in which the right square is a pushout square of path-connected open subsets of $F_j\xi(k, n; \Sigma X)$. All the maps in the diagram are inclusions. We have $F_1\xi(k, n; \Sigma X) \simeq TX$ from Lemma 2.3(ii). Since $(\tilde{w}_j, \tilde{q}_j)$ is a strong NDR-representation of $(F_j\xi(k, n; \Sigma X), F_{j-1}\xi(k, n; \Sigma X))$, the homotopy $\tilde{q}_j$ restricted to P is a strong deformation retraction of P onto $F_{j-1}\xi(k, n; \Sigma X)$, which is assumed to be 1-connected. Hence P is 1-connected. The left square, together with Lemmas 5.7 and 5.8, imply that $\pi_1(P \cap Q) \to \pi_1 Q$ is onto. Therefore, $F_j\xi(k, n; \Sigma X)$ is 1-connected by invoking the Seifert and Van Kampen Theorem again. □

Proposition 5.10 *Under the same conditions as in Proposition* 5.9, $C_{k+n}X$, $F_jC_{k+n}X$ *and* $D_jC_{k+n}X$ *are* 1-*connected.*

Proof. The proof is a simpler modification of that of Proposition 5.9. We induct on j with the following diagram

$$\begin{array}{ccc} \tilde{U} \cap \tilde{V} & \longrightarrow & \tilde{U} \\ \downarrow & & \downarrow \\ \tilde{V} & \longrightarrow & F_jC_{k+n}X \end{array}$$

which is a pushout of path-connected open subsets of $F_jC_{k+n}X$. □

Remark. In fact, the spaces in Proposition 5.10 are highly connected if X is [CT].

Corollary 5.11 *If* $r \geqq 2$ *and* $j \geqq 0$, *then* $\xi(k, n; S^{r+1})$, $F_j\xi(k, n; S^{r+1})$, $D_j\xi(k, n; S^{r+1})$, $C_{k+n}S^r$, $F_jC_{k+n}S^r$ and $D_jC_{k+n}S^r$ *are all* 1-*connected.*

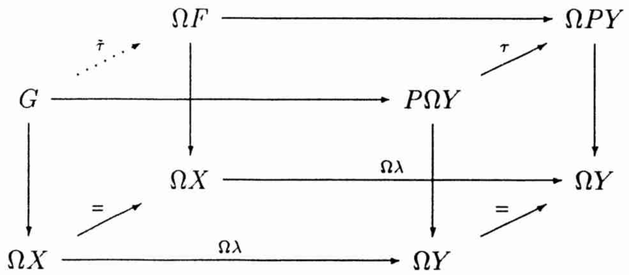
6 The multiplicative structure of $W(k, n; \Sigma X)$

For $k \geqq 2$, $W(k, n; \Sigma X)$ is naturally homeomorphic to $\Omega W(k-1, n; \Sigma^2 X)$, which give $W(k, n; \Sigma X)$ a loop space structure. In this section, we show that the multiplication ϕ_2 in $\xi(k, n; \Sigma X)$ (see Proposition 4.2) is compatible with the loop multiplication in $W(k, n; \Sigma X)$ for $k \geqq 2$. This compatibility can be proved by a direct calculation, but since the notation becomes quite unmanageable, we break it down into a few lemmas. For $k = 1$, we show that it is rarely the case that $W(1, n; \Sigma X)$ is an H-space.

6.1 For a space Y, there is a natural homeomorphism $\tau: P\Omega Y \to \Omega PY$. By definitions, $P\Omega Y = \mathrm{Map}_*(I; \mathrm{Map}_*(S^1; Y))$; $\Omega PY = \mathrm{Map}_*(S^1; \mathrm{Map}_*(I; Y))$. For $f \in P\Omega Y$, $\tau(f)$ is given by $\tau(f)(t)(s) = f(s)(t)$ for $t \in S^1$ ad $s \in I$. The inverse of τ is given by $\tau^{-1}(g)(s)(t) = g(t)(s)$.

