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Graded analytic algebras over a field of positive characteristic

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0 Introduction

The problem of characterizing quasihomogeneous analytic k-algebras is considered in the case of a field k of characteristic 0 and originates in the following criterion of Saito [6]: an isolated complex hypersurface singularity with defining equation f is quasihomogeneous if and only if $f \in j(f)$, where $j(f)$ is the ideal generated by the partial derivatives of f. This result was extended to a field k of characteristic 0 and generalized by Scheja and Wiebe [7, 8, 9].

This paper deals with the corresponding question in the case of an algebraically closed field k of characteristic $p > 0$.

Let $R = A/\mathfrak{a}$ be a quotient of a formal power series ring $A = k[[X_1, \ldots, X_n]]$. We call R G-graded, G being a subgroup of k, if R is a direct sum of k-subspaces of R $\{R_\lambda, \lambda \in G\}$ such that $R_\lambda R_\mu \subset R_{\lambda+\mu}$ for $\lambda, \mu \in G$. In particular, we say $f \in A$ is homogeneous of degree $\alpha \in k$, if there exist coordinates $Y_1, \ldots, Y_n$ and elements $\alpha_1, \ldots, \alpha_n \in k$ not all zero such that for all monomials $Y_1^{i_1} \ldots Y_n^{i_n}$ occurring in f, one has $i_1\alpha_1 + \cdots + i_n\alpha_n = \alpha$. The problem is to find necessary and sufficient conditions for R to be G-graded.

The following conjecture of J. Herzog is proved in Sect. 3: $f \in A$ is homogeneous of nonzero degree if and only if $f \in \mathfrak{m}_A j(f)$, where $\mathfrak{m}_A$ is the maximal ideal of A. In Sect. 2 it is shown that there is a correspondence between gradings and some derivations. It is proved that R is graded if and only if there exists a derivation d of R which induces a nonnilpotent map $\bar{d}$: $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$, $\mathfrak{m}$ being the maximal ideal of R. I borrow ideas from the papers [6, 7, 9].

1 Preliminaries

The notation and assumptions in this section are kept in force throughout.

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Let $(R, \mathfrak{m}, k)$ be an analytic algebra over an algebraically closed field k of characteristic $p > 0$, understood here as a quotient $A/\mathfrak{a}$ of a formal power series ring $A = k[[X_1, \ldots, X_n]]$ by an ideal $\mathfrak{a} \subset A$. Denote: $\mathfrak{m}_A$ the maximal ideal of A; φ the canonical projection $A \to R$; $\mathbb{Z}/p\mathbb{Z}$ the field with p elements.

Let G be an abelian group. As usual, a system of group homomorphisms π_λ: $R \to R$, $\lambda \in G$, is called a (*finite*) (*G*-) *grading* of R, if $\pi_\lambda(R)\,\pi_\mu(R) \subset \pi_{\lambda+\mu}(R)$ for $\lambda, \mu \in G$ and $R = \bigoplus_{\lambda \in G} \pi_\lambda(R)$. On the other hand, in the case of analytic algebras, one can define infinite gradings as follows [7, §1]. A system of group homomorphisms π_λ: $R \to R$, $\lambda \in G$, is called an *infinite* (*G*-) *grading* of R, if for each integer $i > 0$ the π_λ's induce maps π_λ: $R/\mathfrak{m}^i \to R/\mathfrak{m}^i$ which define a finite grading of $R/\mathfrak{m}^i$.

(*1.1*) *Remark.* If $R = \bigoplus_{\lambda \in G} \pi_\lambda(R)$ is G-graded where G is a $\mathbb{Z}/p\mathbb{Z}$-vector space, then R is also $\mathbb{Z}/p\mathbb{Z}$-graded.

Indeed, let $0 \neq \lambda_0 \in G$ and let H be the $\mathbb{Z}/p\mathbb{Z}$-subspace of G such that $G = \mathbb{Z}/p\mathbb{Z}\lambda_0 \oplus H$. Set $R_i = \bigoplus_{\mu \in H} \pi_{i\lambda_0+\mu}(R)$ for $i \in \mathbb{Z}/p\mathbb{Z}$. Then $R = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} R_i$ and this is a grading.

The above argument was suggested by Th. Lehmkuhl.

(*1.2*) Denote by $\mathrm{Der}(R)$ the set of all derivations d: $R \to R$. Set $R^p = \{f^p \mid f \in R\}$, and $R^d = \{f \in R \mid d(f) = 0\}$ for $d \in \mathrm{Der}(R)$. The inclusions of rings $R^d \supset R^p \supset k$ show that every $d \in \mathrm{Der}(R)$ is k-linear.

For a ring S with $R^p \subset S \subset R$, recall that a set of elements $x_1, \ldots, x_n \in R$ is said to be a p-basis of R over S if $\{x_1^{i_1} \ldots x_n^{i_n} \mid 0 \leqq i_j < p, 1 \leqq j \leqq n\}$ is a basis of the free S-module R. If ∂_j denotes "the partial derivative with respect to x_j", defined by $\partial_j(x_i) = \delta_{ij}$ (Kronecker delta), $\partial_j(S) = 0$, then the set of all S-linear derivations of R, denoted by $\mathrm{Der}_S(R)$, is the free R-module on the basis $\partial_1, \ldots, \partial_n$.

(*1.3*) Let $0 \neq d \in \mathrm{Der}(R)$. The associative subring of $\mathrm{End}_{R^d}(R)$, generated by d over the canonical image of R, is denoted $R[d]$. For a left $R[d]$-module M and a $\lambda \in R^d$, set

$$M_\lambda(d) = M_\lambda = \{b \in M \mid db = \lambda b\}\ .$$

From the product rule it follows that M_λ is an R^d-module and $R_\lambda M_\mu \subset M_{\lambda+\mu}$ for $\lambda, \mu \in R^d$. Therefore, if $R = \bigoplus_{\lambda \in G} R_\lambda$ where G is a subgroup of R^d, the ring R is (*G*-) graded, and if $M = \bigoplus_{\lambda \in G} M_\lambda$, then M is a graded R-module. In this case, we call d a (*G*-) *grading derivation* of R (resp. of M). Note that $R_0 = R^d$ and G is a $\mathbb{Z}/p\mathbb{Z}$-subspace of R_0.

