the essential uniqueness of development, is equivalent to the standard inclusion map which misses exactly one point. Let $\tilde{f}_1$ be $\pi \circ \tilde{f}$ where $\pi$ is the covering projection from $\mathbf{S}^n$ to $N$. But $\mathbf{E}^n$ is an infinite covering space of $M$ whereas $\mathbf{S}^n$ is a finite covering space of $N$. This leads to a contradiction since one way we can conclude that for any point $y$ in $\operatorname{im} f$ the number of $\tilde{f}_1$-preimages of $y$ is infinite whereas the other way it is finite.

Next suppose $N$ is hyperbolic. By passing to coverings we may take $M$ and $N$ to be developable. So $f$ extends to $\bar{f}$: $\bar{M} \to \bar{N}$. Now $\partial_0 M$ has only one point say $a$, and $\partial_0 N$ has at least two distinct points say $b$ and $c$. Take two disjoint arcs $\gamma_b$ and $\gamma_c$ starting from a point in $\operatorname{im} f$ and tending towards $b$ and $c$ respectively. Then the maximal partial lifts of $\gamma_b$ and $\gamma_c$ in $M$ must tend towards $a$. But then $\bar{f}(a)$ would belong to both $\gamma_b \cup \{b\}$ and $\gamma_c \cup \{c\}$. This is not possible. q.e.d.

*(2.11)* We end this section by noting some relationships among the ideal boundaries of the developable quotient spaces of a single simply connected Möbius manifold. Let $M$ be a simply connected Möbius manifold, and fix a development map dev: $M^n \to \mathbf{S}^n$ and $\rho$: $\operatorname{Aut}(M, \sigma) \to \mathcal{M}(n)$ the holonomy homomorphism which is equivariant with respect to dev. Let $\kappa$ be the kernel of $\rho$. It is easy to see that $\kappa$ is discrete (with respect to compact-open topology) and acts freely on $M$. Let $M_\kappa$ be the quotient of $M$ by $\kappa$. It is easy to see that all developable quotient spaces of $M$ are covering spaces of $M_\kappa$ and they form a lattice in the usual manner inversely isomorphic to the lattice of subgroups of $\kappa$.

Clearly the components of $\partial_0 M$ lie over those of $\partial_0 M_\kappa$. Let $\alpha$ be a component of $\partial_0 M_\kappa$ and $\iota_\alpha$: $M_\kappa \hookrightarrow M_\kappa \cup \{\alpha\}$ the canonical inclusion. If $\kappa_\alpha$ is the kernel of the homomorphism on the fundamental groups induced by $\iota_\alpha$ then the components of $\partial_0 M$ lying over $\alpha$ form a (not necessarily connected) covering space of $\alpha$ with covering group $\kappa/\kappa_\alpha$.

## 3 The spaces of round balls

*(3.1)* A characteristic notion in the Möbius category is that of a *round ball*, i.e. if we consider $\mathbf{S}^n$ as $\mathbf{E}^n \cup \{\infty\}$ then a round ball is any image of the open unit ball in $\mathbf{E}^n$ under a Möbius transformation, or more generally if $M^n$ is a developable Möbius manifold then any subset of $M^n$ which is mapped by dev on a round ball is also considered as a round ball. Still more generally if $M^n$ is any connected Möbius manifold and $\tilde{M}^n$ is its universal cover then a subset $B$ of $M^n$ is a round ball if it is an image of a round ball in $\tilde{M}^n$. Notice that by the essential uniqueness of the development map it follows that dev *is injective on any round ball in any developable Möbius manifold*. Consequently the two apparently different notions of a round ball on a non-simply connected developable Möbius manifold coincide. Notice also that a round ball is by definition always open. By a **closed round ball** in a developable Möbius manifold $M$ we shall mean the closure of an (open) round ball in $\bar{M}$. We shall have no need to talk about a closed round ball in a non-developable Möbius manifold. A **pointed round ball** is a round ball with a base-point. For any Möbius manifold $M$ we introduce two basic spaces.

(3.1.1) $\quad B(M) = \{$the set of all round balls in $M\}$.

(3.1.2) $\quad B_*(M) = \{$the set of all pointed round balls in $M\}$.

