Werk

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Similarly, we can replace $\Sigma^r D_{2j-1}C_{n+1}S^r$ by $\Sigma^{2r}D_{2j-2}C_{n+1}S^r$ in the cofibre sequence in Theorem 8.2.

Proposition 8.4 *For $r \geqq 3$, $\xi(1, S^r)$ and C_2S^{2r-1} are 2-equivalent.*

Proof. Consider the EHP-sequence that arises from

$$S^r \xrightarrow{E} \Omega S^{r+1} \xrightarrow{H} \Omega S^{2r+1},$$

which is a 2-local fibration. There is a 2-equivalence from $W(1, S^r)$, which is the fibre of E, to $\Omega^2 S^{2r+1}$. The proposition follows from Theorem 5.4 and Proposition 1.8. □

8.5 The Brown–Gitler spectrum $B(j)$ has mod 2 cohomology [BG]

$$H^*(B(j); Z/2) \cong \mathscr{A}/\mathscr{A}\{\chi(Sq^i) | i > j\}$$

where $\mathscr{A}$ is the mod 2 Steenrod algebra and $\chi: \mathscr{A} \to \mathscr{A}$ is the canonical anti-automorphism.

It is shown in [Ma, BC] that at the prime 2, the spaces $D_{2j}C_2S^{2r-1}$ are representatives of the Brown–Gitler spectra $B(j)$, i.e. for each j, there is an isomorphism

$$\tilde{H}^*(D_{2j}C_2S^{2r-1}; Z/2) \cong \tilde{H}^*(B(j); Z/2)$$

with an appropriate dimension shift; and there is a cofibre sequence [CMM] at the prime 2 given by

$$\Sigma^{2r-1}D_{2j-1}C_2S^{2r-1} \to D_{2j}C_2S^{2r-1} \to D_jC_2S^{4r-1}$$

for r much larger than j. Comparison of this sequence with that in Theorem 8.2 with $n = 1$ and replacing r by $2r - 1$ shows that there is a 2-equivalence from $D_{2j}\xi(1; S^{2r})$ to $D_jC_2S^{4r-1}$. These imply the following proposition.

Proposition 8.6 *For r much larger than j, the space $D_{4j}\xi(1; S^{2r})$ is a representative of the Brown–Gitler spectrum $B(j)$ at the prime 2.*

Appendix

In this appendix, we recall the definitions and some of the properties of NDR-pairs and quasifibrations. Most of the results on *NDR*-pairs are from [S, M] and we include the explicit constructions of some of the maps for the convenience of the reader. By inspection of the maps involved, these results on *NDR*-pairs can be extended to strong *NDR*-pairs. We also show that if $(X, *)$ is a strong *NDR*-pair, then (TX, X) and $(\Sigma X, X)$ are both strong *NDR*-pairs and s-pairs (s-pair is defined in §2.3). The results on quasifibrations are from [DT].

A.1 A pair of space (X, A) is said to be an *NDR-pair* if there exist a map $u: X \to I$ and a homotopy $h: I \times X \to X$ such that

(i) $A = u^{-1}(0)$;
(ii) $h(0, x) = x$ for all $x \in X$;
(iii) $h(t, x) = x$ for all $t \in I$ and $x \in A$;
(iv) $h(1, x) \in A$ for all $x \in X$ such that $ux < 1$.

The pair (u, h) is said to represent (X, A) as an NDR-pair. (X, A) is an NDR-pair if and only if the inclusion $A \subsetneq X$ is a cofibration [S, 7.1].

An NDR-pair (X, A) is a *strong NDR-pair* if (u, h) also satisfies

(v) $uh(t, x) < 1$ for all $t \in I$ whenever $ux < 1$.

Condition (v) implies that A is a strong deformation retract of $u^{-1}[0, 1) \subset X$.

Lemma A.2 [S, 6.3] *Let (u, h) and (v, j) represent (X, A) and (Y, B) as (strong) NDR-pairs. Then the product pair*

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

has a (strong) NDR-representation (w, q) with

$$w(x, y) = \min(ux, vy) ;$$

$$q(t, x, y) = \begin{cases} (x, y) & \text{if } x \in A, y \in B ; \\ \left(h(t, x), j\left(\frac{ux}{vy} t, y\right)\right) & \text{if } vy \geqq ux \text{ and } vy > 0 ; \\ \left(h\left(\frac{vy}{ux} t, x\right), j(t, y)\right) & \text{if } ux \geqq vy \text{ and } ux > 0 . \end{cases}$$