(*1.4*) *Remark.* If d is a G-grading derivation of R, then every $0 \neq \lambda \in G$ is a non-zerodivisor in R. In particular, if $\mathfrak{m}$ is homogeneous and if the induced by d derivation of $R/\mathfrak{m}^i$ is grading for every $i \geqq 1$, then $G \setminus \{0\} \subset R^\times \cap R^d = (R^d)^\times$, where $(\,)^\times$ denotes the group of units of $(\,)$.

Indeed, assume for $0 \neq \lambda \in G$ and $f \in R$ one has $\lambda f = 0$. Then expressing $f = \sum_{\mu \in G} f_\mu$ with $f_\mu \in R_\mu$, one obtains $\lambda f_\mu = 0$ for every $\mu \in G$, so that $(\lambda + \mu) f_\mu = \mu f_\mu = d(f_\mu)$. Therefore $f_\mu \in R_\mu \cap R_{\lambda+\mu} = 0$.

(*1.5*) Semisimple derivations are introduced in the following way in [1] in order to study the singularities of rings of constants.

By [1, (1.2.3)], for every $d \in \mathrm{Der}(R)$ there exists a p-polynomial

$$F(X) = X^{p^s} - \sum_{i=0}^{s-1} c_i X^{p^i} \in R^d[X]$$

for which $F(d) = 0$. Any such polynomial will be called an *annihilating p-polynomial* of d [1, (1.1.4)].

A derivation $d \in \mathrm{Der}(R)$ is called *semisimple*, if it has an annihilating p-polynomial $F(X)$ whose coefficient c_0 of the linear term is invertible.

Remark. Assume R is a normal domain with a field of fractions K. Let $F_d(X)$ be the characteristic polynomial of $d \in \mathrm{Der}(R)$, considered as a K^d-linear map of K. Then $F_d(X)$ is the minimal polynomial of d, and is an annihilating p-polynomial [2, (2.4)]. Moreover, d is semisimple if and only if $F_d(X)$ has an invertible coefficient c_0 of the linear term [1, (1.8)].

We say d is a *diagonalizable derivation* of R, if the maximal ideal $\mathfrak{m}$ can be generated by $x_1, \ldots, x_n$ such that $d(x_i) = \alpha_i x_i$ with $\alpha_i \in k$ for $1 \leqq i \leqq n$.

(1.6) Examples. (1) Every diagonalizable $d \in \mathrm{Der}(R)$ is semisimple.

Proof. Let $x_1, \ldots, x_n$ be generators of $\mathfrak{m}$ with $d(x_i) = \alpha_i x_i$, $\alpha_i \in k$, $1 \leqq i \leqq n$. Choose $X_1, \ldots, X_n$ in A such that $\varphi(X_i) = x_i$, and let D be the derivation of A defined by $D(X_i) = \alpha_i X_i$, $1 \leqq i \leqq n$. By [2, (6.7)] D is semisimple with $F_D(X) = \prod_{\alpha \in W}(X - \alpha)$ where $W = (\alpha_1, \ldots, \alpha_n)$ denotes the $\mathbb{Z}/p\mathbb{Z}$–linear span of $\alpha_1, \ldots, \alpha_n$ in k. Since d is induced by D, one has $F_D(d) = 0$, so that d is also semisimple.

(2) The derivation of A from (4.0) below is semisimple but not diagonalizable.

(1.7) Lemma. *Let $D \in \mathrm{Der}(A)$ be such that there exist coordinates $X_1, \ldots, X_n$ in A with $D(X_i) = \lambda_i X_i$, $\lambda_i \in A^D$, $1 \leqq i \leqq n$. Then $F_D(X) = \prod_{\lambda \in W}(X - \lambda)$ is the minimal polynomial of D, where $W = (\lambda_1, \ldots, \lambda_n)$ denotes the $\mathbb{Z}/p\mathbb{Z}$-linear span of $\lambda_1, \ldots, \lambda_n$ in A^D. In particular, D is semisimple if and only if $\prod_{0 \neq \lambda \in W}(-\lambda)$ is invertible.*

Proof. The one given in [2, (6.7)] under the assumption that $\lambda_i \in k$ carries over in the case when $\lambda_i \in A^D$, $1 \leqq i \leqq n$.

2 Grading derivations

(2.1) Proposition. *Let $d \neq 0$ be a semisimple derivation of $R = A/\mathfrak{a}$ and let $F(X)$ be an annihilating p-polynomial of d with invertible coefficient c_0 of the linear term. Then:*

(1) *d is a G-grading derivation of R where G is a finite subgroup of R^d such that $G \setminus \{0\} \subset R^\times$.*

(2) *Every $R[d]$-module is a graded R-module.*

(3) *An ideal $\mathfrak{b} \subset R$ is homogeneous (respectively to the grading defined by d) if and only if $d(\mathfrak{b}) \subset \mathfrak{b}$.*

(4) *If $F(X) \in R^p[X]$ and $d(R) \subset \mathfrak{m}$, then d is induced by a semisimple derivation D of A.*

(5) *If $F(X) \in k[X]$ and $d(R) \subset \mathfrak{m}$, then d is diagonalizable.*

Proof. Since $F'(X) = -c_0 \in R^\times$ and k is algebraically closed, $F(X)$ decomposes over $k[X]$ into a product of p^s distinct linear forms. But $(R^d, \mathfrak{m} \cap R^d, k)$ being complete, by Hensel's lemma $F(X)$ decomposes also in $R^d[X]$: $F(X) = \prod_{i=1}^{p^s}(X - \lambda_i)$ with $\lambda_i - \lambda_j$ invertible for $i \neq j$. Setting $P_i(X) = F(X)/(X - \lambda_i)$, one has $-c_0 = F'(X) = \sum_{i=1}^{p^s} P_i(X)$. Then $\rho_i = -c_0^{-1} P_i(d)$ for $1 \leqq i \leqq p^s$ are projections, and $M = \bigoplus_{i=1}^{p^s} \rho_i(M)$ for every $R[d]$-module M. It remains to show $\rho_i(M) = M_{\lambda_i}$. As $(d - \lambda_i)\rho_i = -c_0^{-1} F(d) = 0$, one has $\rho_i(M) \subseteq M_{\lambda_i}$. On the other hand, $M_{\lambda_i} \cap M_{\lambda_j} = 0$ for $i \neq j$, since if $x \in M_{\lambda_i} \cap M_{\lambda_j}$, then $\lambda_i x = d(x) = \lambda_j x$, and

$\lambda_i - \lambda_j$ being invertible, one obtains $x = 0$. Therefore $M = \bigoplus_{\lambda \in G} M_\lambda$ where $G = \{\lambda \in R^d \mid F(\lambda) = 0\}$, and in particular $R = \bigoplus_{\lambda \in G} R_\lambda$. It remains to note that $F(X)$ being a p-polynomial, the set G is a $\mathbb{Z}/p\mathbb{Z}$-vector space. This proves (1)–(3).

