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On equimultiple ideals

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1 Notation

All our rings will always be commutative, associative, contain an identity, Noetherian, with all local rings having infinite residue fields. An ideal J contained in I is said to be a reduction of I iff $I^{n+1}=I^nJ$ for some non-negative integer n. Furthermore J is said to be a minimal reduction of I iff J has no further proper reductions. If J is fixed then the least integer n for which $I^{n+1}=I^nJ$ holds is said to be the reduction exponent of I with respect to J. The minimum of all the reduction exponents of I with respect to J as J varies over all minimal reductions of I is called the reduction exponent of I. The reduction exponent t of I is said to be independent of its minimal reductions if $I^{t+1}=I^tJ$ holds for all minimal reductions J of I. An ideal I has *generically reduction exponent* t iff I_P has reduction exponent at most t for all minimal primes P of I. This last italicised definition is due to Grothe et al. [GHO, (5.8)] which we have stated here in a slightly adapted form. We will say that an ideal I has *local exponent* t iff I_P has reduction exponent exactly t for all minimal primes P of I. The analytic spread of I is the least number of elements required to generate a minimal reduction of I. Two ideals I and J are said to be analytically disjoint if the analytic spread of $I+J$ equals the analytic spread of I plus the analytic spread of J. An ideal I is said to be equimultiple iff height of I equals its analytic spread. If R is a local ring with maximal ideal M then the fiber cone $F(I)$ is the ring $\bigoplus_{s=0}^{\infty} I^s/I^sM$. The associated graded ring $G(I)$ is the ring $\bigoplus_{s=0}^{\infty} I^s/I^{s+1}$. If I is an ideal which is primary to the maximal ideal in a local ring then $e(I)$ denotes its Hilbert multiplicity. A non-negatively graded ring over an Artinian local ring admits a Hilbert polynomial and we work also with multiplicity of such graded rings. If E is any R-module then $\ell(E)$ denotes length of E as an R-module. If I is any ideal then $\mu(I)$ denotes minimal number of generators of I.

2 Introduction

The basic results of Cowsik and Nori [CN2, Theorem], as well as of Herrmann and Ikeda [HI, Theorem 3.1(a)] may be summarized in essence as follows:

Let R be a Cohen-Macaulay local ring. Let I be an equimultiple ideal which is generically a complete intersection. Then I itself is a complete intersection.

The above result may be suitably interpreted using the language of generically reduction exponents of ideals as follows:

Let R be a Cohen-Macaulay local ring. Let I be an equimultiple ideal which has generically reduction exponent zero. Then I itself has reduction exponent zero.

Thus a question arises: If an equimultiple ideal I has generically reduction exponent t then when does I itself have reduction exponent at most t? If $t=0$ then the above quoted result is a satisfactory answer. If $t=1$ then we give an equally satisfactory answer:

Theorem 1 *Let R be a Cohen-Macaulay local ring. Let I be an integrally closed equimultiple ideal which has generically reduction exponent one. Then I itself has reduction exponent at most one. Furthermore if I has local exponent one then I has reduction exponent one.*

Grothe et al. [GHO, Proposition 5.9] showed that:

If R is Cohen-Macaulay local ring and if I is an ideal such that R is normally Cohen-Macaulay along I and I has generically reduction exponent one then the associated graded ring $G(I)$ is Cohen-Macaulay.

We obtain an analogous result for fiber cone:

Corollary 1.1 *Let R be a Cohen-Macaulay local ring containing an integrally closed equimultiple ideal I which has generically reduction exponent one. Then the fiber cone $F(I)$ is Cohen-Macaulay.*

We combine the above corollary with the basic Lipman and Teissier result [LT, Proposition 5.5] to obtain the following:

Corollary 1.2 *Let R be a regular local ring containing an integrally closed equimultiple ideal I of height two. Then I has reduction exponent zero or one. Also, the fiber cone $F(I)$ is Cohen-Macaulay.*

Hochster [Ho, Theorem 1] gave a criterion from which equality of ordinary and symbolic powers of prime ideals could be deduced in basic situations. Our next result explains the underlying situation for certain equimultiple ideals.