Lemma 6.2 *Let* $\lambda: X \to Y$ *be a pointed map, F be the homotopy fibre of* λ, *and G be the homotopy fibre of* $\Omega\lambda: \Omega X \to \Omega Y$. *Then there is a natural homeomorphism* $\tilde{\tau}: G \to \Omega F$.

Proof. Recall that the homotopy fibre of a map has been explicitly defined in Sect. 5.1. In the following diagram,



the pullback square at the back is gotten by looping the pullback square

$$\begin{array}{ccc} F & \longrightarrow & Py \\ \downarrow & & \downarrow \\ X & \xrightarrow{\lambda} & Y. \end{array}$$

The map τ: $P\Omega Y \to \Omega PY$ induces a unique map $\tilde{\tau}$: $G \to \Omega F$ such that the whole cube is commutative. Arguing with the universal property of pullbacks and the fact that τ is a homeomorphism, we see that $\tilde{\tau}$ is a homeomorphism. □

6.3 We apply Lemma 6.2 to the map $\Omega^{k-2}E^n$: $\Omega^{k-2}\Sigma^k X \to \Omega^{k+n-2}\Sigma^{k+n}X$ with $k \geqq 2$. Its homotopy fibre is $W(k-1, n; \Sigma^2 X)$. The homotopy fibre of $\Omega^{k-1}E^n$ is $W(k, n; \Sigma X)$. Since $W(k, n; \Sigma X)$ is a subspace of $P\Omega^{k+n-1}\Sigma^{k+n}X$, there is a homeomorphism τ: $W(k, n; \Sigma X) \to \Omega W(k-1, n; \Sigma^2 X)$ which is the restriction of τ: $P\Omega^{k+n-1}\Sigma^{k+n}X \to \Omega P\Omega^{k+n-2}\Sigma^{k+n}X$ defined by $\tau(f)(t)(s)(u) = f(s)(u \wedge t)$ for $f \in P\Omega^{k+n-1}\Sigma^{k+n}X$, $t \in S^1$, $s \in I$ and $u \in S^{k+n-2}$.

We shall identify $W(k, n; \Sigma X)$ with $\Omega W(k-1, n; \Sigma^2 X)$ by this τ. The loop-multiplication in $\Omega W(k-1, n; \Sigma^2 X)$ induces a multiplication ψ in $W(k, n; \Sigma X)$. By iteration, $W(k, n; \Sigma X) \cong \Omega^{k-1}W(1, n; \Sigma^{k-1}X)$ is a $(k-1)$-fold loop space.

6.4 For a space Y, there is a natural homeomorphism

$$\tau' \colon \Sigma TY = S^1 \wedge I \wedge Y \to I \wedge S^1 \wedge Y = T\Sigma Y$$

given by $\tau'(t \wedge s \wedge y) = s \wedge t \wedge y$.

Recall that for $k \geqq 2$,

$$\phi_2 \colon \xi(k, n; X, A) \times \xi(k, n; X, A) \to \xi(k, n; X, A)$$

is a filtration preserving multiplication (Proposition 4.2) and that

$$\beta_2 \colon \xi(k, n; X, A) \to \Omega\xi(k-1, n; \Sigma X, \Sigma A)$$

is a weak homotopy equivalence. Let β'_2 be the composite

$$\xi(k, n; TX, X) \xrightarrow{\beta_2} \Omega\xi(k-1, n; \Sigma TX, \Sigma X) \to \Omega\xi(k-1, n; T\Sigma X, \Sigma X)$$

where the last map is induced by τ'. Notice that $\Omega\xi(k-1, n; T\Sigma X, \Sigma X) = \Omega\xi(k-1, n; \Sigma^2 X)$. With slight modifications (taking τ' into consideration), the proof in Proposition 4.9 shows that the following diagram, in which ϕ is the loop multiplication, is commutative.