(4) By assumption $d(\mathfrak{m}) \subset \mathfrak{m}$, so that according to (3) $\mathfrak{m}$ is homogeneous. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ with $d(x_i) = \lambda_i x_i$, $\lambda_i \in G$. Let $\bar{W} = (\lambda_1 \ldots, \lambda_n)$ and let $\mu_1, \ldots, \mu_t$ be a basis of $\bar{W}$. Choose $\beta_i \in A^p$ such that $\varphi(\beta_i) = \mu_i$, $1 \leqq i \leqq t$, and set $W = (\beta_1, \ldots, \beta_t)$. Then W is a $\mathbb{Z}/p\mathbb{Z}$-vector space isomorphic to $\bar{W}$. Let $\alpha_i \in W$ be such that $\varphi(\alpha_i) = \lambda_i$, and let $X_i \in A$ be such that $\varphi(X_i) = x_i$ for $1 \leqq i \leqq n$. Let D be the derivation of A defined by $D(X_i) = \alpha_i X_i$, $1 \leqq i \leqq n$. Then $\varphi(D(f)) = d(\varphi(f))$ for $f \in A$, i.e. D induces d. Since $\alpha_i \in A^p \subset A^D$, $1 \leqq i \leqq n$, and $W \setminus \{0\} \subset A^\times$, by (1.7) D is semisimple.

(5) In this case $\mathfrak{m} = (x_1, \ldots, x_n)$ with $d(x_i) = \alpha_i x_i$ where $\alpha_i \in k$, $1 \leqq i \leqq n$, so that by definition d is diagonalizable.

Remarks. (1) The derivation d being nonzero, one has $R_0 = R^d \neq R$, therefore the grading defined by d is not trivial. Moreover, in the case when R is a normal domain, taking $F(X)$ equal to the minimal polynomial of d (1.5), the proof of [2, (5.4)] shows that $R_\lambda \neq 0$ with $\lambda \in R^d$ if and only if $\lambda \in G$.

(2) Since the derivation $D \in \mathrm{Der}(A)$ from (4) is semisimple, it defines a W-grading of A. As $D(\mathfrak{a}) \subset \mathfrak{a}$, by (3) the ideal $\mathfrak{a}$ is homogeneous. Therefore the quotient R has the induced grading: $R = \bigoplus_{\alpha \in W} A_\alpha(D)/\mathfrak{a}_\alpha(D)$. But as $A_\alpha(D)/\mathfrak{a}_\alpha(D) \subseteq R_{\varphi(\alpha)}(d)$ for $\alpha \in W$, one obtains $R = \bigoplus_{\alpha \in W} R_{\varphi(\alpha)}$ and $A_\alpha(D)/\mathfrak{a}_\alpha(D) = R_{\varphi(\alpha)}$ for $\alpha \in W$.

(2.2) Proposition. *Let $R = \bigoplus_{\lambda \in G} \pi_\lambda(R)$ be a G-grading such that*

(i) $G \subset \pi_0(R)$, *and*
(ii) *every $0 \neq \lambda \in G$ is a nonzero divisor in R.*

Then there exists a unique derivation d of R such that $\pi_\lambda(R) = R_\lambda(d)$ for $\lambda \in G$.

Proof. For $g \in R$ define a map d: $R \to R$ by $d(g) = \sum_{\lambda \in G} \lambda \pi_\lambda(g)$. The π_λ's being group homomorphisms, d is additive. Checking the product rule, one obtains that d is a derivation. It remains to show that $\pi_\lambda(R) = R_\lambda$, $\lambda \in G$. If $f \in \pi_\lambda(R)$, then $f = \pi_\lambda(f)$, so that $d(f) = \lambda \pi_\lambda(f) = \lambda f$ and $f \in R_\lambda$. Conversely, if $f \in R_\lambda$, then expressing $f = \sum_{\mu \in G} \pi_\mu(f)$, one obtains $\sum_{\mu \in G} \lambda \pi_\mu(f) = \lambda f = d(f) = \sum_{\mu \in G} \mu \pi_\mu(f)$. Since $G \subset \pi_0(R)$, it follows $\lambda \pi_\mu(f) = \mu \pi_\mu(f)$ for every $\mu \in G$. By (ii) one obtains $\pi_\mu(f) = 0$ for $\mu \neq \lambda$, hence $f = \pi_\lambda(f) \in \pi_\lambda(R)$.

(2.3) Lemma. *If $d \in \mathrm{Der}(R)$ is semisimple G-grading, then the derivation $d^p \in \mathrm{Der}(R)$ is semisimple G^p-grading with $R_\lambda(d) = R_{\lambda^p}(d^p)$ for $\lambda \in G$, where $G^p = \{\lambda^p \mid \lambda \in G\}$ is isomorphic to G.*

Proof. Let $F_d(X) = X^{p^s} - \sum_{i=0}^{s-1} c_i X^{p^i} = \prod_{\lambda \in G} (X - \lambda)$ be an annihilating p-polynomial of d with $c_0 \in R^\times$. Then $F_{d^p}(X) = X^{p^s} - \sum_{i=0}^{s-1} c_i^p X^{p^i}$ is an annihilating p-polynomial of d^p with $c_0^p \in (R^p)^\times$, hence d^p is semisimple. Since $G \setminus \{0\} \subset R^\times$, the Frobenius homomorphism $G \to G^p$, $\lambda \mapsto \lambda^p$, is an isomorphism. It remains to show $R_\lambda(d) = R_{\lambda^p}(d^p)$ for $\lambda \in G$. If $x \in R_\lambda(d)$, as $\lambda \in R^d$, the equality $d(x) = \lambda x$ implies $d^p(x) = \lambda^p x$, therefore $R_\lambda(d) \subseteq R_{\lambda^p}(d^p)$, $\lambda \in G$. Since $R = \bigoplus_{\lambda \in G} R_{\lambda^p}(d^p)$, it follows that $R_\lambda(d) = R_{\lambda^p}(d^p)$.