Corollary 1.3 *Let R be a Cohen-Macaulay local ring. Let P be an equimultiple prime ideal such that P has generically reduction exponent one. Then all the ordinary powers of P coincide with the corresponding symbolic powers of P.*

If $I^3=I^2J$ and I is an integrally closed M-primary ideal in a Cohen-Macaulay local ring then the author [Sh1, Corollary 4(d), Remark 2(b)] showed that $G(I)$ is Cohen-Macaulay. We exploit this result with a new trick of considering ideals

which are integral over *powers* of J which is generated by a regular sequence to give a deceptively simple proof in low dimensions of the following:

Theorem 2 *Let R be a d-dimensional Cohen-Macaulay local ring with maximal ideal M and $1 \leqq d \leqq 3$. Let J be an M-primary ideal generated by a regular sequence. Let I be an integrally closed ideal containing J^2 such that $I^2 = IJ^2$. Then $G(I)$ is Cohen-Macaulay.*

The above result contains in fact a key to studying graded rings formed out of product of ideals as noted in the following result:

Corollary 2.1 *Let R be a d-dimensional Cohen-Macaulay local ring with maximal ideal M and $1 \leqq d \leqq 3$. Let J be a fixed M-primary ideal generated by a regular sequence. Let I_1, I_2 be two ideals containing J such that $(I_1)^2 = I_1 J$ and $(I_2)^2 = I_2 J$. If $I_1 I_2$ is integrally closed then $G(I_1 I_2)$ is Cohen-Macaulay.*

Our next results are in the direction of the hundred year old Hilbert polynomial. If R is a local Noetherian ring with maximal ideal M and if K is an M-primary ideal then the length $\ell(R/K^r)$ of R/K^r is a polynomial in r for all large r, of degree equalling the dimension of R, called the Hilbert polynomial of K. We note that K is in particular an equimultiple ideal. Is the Hilbert polynomial a special case of a more general polynomial involving two equimultiple ideals? More precisely, we pose the following:

Question. Let I, J be equimultiple ideals in R not necessarily M-primary but such that $I+J$ is M-primary. When is $\ell(R/(I^r, J^s))$ a polynomial in r, s for all large r, s?

We use the homological tools of short and long exact sequences and Tor to obtain the following result:

Theorem 3 *Let R be a local ring with maximal ideal M. Let I and J be any non-zero ideals in R such that I^n/I^{n+1} is R/I-free and J^n/J^{n+1} is R/J-free for all non-negative integers n and $I \cap J = IJ$ with $I+J$ as M-primary. Then $\ell(R/(I^r, J^s))$ is a polynomial in r, s for all large r, s of total degree $d+g$ where d, g are the analytic spreads of I, J respectively. For each $r \geqq 1$, $s \geqq 1$ we have in fact:*

$$\ell(R/(I^r, J^s)) = \ell(R/(I, J))[1 + \mu(I) + \ldots + \mu(I^{r-1})][1 + \mu(J) + \ldots + \mu(J^{s-1})].$$

We note that the condition I^n/I^{n+1} is R/I-free for all non-negative integers n does imply that I is an equimultiple ideal by an interesting result of Herrmann and Orbanz [HO3, Theorem 1]. We also note that in particular the hypothesis $I \cap J = IJ$ in the theorem excludes the cases $I = J$ and $J \subset I = M$. It is natural to study the algebraic properties of the tensor ring $G(I) \otimes_R G(J)$.

Theorem 4 *Let R be a local ring with maximal ideal M. Let I and J be ideals such that $I+J$ is M-primary. Then*
(a) *$G(I) \otimes_R G(J)$ has dimension $d+g$ where d, g are analytic spreads of I, J respectively.*
(b) *If I and J are analytically disjoint ideals, each generated by regular sequences then $G(I) \otimes_R G(J)$ is a Cohen-Macaulay ring of dimension* $\dim . R$.

(c) *If I and J are analytically disjoint ideals such that $I+J=M$, with I containing X such that $I^2=IX$, and J containing Y such that $J^2=JY$, with each of X and Y being generated by regular sequences then $G(I)\otimes_R G(J)$ is Cohen-Macaulay of dimension* $\dim . R$.