Proof. It is proved in [S, 63] that (w, q) represents the product as an NDR-pair. Suppose (u, h) and (v, j) are strong representations and $w(x, y) < 1$. Then either $ux < 1$ and $ux \leqq vy$ or $vy < 1$ and $vy \leqq ux$. Consider the first case. If $vy = 0$, then $ux = 0$. So $x \in A$ and $y \in B$. Therefore $wq(t, x, y) = w(x, y) = \min(ux, vy) = 0$. If $vy > 0$, then $wq(t, x, y) = \min\left(uh(t, x), vj\left(\frac{ux}{vy} t, y\right)\right) < 1$ for all $t \in I$ because $uh(t, x) < 1$. The second case is similar. Therefore (w, q) represents the product as a strong NDR-pair. □

Lemma A.3 [S, 8.4] *Let $f: (X, A) \to (Y, B)$ be a map of pairs such that $f: X \to Y$ is a quotient map and $f|_{(X-A)}: X - A \to Y - B$ is a homeomorphism (i.e., f is a relative homeomorphism). Then every representation (u, h) of (X, A) as a (strong) NDR-pair induces a representation $(\bar{u}, \bar{h})$ of (Y, B) as a (strong) NDR-pair with*

$$\bar{u}(y) = \begin{cases} 0 & \text{if } y \in B ; \\ u(f^{-1}y) & \text{if } y \in Y - B ; \end{cases}$$

$$\bar{h}(t, y) = \begin{cases} y & \text{if } y \in B ; \\ fh(t, f^{-1}y) & \text{if } y \in Y - B ; \end{cases}$$

which makes the following diagrams

$$\begin{array}{ccc} X & \xrightarrow{u} & I \\ f \downarrow & & \| \\ Y & \xrightarrow{\bar{u}} & I \end{array} \qquad \begin{array}{ccc} I \times X & \xrightarrow{h} & X \\ 1 \times f \downarrow & & \downarrow f \\ I \times Y & \xrightarrow{\bar{h}} & Y \end{array}$$

commute.

Lemma A.4 *Let (u, h) represent $(Y, *)$ as a (strong) NDR-pair. Then for any pointed space X, $(X \vee Y, X)$ has a (strong) NDR-representation (v, j) with*

$$v(x, *) = 0, \quad v(*, y) = uy; \quad \text{and}$$

$$j(t, (x, *)) = x, \quad j(t(*, y)) = h(t, y) .$$

Lemma A.5 *Let (u_i, h_i) represent $(X_i, *)$ as a (strong) NDR-pair for $i = 1, \ldots, j$. Then $(\bigvee_{i=1}^{j} X_i, *)$ has a (strong) NDR-representation (u, h) with*

$$u(*, \ldots, *, x_i, *, \ldots, *) = u_i x_i ;$$

and

$$h(t, (*, \ldots, *, x_i, *, \ldots, *)) = h_i(t, x_i) .$$

Lemma A.6 [M, A4] *Let (u, h) represent (X, A) as a (strong) NDR-pair. Then*

$$(X, A)^j = \left(X^j, \bigcup_{i=1}^{j} X^{i-1} \times A \times X^{j-1} \right)$$

has a (strong) NDR-representation (u_j, h_j) with

$$u_j(x_1, \ldots, x_j) = \min(ux_1, \ldots, ux_j) ;$$

$$h_j(t, x_1, \ldots, x_j) = (h(t_1, x_1), \ldots, h(t_j, x_j))$$

where

$$t_i = \begin{cases} t\left(\min\limits_{r \neq i} \left(\dfrac{ux_r}{ux_i} \right) \right) & \text{if for some } r \neq i,\ ux_r < ux_i ; \\ t & \text{if for all } r = 1, \ldots, j, ux_r \geqq ux_i . \end{cases}$$

A.7 In Sect. 2.4, we have introduced s-pairs in order to give $\xi(k, n; X, A)$ a strong filtration. In the following, we give some examples of (strong) NDR-pairs and s-pairs that we have used.