(2.4) Proposition. *Let π_λ: $R \to R$, $\lambda \in G$, be an infinite grading of R such that*

(i) $G \subset \pi_0(R)$, *and*
(ii) $G \setminus \{0\} \subset R^\times$.

Then there exists a (unique) semisimple derivation d of R such that $\pi_\lambda(R) = R_{\lambda^p}(d)$ for $\lambda \in G$. Moreover d is induced by a semisimple derivation D of A. In particular, the grading π_λ, $\lambda \in G$ is finite: $R = \bigoplus_{\lambda \in \bar{W}} \pi_\lambda(R)$ where $\bar{W}$ is a finite subgroup of G.

Proof. The ideal $\mathfrak{m}$ being homogeneous, let $\mathfrak{m} = (x_1, \ldots, x_n)$ with $x_i \in \pi_{\lambda_i}(R)$, $1 \leqq i \leqq n$. Let $\bar{W} = (\lambda_1^p, \ldots, \lambda_n^p)$ and let $W \subset A^p$ be a $\mathbb{Z}/p\mathbb{Z}$-vector space isomorphic by φ to $\bar{W}$ (cf. the proof of (2.1.4)). Let $a_i \in W$ be such that $\varphi(a_i) = \lambda_i^p$ and let $X_i \in A$ be such that $\varphi(X_i) = x_i$. Consider the derivation $D \in \mathrm{Der}(A)$ defined by $D(X_i) = a_i X_i$, $1 \leqq i \leqq n$. For every $j \geqq 1$, the ring $\bar{R} = R/\mathfrak{m}^j$ being finitely G-graded with $G \subsetneq \pi_0(R)$, by (2.2) there exists a unique $d_j \in \mathrm{Der}(\bar{R})$ such that $\pi_\lambda(\bar{R}) = \bar{R}_\lambda(d_j)$. By (2.3) $\bar{R}_\lambda(d_j) = \bar{R}_{\lambda^p}(d_j^p)$ for $\lambda \in G$. Since $\bar{x}_i = x_i \bmod \mathfrak{m}^j \in \pi_{\lambda_i}(\bar{R})$, one has $d_j^p(\bar{x}_i) = \lambda_i^p \bar{x}_i = D(X_i) \bmod \mathfrak{m}^j$, therefore d_j^p is induced by D. It follows that $D(\mathfrak{a} + \mathfrak{m}^j) \subset \mathfrak{a} + \mathfrak{m}^j$ for every j, which implies $D(\mathfrak{a}) \subseteq \mathfrak{a}$. Hence D induces a derivation d of R with $d(x_i) = \lambda_i^p x_i$, $1 \leqq i \leqq n$. Since $\pi_\lambda(R/\mathfrak{m}^j) = (R/\mathfrak{m}^j)_{\lambda^p}(d)$, $\lambda \in G$, for every $j \geqq 1$, one obtains $\pi_\lambda(R) = R_{\lambda^p}(d)$. As $W \setminus \{0\} \subset A^\times$, by (1.7) the derivation D is semisimple, so that d is also semisimple.

Remarks. The condition (ii) is fulfilled if $G \subset k \subset \pi_0(R)$. If (ii) does not hold, by (1.4) there is no derivation defining this grading.

Note that it follows $\pi_0(R) \supset R^p \supset k$.

(*2.5*) *Chevalley decomposition.* We say $d \in \mathrm{Der}(R)$ has a Chevalley decomposition, if $d = d_s + d_n$ where $d_s, d_n \in \mathrm{Der}(R)$ with $d_s(\mathfrak{m}) \subset \mathfrak{m}$, $d_n(\mathfrak{m}) \subset \mathfrak{m}$, and such that the induced by d_n map $\bar{d}_n$: $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is nilpotent, d_s is diagonalizable, and $[d_s, d_n] = 0$.

The ring R being complete, every $d \in \mathrm{Der}(\mathrm{R})$ with $d(\mathfrak{m}) \subset \mathfrak{m}$ has a unique Chevalley decomposition. This is proved in [9, (1.2)] in the case of a field of characteristic 0, using arguments which do not depend on this assumption.

(2.6) Proposition. *The following conditions are equivalent for $R = A/\mathfrak{a}$:*

(i) *there exists a diagonalizable derivation $0 \neq d \in \mathrm{Der}(R)$;*
(ii) *there exists a semisimple derivation $0 \neq d \in \mathrm{Der}(R)$;*
(iii) *there exists a grading derivation $0 \neq d \in \mathrm{Der}(R)$;*
(iv) *R is $\mathbb{Z}/p\mathbb{Z}$-graded;*
(v) *there exists a derivation $0 \neq D \in \mathrm{Der}(A)$ such that $D^p = D$, $D(\mathfrak{m}_A) \subset \mathfrak{m}_A$, and $D(\mathfrak{a}) \subset \mathfrak{a}$;*
(vi) *there exists a derivation $d \in \mathrm{Der}(R)$ with $d(\mathfrak{m}) \subset \mathfrak{m}$ such that it induces a nonnilpotent map $\bar{d}$: $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$.*

Proof. (i) $\Rightarrow$ (ii) by (1.6.1). (ii) $\Rightarrow$ (iii) by (2.1.1). (iii) $\Rightarrow$ (iv) by (1.1).