3 The proofs of the results stated in the introduction

We now sequentially prove the results stated in the introduction above, often proving other results of independent interest.

Lemma 1.4 *Let R be a local ring with maximal ideal M. Let I be any ideal of R. Let J be any minimal reduction of I. Let N be the unique irrelevant maximal ideal of the fiber cone $F(I)$. Then*
(a) $1+\mu(I)-\mu(J)\leqq \ell(F(I)/JF(I))$ *with equality holding if and only if* $I^2=IJ$.
(b) $1+\mu(I)-\mu(J)\leqq e(F(I)_N)$ *provided $F(I)$ is Cohen-Macaulay. In this case equality holds if and only if* $I^2=IJ$.

Proof. Consider (a). $\ell(F(I)/JF(I))=1+\mu(I)-\mu(J)+\delta$ where δ is $\sum \ell(I^s/(I^sM+JI^{s-1}))$ with $2\leqq s$. Now $\delta=0$ if and only if $I^2=I^2M+JI$ if and only if $I^2=IJ$. This completes the proof of (a). If $F(I)$ is Cohen-Macaulay then $\ell(F(I)/JF(I))=e(F(I)_N)$ and proof of (b) follows from (a). □

Lemma 1.5 *Let R be a local ring with maximal ideal M. Let I be any ideal of R. Then*
(a) *If I is integral over a regular sequence and if reduction exponent of I is at most one then the reduction exponent of I is independent of its minimal reductions.*
(b) *If R is a Buchsbaum ring and if reduction exponent of M is at most one then the reduction exponent of M is independent of its minimal reductions.*

Proof. First consider (a). Since I has reduction exponent at most one, we deduce that $I^2=IJ$ for some minimal reduction J of I. The fiber cone $F(I)$ of I is Cohen-Macaulay by [Sh1, Theorem 1]. This means by Lemma 1.4 above that $e(F(I)_N)=1+\mu(I)-\mu(J)$. Let K be any minimal reduction of I. Then $e(KF(I)_N)=\ell(F(I)_N/KF(I)_N)=\ell(F(I)/KF(I))$. Thus we obtain that $1+\mu(I)-\mu(J)=1+\mu(I)-\mu(K)=\ell(F(I)/KF(I))$. By Lemma 1.4 it follows that $I^2=IK$.

We now prove part (b) of our lemma. Since R is Buchsbaum, we may assume that $\ell(R/J)-e(J)=c$ for all systems of parameters ideals J of R where c is a fixed constant. Since reduction number of M is at most one there exists a minimal reduction J of M such that $M^2=MJ$. Then $\ell(R/M^2)$ equals

$$\ell(R/J)+\ell(J/M^2)=\ell(R/J)+\ell(J/JM)=e(J)+c+\mu(J)=e(R)+c+\mu(J).$$

If K is any minimal reduction of M then $\ell(R/M^2)$ also equals

$$\ell(R/K)+\ell(K/KM)-\ell(M^2/KM)=e(K)+c+\mu(K)-\ell(M^2/KM).$$

Since $\mu(J)=\mu(K)$, and $e(R)=e(K)$, it follows that $\ell(K^2/KM)=0$. This means that $K^2=KM$ completing the entire proof. □

We refer the reader to Huckaba [Huc, Theorem 2.1], Trung [T, Theorem 1.2] concerning the topic related to the above lemma. Part (a) of Lemma 1.5 also

follows from [Sh1, Theorem 1] and [HM, Theorem 3.1]. I thank Jack Ratliff concerning the following:

Theorem 1.6 *Let R be a Noetherian Cohen-Macaulay local ring with maximal ideal M. Let I be an integrally closed equimultiple ideal. If J is a minimal reduction of I then $J^s I$* has no embedded primes for all s. In particular, JI has no embedded primes.