Example 1 $(I, 0)$ is a strong NDR-pair with

$$u(s) = \begin{cases} 2s & \text{if } 0 \leqq s \leqq \dfrac{1}{2} , \\ 1 & \text{if } \dfrac{1}{2} \leqq s \leqq 1 . \end{cases}$$

$$h(t, s) = \begin{cases} 0 & \text{if } 0 \leqq s \leqq \dfrac{t}{1+t} , \\ s(1+t) - t & \text{if } \dfrac{t}{1+t} \leqq s \leqq 1 . \end{cases}$$

Example 2 $(I, \partial I)$, where $\partial I = \{0, 1\}$, is a strong *NDR*-pair with

$$u(s) = \begin{cases} 4s & \text{if } 0 \leqq s \leqq \frac{1}{4}, \\ 1 & \text{if } \frac{1}{4} \leqq s \leqq \frac{3}{4}, \\ 4(1-s) & \text{if } \frac{3}{4} \leqq s \leqq 1. \end{cases}$$

$$h(t, s) = \begin{cases} 0 & \text{if } 0 \leqq s \leqq \frac{t}{2(1+t)}, \\ s(1+t) - \frac{1}{2}t & \text{if } \frac{t}{2(1+t)} \leqq s \leqq 1 - \frac{t}{2(1+t)}, \\ 1 & \text{if } 1 - \frac{t}{2(1+t)} \leqq s \leqq 1. \end{cases}$$

Example 3 (TX, X) is a (strong) *NDR*-pair if $(X, *)$ is a (strong) *NDR*-pair. By Lemma A.2, any representation of $(I, \partial I)$ as a (strong) *NDR*-pair makes $(I, \partial I) \times (X, *) = (I \times X, 1 \times X \cup 0 \times X \cup I \times *)$ a (strong) NDR-pair. The quotient map

$$(I \times X, 1 \times X \cup 0 \times X \cup I \times *) \to (I \times X/0 \times X \cup I \times *), 1 \times X) = (TX, X)$$

is a relative homeomorphism. It follows from Lemma A.3. that (TX, X) is a (strong) *NDR*-pair.

Example 4 (TX, X) is an s-pair if $(X, *)$ is a strong *NDR*-pair. Let $(I, 0)$ be represented by the maps defined in Example 1. The relative homeomorphism

$$(I, 0) \times (X, *) = (I \times X, 0 \times X \cup I \times *) \to (I \times X/0 \times X \cup I \times *, *) = (TX, *)$$

induces a representation (w, q) of $(TX, *)$ as a strong NDR-pair. By following the maps explicitly defined in Example 1, Lemmas A.2 and A.3, we see that $q(I \times (1 \times X)) \subset 1 \times X$.

Example 5 Using the same method as in Example 3, it can be shown that if (X, A) is a (strong) *NDR*-pair, then (TX, TA) and $(\Sigma X, \dot{\Sigma} A)$ are also (strong) *NDR*-pairs.

Using the same method as in Example 4, using the representation of $(I, \partial I)$ in Example 2, it can be shown that if $(X, *)$ is a strong *NDR*-pair, then $(\Sigma X, X)$ is an s-pair. Here $x \in X$ is identified with $\frac{1}{2} \wedge x \in \Sigma X$.

A.8 A surjective map $p: E \to B$ is a *quasifibration* if

$$p_*: \pi_i(E, p^{-1}(x), y) \cong \pi_i(B, x) \quad \text{for all } x \in B,\ y \in p^{-1}(x), \text{ and } i \geqq 0.$$

$p^{-1}(x)$ is the fibre over x of the quasifibration. Since B is path-connected, any two fibres are of the same weak homotopy type [DT, 1.10]. If B has a basepoint $*$, let $F = p^{-1}(*)$. As in the case of a fibration sequence, the isomorphisms in the definition yield the long exact homotopy sequence of a quasifibration [DT, 1.4]:

$$\cdots \to \pi_{i+1}(B) \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \cdots$$

A subset $U \subset B$ is said to be *distinguished* with respect to p if $p|_{p^{-1}(U)}: p^{-1}(U) \to U$ is a quasifibration.

The three criteria for quasifibrations in [DT, 2.2, 2.10, 2.15] combine to imply the following proposition [M, 7.2].

Proposition A.9 *Let $p: E \to B$ be a map onto a filtered space B. Then each F_jB is distinguished and p is a quasifibration if*

(i) *F_0B and every open subset of $F_jB - F_{j-1}B$ for $j > 0$ is distinguished.*

(ii) *For each $j > 0$, there is an open subset U of F_jB which contains $F_{j-1}B$ and there are homotopies $h_t: U \to U$ and $H_t: p^{-1}(U) \to p^{-1}(U)$ such that*

(a) *$h_0 = 1$, $h_t(F_{j-1}B) \subset F_{j-1}B$ and $h_1(U) \subset F_{j-1}B$;*

(b) *$H_0 = 1$, $pH_t = h_tp$; and*

(c) *$H_1: p^{-1}(x) \to p^{-1}(h_1(x))$ is a weak homotopy equivalence for all $x \in U$.*

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