(iv) $\Rightarrow$ (v) Let $R = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} \pi_i(R)$ be $\mathbb{Z}/p\mathbb{Z}$-graded. Consider the derivation $\delta \in \mathrm{Der}(R)$ defined by $\delta(f) = \sum_{i \in \mathbb{Z}/p\mathbb{Z}} i\pi_i(f)$, $f \in R$ (cf. (2.2)). Then $\delta^p = \delta$. We shall show that there exists a $0 \neq d \in \mathrm{Der}(R)$ with $d^p = d$ and such that $d(\mathfrak{m}) \subset \mathfrak{m}$. If $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$, then take $d = \delta$. Otherwise, there exists an $x \in \mathfrak{m}$ such that $\delta(x) = u \in R^\times$. (Example 4.2 below shows that this case is possible.) According to [1, (2.1), (2.3)], $(u^{-1}\delta)^p = 0$ and $\{x\}$ is a p-basis of R over R^δ (cf. (1.2)). Consider the derivation $d \in \mathrm{Der}_{R^\delta}(R)$ defined by $d(x) = x$. Then $d^p = d \neq 0$ and $d(\mathfrak{m}) \subset \mathfrak{m}$. By (2.1.5) d is diagonalizable. By (2.1.4) d is induced by $D \in \mathrm{Der}(A)$ such that $F_D(X) = X^p - X$. Hence D is the required derivation.

(v) $\Rightarrow$ (vi) Since D induces a diagonalizable $d \in \mathrm{Der}(R)$ (2.1.5), $\bar{d}$ is nonnilpotent.

(vi) $\Rightarrow$ (i) Let $d = d_s + d_n$ be the Chevalley decomposition of d (2.5). Since $\bar{d}$ is nonnilpotent, $d_s \neq 0$ so that d_s is the required derivation.

Remark. Note that not every grading derivation is semisimple (cf. (4.3) below).

(*2.7*) *Remark.* Under the assumptions of (2.1), assume $d(\mathfrak{m}) \subset \mathfrak{m}$ and let $d = d_s + d_n$ be the Chevalley decomposition of d. Then $R_\lambda(d) = R_{\bar{\lambda}}(d_s)$ for $\lambda \in G$, where $\bar{\lambda} = \lambda \bmod \mathfrak{m}$.

Proof. First we shall show that this is true for $D \in \mathrm{Der}(A)$ semisimple such that $W = \{\lambda \in A^D \mid F_D(\lambda) = 0\} \subset A^p$, $F_D(X)$ being the minimal polynomial of D. Let $X_1, \ldots, X_n$ be coordinates such that $D(X_i) = \lambda_i X_i$, $\lambda_i \in W$, $1 \leqq i \leqq n$ (cf. the beginning of the proof of (2.1.4)). Express $\lambda_i = \alpha_i + g_i$ with $\alpha_i \in k$ and $g_i \in \mathfrak{m}_A \cap A^p$, $1 \leqq i \leqq n$. Set $D_s = \sum_{i=1}^n \alpha_i X_i \partial/\partial X_i$ and $D_n = \sum_{i=1}^n g_i X_i \partial/\partial X_i$. Then $D = D_s + D_n$ is the Chevalley decomposition of D. By (1.7) $F_{D_s}(X) = \prod_{\alpha \in \bar{W}}(X - \alpha)$ is the minimal polynomial of D_s, where $\bar{W} = (\alpha_1, \ldots, \alpha_n)$. Note that D being semisimple, the vector space $\bar{W}$ is isomorphic to W. Let $f \in A$ and express f in the p-basis $X_1, \ldots, X_n$ of A over A^p (1.2): $f = \sum_{0 \leqq j_i < p} f_{j_1, \ldots, j_n} X_1^{j_1} \ldots X_n^{j_n}$ with $f_{j_1 \ldots j_n} \in A^p$. Then one has: $D(f) = \lambda f$, $\lambda \in W$ if and only if $j_1\lambda_1 + \cdots + j_n\lambda_n = \lambda$ for all monomials $X_1^{j_1} \ldots X_n^{j_n}$ occurring in f; if and only if $j_1\alpha_1 + \cdots + j_n\alpha_n = \bar{\lambda}$ for all monomials $X_1^{j_1} \ldots X_n^{j_n}$ in f; if and only if $D_s(f) = \bar{\lambda} f$. Hence $A_\lambda(D) = A_{\bar{\lambda}}(D_s)$ for $\lambda \in W$.

Let now $d \in \mathrm{Der}(R)$ be semisimple. By (2.3) $R_\lambda(d) = R_{\lambda^p}(d^p)$. Since $[d_s, d_n] = 0$, one has $d^p = d_s^p + d_n^p$. Hence we can assume $G \subset R^p$. Then by (2.1.4) d is induced by $D \in \mathrm{Der}(A)$. Let $D = D_s + D_n$ be the Chevalley decomposition of D. Since $D(\mathfrak{a}) \subset \mathfrak{a}$, one has $D_s(\mathfrak{a}) \subset \mathfrak{a}$ and $D_n(\mathfrak{a}) \subset \mathfrak{a}$, so that D_s induces d_s. Then $R_\lambda(d) = A_\lambda(D)/\mathfrak{a}_\lambda(D) = A_{\bar{\lambda}}(D_s)/\mathfrak{a}_{\bar{\lambda}}(D_s) = R_{\bar{\lambda}}(d_s)$ for $\lambda \in W = \{\lambda \in A^p \mid F_D(\lambda) = 0\}$.

3 Homogeneous hypersurfaces

Set $j(f) = (\partial f/\partial X_1, \ldots, \partial f/\partial X_n)$ for the ideal generated by the partial derivatives of $f \in A = k[[X_1, \ldots, X_n]]$.

From now on we use the following terminology.

We call $f \in A$ *homogeneous of degree* $\alpha \in k$, if there exist coordinates $X_1, \ldots, X_n$ and elements $\alpha_1, \ldots, \alpha_n \in k$ not all zero such that for all monomials $X_1^{i_1} \ldots X_n^{i_n}$ occurring in f with nonzero coefficients, one has $i_1\alpha_1 + \cdots + i_n\alpha_n = \alpha$.

We call the quotient A/fA *graded*, if there exists a diagonalizable $0 \neq d \in \mathrm{Der}(A)$ such that $d(f) \in fA$ (cf. (3.1) below).