$$\begin{array}{ccc} \xi(k,n;\Sigma X)\times\xi(k,n;\Sigma X) & \xrightarrow{\phi_2} & \xi(k,n;\Sigma X) \\ \beta'_2\times\beta'_2\downarrow & & \downarrow\beta'_2 \\ \Omega\xi(k-1,n;\Sigma^2 X)\times\Omega\xi(k-1,n;\Sigma^2 X) & \xrightarrow{\phi} & \Omega\xi(k-1,n;\Sigma^2 X). \end{array}$$

Lemma 6.5 *For $n \geqq 2$, the following diagram commutes.*

$$\begin{array}{ccc} E_n(TX,X) & \xrightarrow{\tilde{\alpha}_n} & P\Omega^{n-1}\Sigma^n X \\ \beta'_2\downarrow & & \cong\ \downarrow\tau \\ \Omega E_{n-1}(T\Sigma X,\Sigma X) & \xrightarrow{\Omega\tilde{\alpha}_{n-1}} & \Omega P\Omega^{n-2}\Sigma^n X. \end{array}$$

Proof. Recall that $E_n(TX, X) = \xi(n, 0; \Sigma X)$. This lemma is proved by a direct calculation. We write a little n-cube c as $c = c'\times c''\times c'''$ with $c', c'' : I \to I$, $c''' : I^{n-2} \to I^{n-2}$; an element of TX as $w \wedge x$. For $t \in S^1, s \in I, u \in S^{n-2}$ and $y = [\langle c_1, \ldots, c_j\rangle, w_1 \wedge x_1, \ldots, w_j \wedge x_j] \in E_n(TX, X)$,

$$\Omega\tilde{\alpha}_{n-1}\circ\beta'_2(y)(t)(s)(u)$$

$$= \begin{cases} * & \text{if } t \notin \bigcup_{r=1}^{j} c''_r(\dot{J}) \\ \tilde{\alpha}_{n-1}[\langle c'_{r_1}\times c'''_{r_1}, \ldots, c'_{r_i}\times c'''_{r_i}\rangle, w_{r_1}\wedge v_{r_1}\wedge x_{r_1}, & \text{if } c''_{r_q}(v_{r_q}) = t,\ 1 \leqq q \leqq i, \\ \qquad \ldots, w_{r_i}\wedge v_{r_i}\wedge x_{r_i}](s)(u) & t \notin c''_r(J) \text{ if } r \notin \{r_1, \ldots, r_i\} \end{cases}$$

$$= \begin{cases} d_r w_r \wedge z_r \wedge v_r \wedge x_r & \text{if } c'_r(d_r) = s,\ c''_r(v_r) = t,\ c'''_r(z_r) = u \\ w_r \wedge z_r \wedge v_r \wedge x_r & \text{if } s \geqq c'_r(1),\ c''_r(v_r) = t,\ c'''_r(z_r) = u \\ * & \text{otherwise} \end{cases}$$

$$= \tilde{\alpha}_n(y)(s)(u \wedge t)$$

$$= \tau\circ\tilde{\alpha}_n(y)(t)(s)(u). \quad \square$$

Lemma 6.6 *For $k \geqq 2$, the following diagram commutes.*

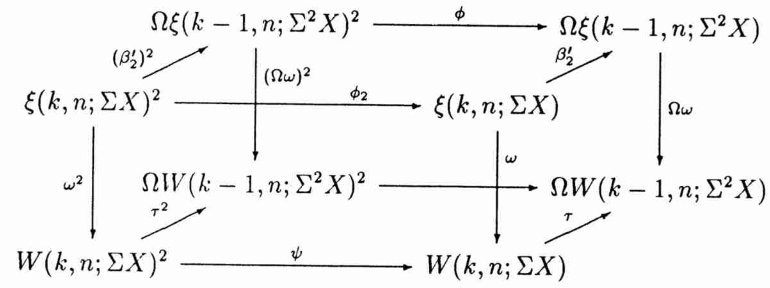
$$\begin{array}{ccc} \xi(k,n;\Sigma X) & \xrightarrow{\omega} & W(k,n;\Sigma X) \\ \beta'_2\downarrow & & \cong\ \downarrow\tau \\ \Omega\xi(k-1,n;\Sigma^2 X) & \xrightarrow{\Omega\omega} & \Omega W(k-1,n;\Sigma^2 X)\,. \end{array}$$