*(3.2)* To introduce the topology and geometry on $B(M)$ and $B^*(M)$ and for computational purposes it is convenient to consider the light-cone model of $(\mathbf{S}^n, \mathcal{M}(n))$ in the $(n+2)$-dimensional Minkowski space. Consider a real vector space $V$ of real dimension $n+2$ equipped with a nondegenerate quadratic form $Q$ of signature $(1, n+1)$. In terms of appropriate coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_{n+1})$.

$$Q(\mathbf{x}) = x_0^2 - x_1^2, -, \ldots, x_{n+1}^2.$$

The associated inner product will be denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$, and $Q(\mathbf{x})$ will be sometimes shortened to $|\mathbf{x}|^2$. The **positive light-cone** is

$$L_+ = \{\mathbf{x} \mid |\mathbf{x}| = 0,\ x_0 > 0\}.$$

Let $O(Q)$ denote the orthogonal group which preserves $Q$. It has four components. Let $O_+(Q)$ denote the two out of these four components which preserve $L_+$. It is easy to see that $O_+(Q)$ acts transitively on $L_+$, in particular it acts transitively also on the space of rays contained in $L_+$. As is wellknown $O_+(Q) \approx \mathcal{M}(n)$ and the action of $O_+(Q)$ on the space of rays in $L_+$ is equivalent to that of $\mathcal{M}(n)$ on $\mathbf{S}^n$, cf. [T, Chap. 1]. Let

$$\mathbf{L}_+ = \{\mathbf{x} \mid |\mathbf{x}|^2 \geqq 0,\ x_0 > 0\},$$
$$\mathbf{H}^{n+1} = \{\mathbf{x} \mid |\mathbf{x}|^2 = 1,\ x_0 > 0\},$$
$$\mathbf{D}^{n+1} = \{\mathbf{x} \mid -|\mathbf{x}|^2 < 1,\ x_0 = 0\}.$$

The space $\mathbf{H}^{n+1}$ which may be identified with the space of rays in $\operatorname{int} \mathbf{L}_+$ is a model of $n+1$-dimensional Riemannian hyperbolic geometry. Indeed $-Q|_{\mathbf{H}}^{n+1}$ induces a complete Riemannian metric of constant curvature $-1$. Also $O_+(Q)$ preserves $\mathbf{H}^{n+1}$, acts transitively on it, and may be identified with the full group of isometries of $\mathbf{H}^{n+1}$. As is wellknown $\mathbf{D}^{n+1}$ with the metric $\dfrac{2|d\mathbf{x}|}{1-|\mathbf{x}|^2}$ is also a model of $n+1$-dimensional Riemannian hyperbolic geometry. The radial projection from $(-1, 0, \ldots, 0)$ maps $\mathbf{D}^{n+1}$ isometrically onto $\mathbf{H}^{n+1}$. If $\mathbf{x}$ and $\mathbf{y}$ are two points of $\mathbf{H}^{n+1}$ then the hyperbolic distance among them is $\cosh^{-1}(\langle \mathbf{x}, \mathbf{y} \rangle)$.

**(3.3) Proposition.** (i) *$B(\mathbf{S}^n)$ may be considered as the "De Sitter space"*

(3.3.1) $$\{\mathbf{x} \mid |\mathbf{x}|^2 = -1\}.$$

*As a homogeneous space it is $\approx \mathrm{SO}_0(1, n+1)/\mathrm{SO}_0(1, n)$ where $\mathrm{SO}_0(1, n+1)$ denotes the identity component of $O(Q)$ and $\mathrm{SO}_0(1, n)$ denotes the identity component of the subgroup of $O(Q)$ which fixes the $x_{n+1}$-axis.*

(ii) *$B_*(\mathbf{S}^n)$ may be considered as the "Stiefel manifold of Lorentzian 2-frames"*

(3.3.2) $$\{(\mathbf{x}, \mathbf{y}) \mid |\mathbf{x}|^2 = -1,\ |\mathbf{y}|^2 = 1,\ \langle \mathbf{x}, \mathbf{y} \rangle = 0\}.$$

*It may also be considered as the unit tangent bundle to $\mathbf{H}^{n+1}$. As a homogeneous space it is $\approx \mathrm{SO}_0(1, n+1)/\mathrm{SO}(n)$. Here $\mathrm{SO}(n)$ is the identity component of the subgroup which fixes the $x_0$-th and the $x_{n+1}$-st axis.*

