case $u_0(x)<s_-(x,\lambda)$ is similar to that in (ii). However, if $u_0(x)>s_-(x,\lambda)$, we can find a $\gamma(\gamma<\lambda)$ such that $u_0(x)\geqq\bar{u}_0(x)=s_-(x,\gamma)$, then $u(x,t;u_0)\geqq u(x,t;\bar{u}_0)$. Because $u_t(x,t;\bar{u}_0)\geqq 0$ and $u(0,t;\bar{u}_0)>s_-(0,\lambda)$, $\lim_{t\to\infty} u(x,t;\bar{u}_0)=s_+(x,\lambda)$, thus $\lim_{t\to\infty} u(x,t;u_0)=s_+(x,\lambda)$.

Theorem 3.3 *Suppose $p>2$.*

(i) *For $0<\lambda<\lambda_1(p)$, if $0\leqq u_0(x)<s_s(x,\lambda_2(p))$ on $(-1,1)$, the solution tends to zero as $t\to\infty$.*

(ii) *For $\lambda=\lambda_1(p)$, if $0\leqq u_0(x)<s(x,\lambda_1(p))$, $\lim_{t\to\infty} u(x,t)=0$; whereas $\lim_{t\to\infty} u(x,t)=s(x,\lambda_1(p))$ if $s(x,\lambda_1(p)\leqq u_0(x)<s_s(x,\lambda_2(p))$.*

(iii) *For $\lambda_1(p)<\lambda<\lambda_2(p)$, if $0\leqq u_0(x)<s_-(x,\lambda)$, $\lim_{t\to\infty} u(x,t)=0$; while $\lim_{t\to\infty} u(x,t)=s_+(x,\lambda)$ if $s_-(x,\lambda)<u_0(x)<s_s(x,\lambda_2(p))$.*

(iv) *For $\lambda=\lambda_2(p)$, if $0\leqq u_0(x)<s(x,\lambda_2(p))$, $\lim_{t\to\infty} u(x,t)=0$; if $s(x,\lambda_2(p))<u_0(x)<s_s(x,\lambda_2(p))$, the solution blows up in infinite time and $\lim_{t\to\infty} u(x,t)=s_s(x,\lambda_2(p))$.*

(v) *For $\lambda>\lambda_2(p)$, if $0\leqq u_0(x)<s(x,\lambda)$, u approaches zero as $t\to\infty$; if $u_0(x)>s(x,\lambda)$, u blows up in finite time.*

Proof. (i) Since $u_0(x)<s_s(x,\lambda_2(p))$, there is a $\lambda_0(\lambda_1(p)<\lambda_0<\lambda_2(p))$ such that $u_0(x)\leqq\bar{u}_0=s_+(x,\lambda_0)$. Then by the comparison theorem, $u(x,t;u_0)\leqq u(x,t;\bar{u}_0)$. Because $u_t(x,t;\bar{u}_0)\leqq 0$, $u(x,t;\bar{u}_0)$ tends to the null stationary solution of (P), and the same is true for $u(x,t;u_0)$.

(ii) Proof for the case $u_0(x)<s(x,\lambda_1(p))$ is the same like that in (ii) of Theorem 3.2. If $s(x,\lambda_1(p))\leqq u_0<s_s(x,\lambda_2(p))$, a number $\sigma(\lambda_1(p)<\sigma<\lambda_2(p))$ can be determined to make $u_0(x)\leqq\tilde{u}_0(x)=s_+(x,\sigma)$. As a consequence, $\lim_{t\to\infty} u(x,t;u_0)=\lim_{t\to\infty} u(x,t;\tilde{u}_0)=s(x,\lambda_1(p))$.

(iii) Proof for the case $u_0(x)<s_-(x,\lambda)$ is omitted If $s_-(x,\lambda)<u_0(x)<s_s(x,\lambda_2(p))$, there are a $\gamma(\gamma<\lambda)$ and a $\sigma(\sigma>\lambda)$ such that $\bar{u}_0(x)=s_-(x,\gamma)\leqq u_0(x)\leqq\tilde{u}_0(x)=s_+(x,\sigma)$. Then, by the comparison principle,

$$u(x,t;\bar{u}_0)\leqq u(x,t;u_0)\leqq u(x,t;\tilde{u}_0).$$

On the other hand, $u(x,t;\bar{u}_0)$ is monotonically increasing while $u(x,t;\tilde{u}_0)$ is monotonically decreasing. Hence, $\lim_{t\to\infty} u(x,t;\bar{u}_0)=\lim_{t\to\infty} u(x,t;\tilde{u}_0)=s_+(x,\lambda)$, the conclusion follows.

(iv) For the case $s(x,\lambda_2(p))<u_0(x)$, we can select a $\gamma(\gamma<\lambda_2(p))$ to ensure $u_0(x)\geqq\bar{u}_0(x)=s_-(x,\gamma)$. Then $u(x,t;u_0)$ is bounded from below by $u(x,t;\bar{u}_0)$. Since $u_t(x,t;\bar{u}_0)\geqq 0$, $u(x,t;\bar{u}_0)$ blows up in infinite time, and so does $u(x,t;u_0)$.

(v) If $u_0(x)>s(x,\lambda)$, set $\bar{u}_0(x)=s(x,\gamma)$ with $\gamma<\lambda$ such that $u_0(x)\geqq\bar{u}_0(x)$. Then $u_t(x,t;\bar{u}_0)\geqq 0$ and $u(x,t;u_0)\geqq u(x,t;\bar{u}_0)$. Because $u(0,t;\bar{u}_0)\geqq\bar{u}_0(0)>s(0,\lambda)$, $u(x,t;\bar{u}_0)$ must blow up in finite time, which implies that $u(x,t;u_0)$ can only exist locally.

Remark 3.1 In view of Theorem 4 in [5], we can treat Theorem 3.3(v) as its counterpart.

Remark 3.2 By the strong maximum principle, Theorems 3.2 and 3.3 remain valid if each "$<(>)$" in the conditions on the initial data is replaced by "$\leqq(\geqq)$".

Acknowledgement. The author would like to thank the referee for several helpful comments.

References

1. Aronson, D., Crandall, M.G., Peletier, L.A.: Stabilization of solutions of a degenerate nonlinear diffusion problem. Nonlinear Anal., Theory Methods Appl. **6**, 1001–1022 (1982)
2. Chipot, M., Weissler, F.B.: Some blowup results for a nonlinear parabolic equation with a gradient term. SIAM J. Math. Anal. **20**, 886–907 (1989)
3. Deng, K.: Quenching for solutions of a plasma type equation. Nonlinear Anal., Theory Methods Appl. **18**, 731–742 (1992)
4. Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Commun. Math. Phys. **68**, 209–243 (1979)
5. Kawohl, B., Peletier, L.A.: Observation on blow-up and dead cores for nonlinear parabolic equations. Math. Z. **202**, 207–217 (1989)
6. Levine, H.A.: Quenching, nonquenching, and beyond quenching for solutions of some parabolic equations. Ann. Mat. Pura Appl. **155**, 243–260 (1989)
7. Schaaf, R.: Global solution branches of two point boundary value problems. (Lect. Notes Math., vol. 1458) Berlin Heidelberg New York: Springer 1990
8. Smoller J., Wasserman, J.: Global bifurcation of steady-state solutions. J. Differ. Equations **39**, 269–290 (1981)
9. Zelenyak, T.I.: Stabilisation of solutions of boundary value problems for a second-order equation with one space variable. Differ. Equations **4**, 17–22 (1968)