Proof. The diagram in this lemma naturally injects into the diagram in Lemma 6.5. □

Theorem 6.7 *For $k \geqq 2$, the following diagram commutes.*

$$\begin{array}{ccc} \xi(k,n;\Sigma X)\times\xi(k,n;\Sigma X) & \xrightarrow{\phi_2} & \xi(k,n;\Sigma X) \\ \omega\times\omega\downarrow & & \downarrow\omega \\ W(k,n;\Sigma X)\times W(k,n;\Sigma X) & \xrightarrow{\psi} & W(k,n;\Sigma X)\,. \end{array}$$

Proof. Consider the following diagram:



The top square commutes from Sect. 6.4. The bottom square commutes from the definition of ψ. The squares on the right side and the left side commute from Lemma 6.6. The back square commutes from the naturality of loop multiplication. The map τ is one-to-one. All these imply that the square at the front commutes. □

The following proposition shows that if $W(1, n; \Sigma X)$ has an H-space structure, then the mod p homology of X is that of a sphere if $\tilde{H}_*(X; Z/p) \neq 0$.

Proposition 6.8 *If for some prime* p, $\sum_{i>0} \dim_{Z/p} H_i(X; Z/p) > 1$, *then for* $n > 0$, $W(1, n; \Sigma X)$ *is not an* H*-space.*

Proof. We make use of the Samelson product and the fact that the suspension of a Whitehead product is nullhomotopic in this proof. In the following, $H_*(-)$ stands for $H_*(-; Z/p)$, W stands for $W(1, n; \Sigma X)$.

Consider the fibration sequence

$$\Omega W \xrightarrow{\lambda} \Omega\Sigma X \xrightarrow{\Omega E^n} \Omega^{n+1}\Sigma^{n+1}X .$$

To show that W is not an H-space, it suffices to show that $H_*\Omega W$ is not a commutative algebra.

There is a map $\bar{\kappa}: \Omega Y \times \Omega Y \to \Omega Y$ given by $\bar{\kappa}(f, g) = ((f \circ g) \circ f^{-1}) \circ g^{-1}$ which when restricted to $\Omega Y \vee \Omega Y$ is nullhomotopic. Thus it induces a map $\kappa: \Omega Y \wedge \Omega Y \to \Omega Y$. Define a map ad^2 as the composite

$$X \wedge X \xrightarrow{E \wedge E} \Omega\Sigma X \wedge \Omega\Sigma X \xrightarrow{\kappa} \Omega\Sigma X$$

and inductively define $\mathrm{ad}^j: X^{[j]} \to \Omega\Sigma X$ as $\kappa \circ (E \wedge \mathrm{ad}^{j-1})$ where $X^{[j]}$ is the j-fold smash product of X. Let $\bar{X}$ denote $\bigvee_{j \geqq 2} X^{[j]}$. Collecting the ad^j together yields a map $\mathrm{ad} = \bigvee_{j \geqq 2} \mathrm{ad}^j: \bar{X} \to \Omega\Sigma X$. Let $\widetilde{\mathrm{ad}}^{\,j}$ and $\widetilde{\mathrm{ad}}$ be the adjoints of ad^j and ad respectively. Let $\phi: \Omega\Sigma\bar{X} \to \Omega\Sigma X$ denote the multiplicative extension (see Sect. 3.1) of ad. Then $\phi = \Omega(\widetilde{\mathrm{ad}})$. It follows from the inductive definition of ad^j and the fact that $E \circ \widetilde{\mathrm{ad}}^2$ is nullhomotopic [A, Proposition 3.2] that $\Omega E^n \circ \phi$ is nullhomotopic. Therefore, there is a lift $\ell: \Omega\Sigma\bar{X} \to \Omega W$ such that $\lambda \circ \ell = \phi$.

Consider the following commutative diagram.