(*3.1*) *Remark.* The following are equivalent:

(i) $f \in A$ is homogeneous of degree $\alpha \in k$;

(ii) there exists a diagonalizable $0 \neq d \in \mathrm{Der}(A)$ such that $d(f) = \alpha f$ with $\alpha \in k$;

(iii) there exists a derivation $0 \neq D \in \mathrm{Der}(A)$ with $D(A) \subset \mathfrak{m}_A$ and $D^p = D$ such that $D(f) = jf$ for some $j \in \mathbb{Z}/p\mathbb{Z}$.

Indeed, if (i) holds, setting $d = \sum_{i=1}^n \alpha_i X_i \partial/\partial X_i \in \mathrm{Der}(A)$, one obtains (ii). (ii) $\Rightarrow$ (i) follows from the product rule for d.

(ii) $\Rightarrow$ (iii) Since d is grading (2.6.iii), by (1.1) it defines a $\mathbb{Z}/p\mathbb{Z}$-grading of A with respect to which $\mathfrak{m}$ is homogeneous. This $\mathbb{Z}/p\mathbb{Z}$-grading defines a derivation D of

A with $D^p = D$ (cf. (2.6) the proof of (iv) $\Rightarrow$ (v)) for which $D(f) = jf$, $j \in \mathbb{Z}/p\mathbb{Z}$, and $D(A) \subset \mathfrak{m}_A$.

(iii) $\Rightarrow$ (ii) follows from (2.1.5).

(3.2) Theorem. *A power series $f \in A$ is homogeneous of nonzero degree if and only if $f \in \mathfrak{m}_A j(f)$.*

Remarks. (1) In characteristic 0, if $f \in A$ is with isolated singularity and $f \in j(f)$, then $f \in \mathfrak{m}_A j(f)$ [6, (4.2)]. However, (4.2) below shows that this is not true in characteristic p.

(2) Example 4.2 also shows that A/fA can be graded with f nonhomogeneous, which is not possible in characteristic 0 [7, (5.4)].

(3.3) Lemma. *Let $d \in \mathrm{Der}(A)$ be such that it induces a nilpotent map $\bar{d}$: $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$. If $d(f) = \lambda f$ with $\lambda \in k$ and $0 \neq f \in A$, then $\lambda = 0$.*

Proof. Let $X_1, \ldots, X_n$ be coordinates in A. Decompose $d = d_1 + d_2$ with $d_1, d_2 \in \mathrm{Der}(A)$ such that $d_1(X_i)$, $1 \leqq i \leqq n$, are linear forms, and $d_2(A) \subset \mathfrak{m}_A^2$. Express $f = \sum_{j \geqq s} f_j$ with f_j homogeneous polynomial of degree j ($\deg X_i = 1$) and $f_s \neq 0$. Then the equality $d(f) = \lambda f$ implies $d_1(f_s) = \lambda f_s$. Since $\bar{d}$ is nilpotent, d_1 induces a nilpotent map in $\mathfrak{m}_A/\mathfrak{m}_A^2$. Therefore $d_1^{p^t}(X_i) = 0$ for some $t \geqq 0$ for $1 \leqq i \leqq n$. Since $d_1^{p^t}$ is a derivation of A, it follows that $d_1^{p^t} = 0$. Hence $0 = d_1^{p^t}(f_s) = \lambda^{p^t} f_s$ which implies $\lambda = 0$.

(3.4) Lemma [6, (3.2)] *Let $d = d_s + d_n$ be the Chevalley decompositin of $d \in \mathrm{Der}(A)$. Let $f \in A$ and $\lambda \in k$. Then $d(f) = \lambda f$ if and only if $d_s(f) = \lambda f$, $d_n(f) = 0$.*

Proof. The proof given in [6, (3.2)] for the case of characteristic 0 carries over in the case of characteristic p. Indeed, let $d(f) = \lambda(f)$. Express $f = \sum_{\mu \in k} f_\mu$ with $f_\mu \in A_\mu(d_s)$. Then $0 = \sum_\mu d(f_\mu) - \lambda f_\mu = \sum_\mu (\mu - \lambda) f_\mu + d_n(f_\mu)$. But since $[d_s, d_n] = 0$, $d_n(f_\mu)$ is homogeneous of degree μ. Therefore $(\mu - \lambda) f_\mu + d_n(f_\mu) = 0$ for every μ. If $f_\mu \neq 0$, by (3.3) $\lambda = \mu$. Hence $f = f_\lambda$, so that $d_s(f) = \lambda f$.

Proof of (3.2) If f is homogeneous of degree $\alpha \neq 0$, then $d(f) = \alpha f$ for $d \in \mathrm{Der}(A)$ with $d(A) \subset \mathfrak{m}_A$, therefore $f = \alpha^{-1} d(f) \in \mathfrak{m}_A j(f)$.

Assume $f \in \mathfrak{m}_A j(f)$. Then $f = \sum_{i=1}^n a_i \partial f/\partial X_i$ with $a_i \in \mathfrak{m}_A$, $1 \leqq i \leqq n$. Setting $d = \sum_{i=1}^n a_i \partial/\partial X_i$, one has $d(f) = f$ and $d(A) \subset \mathfrak{m}_A$. Let $d = d_s + d_n$ be the Chevalley decomposition of the derivation d. By (3.4) $d_s(f) = f$, i.e. f is homogeneous of degree 1.

Remark. Note that the equality $d(f) = f$ with $f \neq 0$, $d \in \mathrm{Der}(A)$ does not imply that d is semisimple (cf. (4.6) below).

(3.5) Denote by $\mu(\mathfrak{b})$ the minimal number of generators of an ideal $\mathfrak{b} \subset A$.

Proposition. *Let $f \in A = k[[X_1, \ldots, X_n]]$.*

(1) *If $\mu(j(f)) < n$, then f is homogeneous of degree 0.*
(2) *If $\mu((f, j(f)) \leqq n$, then A/fA is graded.*

Proof. (1) Let $\partial f/\partial X_1 \in (\partial f/\partial X_2, \ldots, \partial f/\partial X_n)$. Then $\partial f/\partial X_1 = \sum_{i=2}^n a_i \partial f/\partial X_i$, $a_i \in A$, so that setting $d = X_1 \partial/\partial X_1 - \sum_{i=2}^n a_i X_i \partial/\partial X_i$, one has $d(f) = 0$. Let $d = d_s + d_n$ be the Chevalley decomposition of d. By (3.4) $d_s(f) = 0$. It remains to note that $\bar{d}$: $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is nonnilpotent, so that $d_s \neq 0$.