Proof. Since R is Cohen-Macaulay it follows that J^s has no embedded primes for all s since J is generated by a regular sequence. Since I has height equal to its analytic spread, the integral closure $(I^{s+1})'$ of I^{s+1} has no embedded primes for each s by Ratliff's theorem [R, Theorem 2.12]. Thus $J^s \cap (I^{s+1})'$ has no embedded primes follows from primary decomposition arguments since J^s and $(I^{s+1})'$ have the same minimal primes. However $J^s \cap (I^{s+1})' = J^s J$ holds by Huneke and Itoh [Hu, Theorem; I2, Theorem 1]. Therefore the claim of the theorem follows. □

Proof of Theorem 1 Let J be any minimal reduction of I. Therefore J is generated by a regular sequence. Hence J_P is generated by a regular sequence in R_P. Also J_P is a minimal reduction of I_P. Since I has generically reduction exponent one, it follows from Lemma 1.5(a) above that $I_P^2 = I_P K$ for all minimal reductions K of I_P. In particular we have that $(I_P)^2 = I_P J_P$ for each minimal prime P of I. There is a natural map of I^2/IJ into R/J with kernel $J \cap I^2/JI$. This map is an injection because $J \cap I^2 = IJ$ by Huneke-Itoh theorem [Hu, Theorem; I2, Theorem 1]. We next adapt a technique to show equality of ideals, found in the proof of Cowsik and Nori [CN2, Theorem], as well as in the proof of Herrmann and Ikeda [HI, Theorem 3.1(a)]. If I^2 is not equal to JI then let Q be any associated prime of I^2/IJ. This means that R/Q injects into I^2/IJ and hence into R/IJ. Since IJ has no embedded primes by Theorem 1.6, we obtain that $Q = P$ for some minimal prime P of I. This means that I_P^2 is not equal to to $I_P J_P$ for some minimal prime P of I. This is not possible and hence $I^2 = IJ$. This shows that I has reduction exponent at most one. If I has local exponent one then since in this case I cannot have reduction exponent zero. I must have reduction exponent one. This completes our entire proof. □

Proof of Corollary 1.1 The conclusion of Theorem 1 implies that I is generated by a regular sequence or I has reduction exponent one. Hence the fiber cone $F(I)$ is Cohen-Macaulay follows from Shah [Sh1, Theorem 1]. □

Proof of Corollary 1.2 Now I_P is integrally closed ideal in a two dimensional regular local ring R_P for any minimal prime P of I. It follows from Lipman and Teissier [LT, Proposition 5.5] that I_P is generated by a regular sequence or I_P has reduction exponent one. This means that I has generically reduction exponent one. This means that I has reduction exponent at most one. Hence $F(I)$ is Cohen-Macaulay follows from Corollary 1.1. □

Proof of Corollary 1.3 Let J be any minimal reduction of P. Then by Theorem 1 we obtain that $P^2 = PJ$. In particular $P^{n+1} = J^n P$. By Theorem 1.6, P^{n+1} has no embedded primes for all n. Thus the claim of the corollary follows from this. □

Corollary 1.7 *Let R be a Cohen-Macaulay local ring. Let P be an equimultiple prime ideal such that P has generically reduction exponent one. Then P has reduction exponent at most one.*

Corollary 1.8 *Let R be a Cohen-Macaulay local ring. Let I be an integrally closed equimultiple ideal which has generically reduction exponent one. Then reduction exponent of I is independent of any of its minimal reductions.*

I thank the Referee for giving a proof of Theorem 1 which does not use Theorem 1.6 and the proof is as follows: Now $I_P^2 = I_P J_P$ for all minimal primes of I as in the author's proof of Theorem 1. Also $I^2 \cap J = IJ$ by [Hu, Theorem; I2, Theorem 1]. If there exists an associated prime Q of I^2/IJ then R/Q must embed in I^2/IJ and hence in R/J also. Hence Q must be a minimal prime over J and hence also a minimal prime over I. This gives a contradiction since $I_P^2 = I_P J_P$ for all minimal primes over I, completing the Referee's proof of the Theorem 1.

We note that a slightly different proof of Theorem 1 can also be sketched as follows: Let J be any minimal reduction of I. It follows that J_P is a minimal reduction of I_P in Cohen-Macaulay ring R_P. Since reduction exponent of I_P is at most one it follows from Vala and Valabrega [VV, Theorem 3.1] that the associated graded ring of R_P with respect to I_P is Cohen-Macaulay. It follows from Huckaba [Huc, Theorem 2.1], Trung [T, Theorem 1.2] that $I_P^2 = I_P K$ holds for all minimal reductions K of I_P. In particular we have that $(I_P)^2 = I_P J_P$ for each minimal prime P of I. The remaining part of the argument is similar to the proof of Theorem 1.