*Proof.* (i) A round ball in $\mathbf{S}^n$ canonically defines a half-space in $\mathbf{H}^{n+1}$ bounded by a totally geodesic hypersurface which in turn canonically defines a half-space

in $V$ bounded by a hyperplane through the origin which cuts $L_+$ transversely. It is easy to see that the induced metric on this hyperplane is of type $(1, n)$ and so its outer unit normal vector at the origin of $V$ has length $-1$. This shows the first part. For the second part observe that $SO_0(1, n+1)$ acts transitively on the set of half-spaces bounded by the hyperplanes through the origin with the induced metric of type $(1, n)$ and the identity component of the stabilizer subgroup of the half-space $x_{n+1} \leqq 0$ is obviously $SO_0(1, n)$.
(ii) Let $(B, p)$ be a pointed round ball. As in (i) $B$ corresponds to a half-space in $V$ bounded by a hyperplane $\pi$ through the origin which cuts $L_+$ transversely. Let $\mathbf{x}$ be the outer unit normal vector to $\pi$ through the origin. Now the point $p$ in $B$ corresponds to a ray $l$ lying in $L_+$. Let $\sigma$ be the 2-dimensional subspace of $V$ spanned by $\mathbf{x}$ and $l$. It is easy to see that the induced metric on $\sigma \cap \pi$ is positive definite and so it contains a unique unit vector $\mathbf{y}$ lying in $\operatorname{int} \mathbf{L}_+$. In this way we have associated to a pointed round ball $(B, p)$ a pair $(\mathbf{x}, \mathbf{y})$ with $|\mathbf{x}|^2 = -1$, $|\mathbf{y}|^2 = 1$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Conversely given such a pair $(\mathbf{x}, \mathbf{y})$ let $\pi$ be the hyperplane through the origin orthogonal to $\mathbf{x}$. In the half-space bounded by $\pi$ which does not contain $\mathbf{x}$ there is a unique ray, namely the one defined by positive multiples of $\mathbf{y} - \mathbf{x}$, which lies in $L_+$. This data in turn canonically determines a pointed ball in $\mathbf{S}^n$. This proves the first part.

For the second part let a pointed ball $(B, p)$ be given. Then $B$ determines a half-space in $\mathbf{H}^{n+1}$ bounded by a totally geodesic hypersurface say $h$. Now the pair $(B, p)$ determines a unit tangent vector to $\mathbf{H}^{n+1}$, namely the outer unit normal vector to $h$ at the foot of the perpendicular from $p$ to $h$. Conversely given a unit tangent vector to $\mathbf{H}^{n+1}$ by reversing the above process we obtain a pointed ball in $\mathbf{S}^n$.

Lastly it is again obvious that $SO_0(1, n+1)$ acts transitively on the set of pointed balls or equivalently on the set of pairs $(\mathbf{x}, \mathbf{y})$ with $|\mathbf{x}|^2 = -1$, $|\mathbf{y}|^2 = 1$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Moreover the identity component of the stabilizer subgroup of the pair $\mathbf{x} = (0, 0, \ldots, 1)$ and $\mathbf{y} = (1, 0, \ldots, 0)$ is $SO(n)$. q.e.d.

*(3.4)* The topologies on $B(\mathbf{S}^n)$ and $B_*(\mathbf{S}^n)$ are those of the corresponding homogeneous spaces. They are in fact differentiable manifolds of dimension $n+1$ and $2n+1$. Now let $M$ be a developable Möbius manifold. Then dev: $M \to \mathbf{S}^n$ induces the maps of sets $B(M) \to B(\mathbf{S}^n)$ and $B_*(M) \to B_*(\mathbf{S}^n)$ which will be again denoted by dev. These maps are locally injective. We topologize $B(M)$ resp. $B_*(M)$ so that dev is a local homeomorphism onto its image in $B(\mathbf{S}^n)$ resp. $B_*(\mathbf{S}^n)$ resp. Let $B_0(M)$ resp. $B_{0*}(M)$ be the subset of $B(M)$ resp. $B_*(M)$ consisting of those balls whose closures in $\bar{M}$ do not have a point on the ideal boundary. The image of $B_0(M)$ resp. $B_{0*}(M)$ under dev is an open subset, and so in fact it may be used to introduce the structure of differentiable manifolds on $B_0(M)$ resp. $B_{0*}(M)$ respectively so that dev is a local diffeomorphism. If $M$ is not developable let $p: M_1 \to M$ be its some developable cover. Then $p$ induces a locally injective map on the corresponding spaces of balls and we use it to topologize $B(M)$ resp. $B_*(M)$ so that the maps induced by $p$ are local homeomorphisms. The following proposition summarizes the basic properties of these spaces.