(2) If $f \in \mathfrak{m}_A j(f)$, by (3.2) f is homogeneous, so that A/fA is graded. Let $f \notin \mathfrak{m}_A j(f)$. We can assume that $(f, j(f)) = (f, \partial f/\partial X_2, \ldots \partial f/\partial X_n)$. Then $\partial f/\partial X_1 = af + \sum_{i=2}^{n} a_i \partial f/\partial X_i$ with $a, a_i \in A$. Set $D = X_1 \partial/\partial X_1 - \sum_{i=2}^{n} a_i X_1 \partial/\partial X_i$. Then $D(f) = aX_1 f$, so that D induces $d \in \mathrm{Der}(A/fA)$. Let $d = d_s + d_n$ be the Chevalley decomposition of d. Since $\bar{d}$ is nonnilpotent, $d_s \neq 0$, so that A/fA is graded.

(*3.6*) *Remarks.* (1) If $f \in \mathfrak{m}_A j(f)$, then for every $u \in A^\times$ one has $g = uf \notin \mathfrak{m}_A j(g)$. Indeed, assuming $g \in \mathfrak{m}_A j(g)$, there exists a $d \in \mathrm{Der}(A)$ with $d(A) \subset \mathfrak{m}_A$ such that $d(g) = g$ (cf. the proof of (3.2)). Therefore $uf = d(u)f + ud(f)$, so that $(1 - u^{-1}d(u))f = d(f)$. Since $(1 - u^{-1}d(u)) \in A^\times$, one obtains $f \in \mathfrak{m}_A j(f)$ which is a contradiction.

(2) Let $\mu((f, j(f)) < n$, $\mu(j(f)) = n$ and $f \notin \mathfrak{m}_A j(f)$. We can assume $(f, j(f)) = (f, \partial f/\partial X_1, \ldots, \partial f/\partial X_{n-2})$. Set $u = 1 + X_{n-1}$. Then $\mu(j(g)) < n$, so that by (3.5.1) g is homogeneous of degree 0.

Note that by (4.4) below this case is possible.

(3) Example (4.5) below shows that conditions (3.5.1) and (3.5.2) are not necessary.

(*3.7*) *Remark.* Assume that $f \in A$ is with isolated singularity.

In characteristic 0, A/fA is quasihomogeneous if and only if $A/(f, j(f))$ is Gorenstein. This follows from [5, (6.1); 3; 6], cf. also [4].

In characteristic p, if f is homogeneous of nonzero degree, then by (3.2) $A/(f, j(f)) = A/j(f)$ which is a complete intersection. However, (4.5) below shows that A/fA graded does not imply $A/(f, j(f))$ Gorenstein. On the other hand, if $A/(f, j(f))$ is Gorenstein, then A/fA is graded: indeed, since $\dim A/(f, j(f)) = 0$, by [3, (1.2)] $\mu((f, j(f)) \leqq n$, so that by (3.5.2) A/fA is graded. Note that in this case f can be nonhomogeneous (cf. (4.2) below).

4 Examples

(*4.0*) Let $\alpha_1, \ldots, \alpha_n \in k$ be linearly independent over $\mathbb{Z}/p\mathbb{Z}$, and let d be the derivation of $A = k[[X_1, \ldots, X_n]]$ defined by $d(X_i) = (\alpha_i + X_i^p)X_i$, $1 \leqq i \leqq n$. Then d is semisimple but not diagonalizable.

Indeed, by (1.7) d is semisimple with $F_d(X) = \prod_{\lambda \in W}(X - \lambda)$, $W = (\alpha_1 + X_1^p, \ldots, \alpha_n + X_n^p)$. If we assume that there exist coordinates $Y_1, \ldots, Y_n$ such that $d(Y_i) = \beta_i Y_i$ with $\beta_i \in k$, $1 \leqq i \leqq n$, then again by (1.7) $F_d(X) = \prod_{\beta \in V}(X - \beta)$ where $V = (\beta_1, \ldots, \beta_n)$ which is a contradiction.

From now on we consider the case $n = 2$, and we set:

$$A = k[[X, Y]]; \quad f_X = \partial f/\partial X, \quad f_Y = \partial f/\partial Y \quad \text{for } f \in A.$$

(4.1) Proposition. *Let $f \in A$ and let t be the greatest common divisor of f_X and f_Y. Then f is homogeneous of degree 0 if and only if one of the following conditions holds:*

(i) $\mu(j(f)) \leqq 1$;
(ii) $f_X = rt$, $f_Y = st$ *with* $(r, s) = \mathfrak{m}_A$.

Proof. Assume f is homogeneous of degree 0. Then there exist a $d \in \mathrm{Der}(A)$ and coordinates X_1, X_2 such that $d(X_1) = \alpha X_1$, $d(X_2) = \beta X_2$ with $\alpha, \beta \in k$, and

$d(f) = 0$. Let $\mu(j(f)) = 2$. Then from $0 = d(f) = \alpha X_1 f_{X_1} + \beta X_2 f_{X_2}$, $\alpha, \beta \neq 0$ it follows (ii). Note that condition (ii) does not depend on the choice of coordinates.

By (3.5.1) condition (i) implies that f is homogeneous of degree 0. Assume (ii) holds. Let $d \in \mathrm{Der}(A)$ be the derivation defined by $d(X) = s$, $d(Y) = -r$. Then $d(f) = 0$. Let $d = d_s + d_n$ be the Chevalley decomposition of d. Since $\bar{d}$ is not nilpotent, $d_s \neq 0$. By (3.4) $d_s(f) = 0$, so that f is homogeneous of degree 0.

(*4.2*) Let $f = Y^2 + X^2 Y + X^3$, and let $p = 3$.

Then f is with isolated singularity and $f = (2 + X) f_X + (X + 2Y) f_Y \in j(f) \setminus \mathfrak{m}_A j(f)$.