We also note that: *If R is a regular local ring with maximal ideal M and* $\dim . R \geqq 1$ *and if I is any M-primary ideal such that $e(I) \leqq 2$ then I is generated by a regular sequence.* This can be seen as follows: Let J be any minimal reduction of I. Then J is generated by a regular sequence and $e(I) = e(J)$. Since R is also Cohen-Macaulay we have that $e(J) = \ell(R/J) \leqq 2$. This means that I must equal M or J. In either case. I is generated by a regular sequence. Thus the results of Cowsik and Nori as well as of Herrmann and Ikeda quoted in the introduction may also be summarized in essence as follows: *if R is Cohen-Macaulay and if I is an equimultiple ideal such that R_P is regular for all minimal primes P of I and that I has generically multiplicity at most two then I must be a complete intersection.*

The following three nice examples were communicated by Bill Heinzer.

Example 1.9 (Bill Heinzer) Let $R = K[\![x^3, x^4 y, x^5 y^3, y^3, y^4]\!]$ and let $P = (x^3, x^4 y, x^5 y^3) R$ with K being a field. Then P is a height one prime ideal in R with analytic spread one. The reduction exponent of P_P is one. The reduction exponent of P is greater than one but using the Macaulay computer program it is seen that R is not Cohen-Macaulay.

Example 1.10 (Bill Heinzer) Let $R = K[\![x^2, x^3 y, y^3, y^5]\!]$ and let $P = (x^2, x^3 y) R$ with K being a field. Then P is a height one prime ideal in R with analytic spread one. The reduction exponent of P_P is one. The reduction exponent of P is greater than one but R is not Cohen-Macaulay.

Example 1.11 (Bill Heinzer) Let $R = K[\![x^2, x^3 y, y]\!]$ with K a field. Let $P = (x^2, x^3 y) R$. Then P has analytic spread one and also P has height one. Since P_P has multiplicity two it follows that P_P has reduction number one and hence that P has generically reduction exponent one. By Corollary 1.3, it follows that all powers of P coincide with all the corresponding symbolic powers of P.

Example 1.12 Let K be a field. Let $R = K[\![x^2, x^3 y^3, y^2]\!]$. Let $P = (x^2, x^3 y^3) R$. Then height of P is one. We claim that analytic spread of P is also one: it

is clear that analytic spread of P must be one or two. If it is two then the fiber cone of P must be a polynomial ring. However the image of $x^3 y^3$ is easily seen to be nilpotent in the fiber cone and therefore analytic spread of P must necessarily be one. Thus P is equimultiple. Set $J=(x^2)R$. Then J forms a minimal reduction of P. Hence JP has no embedded primes by Theorem 1.6. Also $P^2=PJ$ and hence it follows from the Corollary 1.3 that P^n has no embedded primes for each n.

Proof of Theorem 2 Let J be generated by $x_1, \ldots, x_d$. Let Q be generated by $x_1^2, \ldots, x_d^2$. Then it is not hard to check that $J^4=J^2Q$. Now $I^2=IJ^2$ implies that $I^3=I^2J^2=IJ^4$. Thus it follows that $I^3=IJ^2Q=I^2Q$. Now the conclusion of the theorem follows immediately from Shah [Sh1, Corollary 4(d), Remark 2(b)]. □

Proof of Corollary 2.1 The hypotheses imply that $(I_1I_2)^2=(I_1I_2)J^2$. The conclusion now follows from Theorem 2. □

Lemma 3.1 *Let R be a Noetherian ring containing ideals I and J such that $I\cap J=IJ$. Then the following two statements hold:*
(1) *If I^n/I^{n+1} is R/I-free for all $n\geqq 0$ then $I^n\cap J=I^nJ$ for all $n\geqq 0$.*
(2) *If I^n/I^{n+1} is R/I-free for all $n\geqq 0$ and also if J^n/N^{n+1} is R/J-free for all $n\geqq 0$ then $I^r\cap J^s=I^rJ^s$ for all $r\geqq 0$, $s\geqq 0$.*