**Proposition.** *Let $M^n$ be a developable Möbius manifold.*
(i) *The base-point-forgetting map $(B, p) \mapsto B$ of $B_*(M) \to B(M)$ is a fibration with fiber $\mathbf{H}^n$.*
(ii) *$B_*(M)$ can be canonically identified with an open subset of $B(M) \times M$.*

(iii) *There is a (noncanonical) homeomorphism* $\phi$ *of* $B(M)$ *into* $M \times \mathbf{R}$. *The image of* $B_0(M)$ *under* $\phi$ *is an open subset whereas the image of* $B(M)$ *is only locally closed. In particular* $B(M)$ *is locally compact.*
(iv) *For* $M \cong \mathbf{S}^n$ *or* $\mathbf{E}^n$ *the map* $(B, p) \mapsto p$ *of* $B_*(M) \to M$ *is a fibration with fiber* $B(\mathbf{E}^n)$. *In general* $B_{0*}(M)$ *is an open subset of the total space of the pullback of the bundle* $B_*(\mathbf{S}^n) \to \mathbf{S}^n$ *under* dev *whereas* $B_*(M)$ *is a locally closed subset, hence it is locally compact. Or in view of* (ii) *and* (iii) $B_{0*}(M)$ *may be identified with an open subset of* $M \times M \times \mathbf{R}$ *and* $B_*(M)$ *lies in its closure as a locally compact subset.*

*Proof.* The part (i) is clear. The part (ii) is also clear by a dimension count. As for (iii) let $g_0$ be a standard metric on $\mathbf{S}^n$, and $g = \mathrm{dev}^*(g_0)$ the induced metric on $M$. With respect to $g$ the **center** and the **radius** of a round ball has a meaning. The map $\phi$ simply associates to a round ball its center and radius. The assertion (iii) is now clear; the only point to note is that a round ball by definition has a positive radius so the image of $\phi$ does not contain any point of $M \times \{0\}$, whereas $M \times \{0\}$ clearly lies in the closure of $\operatorname{im} \phi$. As for (iv) notice that the round balls in $\mathbf{S}^n$ or $\mathbf{E}^n$ containing a given point $p$ are, by taking interiors of the complements, in a natural $1-1$ correspondence with the round balls in $\mathbf{S}^n$ not containing the given point $p$. The latter set is clearly $B(\mathbf{E}^n)$. So the assertion is true for $\mathbf{S}^n$ or $\mathbf{E}^n$. The assertion for general $M$ may be left to the reader. q.e.d.

*(3.5)* Let $M$ be a developable Möbius manifold. As noted in (3.3) the space $B(\mathbf{S}^n)$ may be identified with the De Sitter space of dimension $n+1$. As is wellknown the De Sitter space is a linear model for a complete Lorentz manifold of constant negative curvature $-1$. It is diffeomorphic to $\mathbf{S}^n \times \mathbf{R}$. Via the development map this structure may be pulled back to $B(M)$. It is independent of the choice of a development map. In particular a tangent vector to $B(M)$ being **timelike**, or **lightlike**, or **spacelike** has a meaning, namely it is a vector $\mathbf{v}$ such that $\langle \mathbf{v}, \mathbf{v} \rangle$ is *positive*, or *zero*, or *negative* respectively. A piecewise smooth curve in $B(M)$ is called **timelike**, or **lightlike**, or **spacelike** if all of its tangent vectors are such.

*(3.6)* The geodesics of the De Sitter space are wellknown, cf. [On] for the description of these spaces. The geodesics of the De Sitter space are given by its intersections with the two-dimensional linear subspaces of the $(n+2)$-dimensional Minkowski space, cf. (3.2). In terms of the round balls in $\mathbf{S}^n$ they may be described as follows. A complete family of concentric round balls (w.r.t. a standard metric on $\mathbf{S}^n$) and its $\mathscr{M}(n)$-translates are precisely the timelike geodesics. The complete families each consisting of round balls which touch each other at a fixed point are precisely the lightlike geodesics. And finally the complete families each consisting of round balls passing through a fixed round $\mathbf{S}^{n-2}$ are precisely the spacelike geodesics. If $M$ is a developable Möbius manifold, and $B(M)$ is equipped with its canonical Lorentz structure as above then these considerations carry over to it in a local fashion. Thus for example a curve of round balls in $B(M)$ passing through a fixed round $\mathbf{S}^{n-2}$ is a possibly reparametrized spacelike geodesic segment.

*(3.7)* We note here a simple method of doing elementary computations in the space $B(\mathbf{S}^n)$. For this purpose it will be convenient to take $\mathbf{S}^n$ as the unit sphere $|\mathbf{x}|^2 = 1$ in $\mathbf{E}^{n+1}$. A round ball in $\mathbf{S}^n$ has a boundary which is a transverse