By (3.2) and (4.1) f is not homogeneous.

The quotient A/fA is graded (3.5.2): take $d \in \mathrm{Der}(A)$ with $d(X) = X - X^2$, $d(Y) = XY - X^2$. Then $d^3 = d$ and $d(f) = -Xf$, so that d induces a diagonalizable derivation of A/fA. Note that taking $g = uf$ with $u = 1 + X + X^2$, one has $d(g) = 0$, therefore the ideal $fA = gA$ is homogeneous.

On the other hand, consider $D \in \mathrm{Der}(A)$ such that $D(X) = 2 + X$, $D(Y) = X + 2Y$. Then $D(f) = f$ and $D^p = D$, so that D induces a derivation of A/fA. Therefore $\{X \bmod fA\}$ is a p-basis of A/fA over $(A/fA)^D$ (cf. (2.6), (iv) $\Rightarrow$ (v)).

(*4.3*) Let $d \in \mathrm{Der}(A)$: $d(X) = X^{p+1}$, $d(Y) = Y^{p+1}$. Then d is a grading derivation of A, but not semisimple.

Since $d(A) \subset \mathfrak{m}^2$ d is not semisimple.

By (1.7) the minimal polynomial of d $F_d(z) = \prod_{\lambda \in W}(z - \lambda)$ where $W = (X^p, Y^p)$. Expressing any $f \in A$ in the p-basis X, Y: $f = \sum_{0 \leq i,j < p} f_{ij} X^i Y^j$ with $f_{ij} \in A^p$, one sees that $f = \sum_{\lambda \in W} f_\lambda$ where $f_\lambda = \sum_{iX^p + jY^p = \lambda} f_{ij} X^i Y^j$ belongs to A_λ. Therefore $A = \bigoplus_{\lambda \in W} A_\lambda$.

Note that for the ideal $\mathfrak{a} = ((X - Y)^2)$ one has $d(\mathfrak{a}) \subset \mathfrak{a}$, however $\mathfrak{a}$ is not homogeneous with respect to d.

Taking $f = XY$, since f is homogeneous respectively to d, $R = A/fA$ has the induced grading. However the induced by d derivation of R is not grading: $y \in R_{y^p} \cap R_{x^p + y^p}$ where $y = Y \bmod fA$, $x = X \bmod fA$.

(*4.4*) Let $0 \neq g \in A^p$, $u = 1 + Y^p X + X^p Y$, $f = ug$. Then $f \notin j(f)$, $\mu(j(f)) = 2$, and $(f, j(f)) = (g)$.

(*4.5*) Let $g = X^p + XY^{p-1}$, $p > 2$.

By (4.1.ii) g is homogeneous of degree 0. Moreover $\mu((g, g_X, g_Y)) = 3$ and g is with isolated singularity (i.e. (g, g_X, g_Y) is $\mathfrak{m}_A$-primary). By [3, (1.2)] $A/(g, g_X, g_Y)$ is not Gorenstein.

(*4.6*) Let $d \in \mathrm{Der}(A)$: $d(X) = (1 + X^p)X$, $d(Y) = X^p Y$. Then by (1.7) $F_d(z) = \prod_{\lambda \in W}(z - \lambda)$ where $W = (1 + X^p, X^p)$, therefore d is not semisimple. However for $f = XY^{p-1}$ one has $d(f) = f$.

(*4.7*) $R = A/fA$ is not graded in the following cases:

(1) $f = X^3 + Y^3 X + X^3 Y + X^2 Y^2$, $p = 3$;
(2) $f = X^3 + XY^{q-1} + Y^{q+1}$, $p > 3$, $q \equiv |5(p)$, $q \geqq 6$;
(3) $f = X^5 + Y^5 + X^3 Y^3$, $p \neq 5$.

We shall show in the case (1). The other cases are proved similarly.

One has $f_X = 2Y^2 t, f_Y = X^2 t$ where $t = X + 2Y$. Assume $d \in \mathrm{Der}(A)$ is such that $d(f) = af$ for some $a \in A$ and $d(A) \subset \mathfrak{m}_A$. Then $af = d(f) = 2Y^2 t d(X) + X^2 t d(Y)$

$\in(t)$. Since $f \notin (t)$, one has $a = bt$ for some $b \in A$, so that $2Y^2 d(X) + X^2 d(Y) = bf$. Express $d(X) = \sum_{i \geqq 1} r_i$, $d(Y) = \sum_{i \geqq 1} s_i$, $b = \sum_{i \geqq 0} b_i$ with r_i, s_i, b_i homogeneous polynomials of degree i ($\deg X = \deg Y = 1$). Then one obtains $2Y^2 r_1 + X^2 s_1 = b_0 X^3$ which implies $r_1 = 0$ and $s_1 = b_0 X$. Hence $\bar{d}$ is nilpotent, and therefore by (2.6.vi) R is not graded.

(*4.8*) Let $f = X^p Y + X Y^{p+1}$.

Then f is with isolated singularity.

Since $f = Yf_Y$, f is homogeneous of degree 1 with respect to $d \in \mathrm{Der}(A)$: $d(X) = 0$, $d(Y) = Y$, so that A/fA is graded. In characteristic 0, if R is an analytic k-algebra with $\dim_k \mathfrak{m}/\mathfrak{m}^2 \leqq 2$, and if there exists a nonzero diagonalizable derivation of R, then there exists a diagonalizable $\delta \in \mathrm{Der}(R)$ such that $\bar{\delta}$ is an isomorphism [7, (3.3)]. However, in characteristic p, taking $R = A/fA$, for every $\delta \in \mathrm{Der}(R)$ with $\delta(\mathfrak{m}) \subset \mathfrak{m}$ one has $\mathrm{rk}\,\bar{\delta} < 2$. Indeed, let $D \in \mathrm{Der}(A)$ be such that $D(f) = af$, $a \in A$, and $D(A) \subset \mathfrak{m}_A$. Then $aYf_Y = af = f_X D(X) + f_Y D(Y)$. Since f_X, f_Y is a regular sequence, one has $D(X) \in (f_Y) \subset \mathfrak{m}_A^2$, therefore $\mathrm{rk}\,\bar{D} < 2$.

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