Proof. Consider (1). We may assume inductively that $I^n\cap J=I^nJ$ for all $n\leqq N$. Consider the short exact sequence:

$$0\to I^N/I^{N+1}\to R/I^{N+1}\to R/I^N\to 0.$$

Since I^N/I^{N+1} is R/I-free we obtain that it is isomorphic to a finite direct sum of R/I-modules. Hence we obtain the exact sequence:

$$0\to \bigoplus R/I\to R/I^{N+1}\to R/I^N\to 0.$$

Now tensor this short exact sequence with R/J. This gives rise to a long exact sequence a part of which is:

$$\ldots\to \mathrm{Tor}_1(\bigoplus R/I, R/J)\to \mathrm{Tor}_1(R/I^{N+1}, R/J)\to \mathrm{Tor}_1(R/I^N, R/J)\to\ldots$$

But our hypothesis $I\cap J=IJ$ implies that $\mathrm{Tor}_1(\bigoplus R/I, R/J)=0$. The inductive hypothesis $I^N\cap J=I^NJ$ implies that $\mathrm{Tor}_1(R/I^N, R/J)=0$. Hence $\mathrm{Tor}_1(R/I^{N+1}, R/J)=0$. This means $I^{N+1}\cap J=I^{N+1}J$ completing the proof of (1).

Consider (2). Now the equation $I^r\cap J=I^rJ$ holds for any $r\geqq 0$ from (1). Fix any such r. Since J^n/J^{n+1} is R/J-free for every $n\geqq 0$, it follows from (1) again that $I^r\cap J^s=I^rJ^s$ holds for every $s\geqq 0$ completing the proof of (2) and our lemma also. □

Proof of Theorem 3 Now I^{r-1}/I^r is a direct sum of finite number of copies of R/I. By tensoring each of them with R/M it follows that the number of

copies of R/I is in fact $\mu(I^{r-1})$ which is the minimal number of generators of I^{r-1}. Thus there is a short exact sequence:

$$0 \to \bigoplus_{1}^{\mu(I^{r-1})} R/I \to R/I^r \to R/I^{r-1} \to 0.$$

Tensor this short exact sequence with R/J^s which induces a long exact sequence a part of which is:

$$\to \mathrm{Tor}_1(R/I^{r-1}, R/J^s) \to \bigoplus_{1}^{\mu(I^{r-1})} R/(I, J^s) \to R/(I^r, J^s) \to R/(I^{r-1}, J^s) \to 0.$$

From the above Lemma 3.1 it follows that $I^{r-1} \cap J^s = I^{r-1} J^s$ implying that the left hand side Tor is 0. The resulting short exact sequence implies that:

$$\ell(R/(I^r, J^s)) = \ell(R/(I^{r-1}, J^s)) + \mu(I^{r-1})\ell(R/(I, J^s)).$$

Thus inductively we obtain the following length equation:

$$\ell(R(I^r, J^s)) = \ell(R/(I, J^s)) + [\mu(I^{r-1}) + \ldots + \mu(I)]\ell(R/(I, J^s)).$$

This furthermore simplies to:

$$\ell(R/(I^r, J^s) = \ell(R/(I, J^s))[\mu(I^{r-1}) + \ldots + \mu(I) + 1].$$

Since J^s/J^{s+1} is R/J-free for all $s \geqq 0$ we may once again use inductive procedure to simplify $\ell(R/(I, J^s))$ and obtain:

$$\ell(R/(I^r, J^s)) = \ell(R/(I, J))[\mu(J^{s-1}) + \ldots + \mu(J) + 1][\mu(I^r) + \ \ldots \ + \mu(I) + 1].$$

Now the expressions in the brackets are the Hilbert functions of the fiber cones of J and I respectively (see Shah [Sh1]). Thus for large r, s they are each polynomials in r, s of degrees g, d respectively. Hence their product is also a polynomial in r, s for all large r, s of total degree $d+g$. Rearranging the terms completes the proof. □

Example 3.2 Let $R = K[\![x_1, \ldots, x_d, y_1, \ldots, y_g]\!]$ be a regular power series ring over a field K. Set $I = (x_1^{a_1}, \ldots, x_d^{a_d})$ and $J = (y_1^{b_1}, \ldots, y_g^{b_g})$. Set $m = a_1 \ldots a_d b_1 \ldots b_g$. Then for all r, s we have the following:

$$\ell(R/(I^r, J^s)) = m\binom{r+d-1}{d}\binom{s+g-1}{g}.$$

This follows immediately from Theorem 3 and from the fact that $\ell(R/(I, J))$ is nothing but the multiplicity of $I+J$ which is m.

Example 3.3 Let I and J be analytically disjoint ideals which are each generated by regular sequences and such that $I+J$ is a system of parameter ideal in

a Cohen-Macaulay local ring R. Set height of I to be d and height of J to be g. Then for all $r \geqq 1$, $s \geqq 1$ the following holds:

$$\ell(R/(I^r, J^s)) = \ell(R/(I, J))\binom{r+d-1}{d}\binom{s+g-1}{g}.$$

This follows from Theorem 3.

Example 3.4 (Referee) Let $R = K[\![x, y, z]\!]/(x^2, xy)$ where $K[\![x, y, z]\!]$ is a power series ring over a field. Set $I = (x, y)R$ and set $J = (z)R$. Then I, J satisfy the conditions of Theorem 3 and also that I is *not* generated by a regular sequence.

Proof of Theorem 4 Consider (a). We write an expression for the tensor ring as follows:

$$G(I) \otimes_R G(J) = \bigoplus_{n=0}^{\infty} \Big(\bigoplus_{r+s=n} (I^r/I^{r+1} \otimes_R J^s/J^{s+1}) \Big).$$

The degree zero component is $R/(I, J)$ which has a finite length. Hence $M/(I, J)$ is nilpotent in the tensor ring. Thus the dimension of the ring $G(I) \otimes_R G(J)$ equals the dimension of the ring $G(I) \otimes_R G(J) \otimes_R R/M$. However the latter ring is nothing but $F(I) \otimes_R F(J)$ for which we now write an expression as follows:

$$F(I) \otimes_R F(J) = \bigoplus_{n=0}^{\infty} \Big(\bigoplus_{r+s=n} (I^r/I^r M \otimes_R J^s/J^s M) \Big).$$

Now $I^r/I^r M \otimes_R J^s/J^s M$ has length $\mu(I^r)\mu(J^s)$ which for large r, s is a polynomial of total degree $d+g-2$. Hence $\sum_{r+s=n} \mu(I^r)\mu(J^s)$ is a polynomial in n for all large n of degree $d+g-1$. Finally $\sum_{n=0}^{n=N} \big(\sum_{r+s=n} \mu(I^r)\mu(J^s) \big)$ is a polynomial in N for all large N of degree $d+g$. This latter polynomial is also the Hilbert polynomial of $F(I) \otimes_R F(J)$. But the degree of this polynomial gives the dimension of $G(I) \otimes_R G(J)$ and hence our claim for (a) is established.

Consider (b). Since $R/(I, J)$ has finite length we can calculate the multiplicity of this tensor ring. Now I^r/I^{r+1} is isomorphic to $\mu(I^r)$ copies of R/I and since J^s/J^{s+1} is isomorphic to $\mu(J^s)$ copies of R/J it follows that

$$I^r/I^{r+1} \otimes_R J^s/J^{s+1} \approx \bigoplus_{\mu(I^r)\mu(J^s)} (R/(I, J)).$$

However $\mu(I^r) = \binom{r+d-1}{d-1}$ and $\mu(J^s) = \binom{s+g-1}{g-1}$. So it follows that degree of $\mu(I^r)$ is $d-1$ and degree of $\mu(J^s)$ is $g-1$ where d, g are the analytic spreads of I, J respectively. Thus the degree of the polynomial $\ell(I^r/I^{r+1} \otimes J^s/J^{s+1})$ is $d+g-2$. Now from Hironaka [Hi, p. 187] we get the binomial identity:

$$\sum_{r+s=n} \binom{r+d-1}{d-1}\binom{s+g-1}{g-1} = \binom{n+d+g-1}{d+g-1}.$$

Thus we obtain that

$$\sum_{r+s=n} \ell(I^r/I^{r+1} \otimes_R J^s/J^{s+1}) = \ell(R/(I,J)) \binom{n+d+g-1}{d+g-1}.$$

Finally from Matsumura [M, p. 97] we have that

$$\sum_{n=0}^{N} \binom{n+d+g-1}{d+g-1} = \binom{d+g+N}{d+g}.$$

Thus combining the above expressions we obtain that

$$\sum_{n=0}^{N} \Big(\sum_{r+s=n} \ell(I^r/I^{r+1} \otimes J^s/J^{s+1}) \Big) = \ell(R/(I,J)) \binom{d+g+N}{d+g}.$$

Thus we have deduced in fact the Hilbert polynomial of the tensor ring $G(I) \otimes_R G(J)$ and that it has degree $d+g$ and leading normalised coefficient $\ell(R/(I+J))$. This means that the dimension of the tensor ring is $\dim . R$ and also that the multiplicity of the tensor ring is $\ell(R/(I,J))$. However if we kill the images of the minimal generators of I, J in the tensor ring then we get the quotient to be $R/(I,J)$ which has the same length as the above multiplicity. It follows that $G(I) \otimes_R G(J)$ is Cohen-Macaulay of $\dim . R$, completing the proof of (b).

We however note the following: $G(I) = R/I[x_1, \ldots, x_d]$ and also that $G(J) = R/J[y_1, \ldots, y_g]$ where the x_i's are algebraically independent variables over R/I and y_j's are algebraically independent variables over R/J. Thus the proof of (b) shows that

$$G(I) \otimes G(J) = R/(I,J)[x_1, \ldots, x_d, y_1, \ldots, y_g]$$

where the indicated variables are algebraically independent over $R/(I,J)$.

Consider (c). Since $I+J=M$ it follows that $G(I) \otimes_R G(J)$ is same as $G(I) \otimes_R G(J) \otimes_R R/M$, and which in turn is same as $F(I) \otimes_R F(J)$. We write out $F(I) \otimes_R F(J)$ explicitly again:

$$F(I) \otimes_R F(J) = \bigoplus_{n=0}^{\infty} \Big(\bigoplus_{r+s=n} (I^r/I^r M \otimes_R J^s/J^s M) \Big)$$

It follows from Shah [Sh1, Theorem 7] that

$$\mu(I^r) = \binom{r+d-1}{d-1} + (\mu(I)-d) \binom{r+d-2}{d-1}.$$

This polynomial has multiplicity $1+\mu(I)-d$. Also,

$$\mu(J^s) = \binom{s+g-1}{g-1} + (\mu(J)-g) \binom{s+g-2}{g-1}.$$

This polynomial has multiplicity $1+\mu(J)-g$. Hence from this it follows that the multiplicity of $F(I) \otimes_R F(J)$ is $(1+\mu(I)-d)(1+\mu(J)-g)$ (It is possible to

write out the Hilbert function of $F(I)\otimes_R F(J)$ explicitly, but is not needed here). On the other hand however, if we mod $F(I)\otimes_R F(J)$ by the images of X and Y in the total degree one component of $F(I)\otimes_R F(J)$, the resulting length is $1+\mu(I)-d+\mu(J)-g+(\mu(I)-d)(\mu(J)-g)$, which simplifies to $(1+\mu(I)-d)(1+\mu(J)-g)$. It follows that $F(I)\otimes_R F(J)$ is Cohen-Macaulay. □

Example 4.1 Let $R=k[\![x^2, x^3y, y]\!]$. Set $P=(x^2, x^3y)R$ and $Q=(x^3y, y)R$. Then by Example 1.11, it follows that P has reduction exponent one, and since $Q^2=Q(y)R$, it follows that Q has reduction exponent one. Since $P+Q=M$ where $M=(x^2, x^3y, y)R$, it follows from Theorem 4 that $G(P)\otimes_R G(Q)$ is Cohen-Macaulay of dimension 2. □

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