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A converse inequality of Hölder type

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1 The inequality

We will show, e.g., the inequality ($0<p<q\leqq\infty$, $0<\theta<1$)

$$\left(\int_0^\infty [t^{-\theta} f(t)]^q \, dt/t\right)^{1/q} \leqq C_{\theta,q}/C_{\theta,p}\left(\int_0^\infty [t^{-\theta} f(t)]^p \, dt/t\right)^{1/p}$$

for functions f on $\mathbf{R}_+$ satisfying the condition

$$0\leqq f(s)\leqq \max(1, s/t) f(t)$$

where

$$C_{\theta,p}=(p\theta(1-\theta))^{-1/p}.$$

Equality holds for the function $f(t)=\min(1,t)$.

Converse inequalities of Hölder type have been extensively treated in the literature. Among the authors are Frank, Pick, Favard, Berwald, Prékopa, Borell. For the earlier papers, we refer to the bibliographical notes in the monograph [B-B]. A later reference is [Bo], where extensive generalizations are made and where there is a connection to the isoperimetric problem. However, the results in [Bo] do not seem to be immediately transferable to our case.

Our present version is well-known in the theory of interpolation spaces, where it takes the form ($0<p<q\leqq\infty$, $0<\theta<1$)

$$\bar{A}_{\theta,p;K}\subset \bar{A}_{\theta,q;K}$$

In this note, based on an old (circa 1975) manuscript by J. Bergh, we determine the best constant in the corresponding norm inequality: this coincides with the inequality in the first paragraph.

Theorem 1 *Let* $0<p<q\leqq\infty$, $0<\theta<1$, *and let the function* f *on* $\mathbf{R}_+$ *satisfy*

$$0\leqq f(s)\leqq \max(1, s/t) f(t). \tag{1}$$

Then there is a number $\xi > 0$, such that

$$\int_0^\infty [t^{-\theta} f(t)]^p \, dt/t = \int_0^\infty [t^{-\theta} \xi \min(1,t)]^p \, dt/t = \xi^p C_{\theta,p}^p \tag{2}$$

and

$$\int_0^\infty [t^{-\theta} f(t)]^q \, dt/t \leqq \int_0^\infty [t^{-\theta} \xi \min(1,t)]^q \, dt/t = \xi^q C_{\theta,q}^q \tag{3}$$

where

$$C_{\theta,p} = (p\theta(1-\theta))^{-1/p}.$$

Equality holds for the function $f(t) = \min(1,t)$.

This theorem yields at once the inequality in the first paragraph.

Remark 1 In (2) and (3) the power functions

$$(\cdot)^p \quad \text{and} \quad (\cdot)^q$$

may be replaced by more general functions ϕ and ψ respectively. Here both ϕ and ψ are positive, ϕ is strictly increasing and $\psi \circ \phi^{-1}$ is convex. This generalization is obvious from the proof of the theorem we present below. Note, however, that the constants have to be changed accordingly.

Remark 2 Christer Borell has, in a personal communication, given a second proof of Theorem 1 using the Brunn-Minkowski inequality.

Proof. The left sides in (2) and (3) are rearrangement-invariant (with respect to the measure dt/t). Put, now, for $t = t_1$ and $t = t_2$,

$$g(t) = \sup\{y; |\{s; s^{-\theta} f(s) > y\}| > |(t_1, t_2)|, t_1 < 1 < t_2, t_1^{1-\theta} = t_2^{-\theta}\}$$

where $|E| = \int_E dt/t$. The function $g(t)$ is covariant with $t^{-\theta} \min(1,t)$ and equi-measurable with $t^{-\theta} f(t)$.

The set $E_0 = \{t; g(t) < t^{-\theta} \xi \min(1,t)\}$ is an interval – we postpone the proof of this fact.

From (1) and (2) we deduce that $t^{-\theta} f(t) \leqq \xi$ and then $g(t) \leqq \xi$ follows. We now get

$$\int_{x_1}^{x_2} [g(t)]^p \, dt/t \leqq \int_{x_1}^{x_2} [t^{-\theta} \xi \min(1,t)]^p \, dt/t \tag{4}$$

for all x_1 and x_2 with $x_1 < 1 < x_2$ and $x_1^{1-\theta} = x_2^{-\theta}$. If this were not the case then $x_i \notin E_0$, $i = 1, 2$, which would violate (1) since E_0 is an interval containing 1 and $g(t)$ is equi-measurable with $t^{-\theta} f(t)$.

The desired inequality (3) now follows from (4) with the aid of a method in [H-L-P]: the inequality (4) gives at once

$$\int_0^\infty ([g(t)]^p - [t^{-\theta}\xi \min(1,t)]^p)\, \Phi(t)\, dt/t \leqq 0 \tag{5}$$

when $\Phi(t)$ is covariant with $t^{-\theta}\min(1,t)$. ($\Phi(t)$ is the limit function of a sequence of simple functions, all covariant with $t^{-\phi}\min(1,t)$.)

But

$$\begin{aligned}[g(t)]^q - [t^{-\theta}\xi \min(1,t)]^q &= ([g(t)]^p)^{q/p} - ([t^{-\theta}\xi \min(1,t)]^p)^{q/p} \\ &= \Psi([g(t)]^p, [t^{-\theta}\xi \min(1,t)]^p) \\ &\quad \cdot ([g(t)]^p - [t^{-\theta}\xi \min(1,t)]^p)\end{aligned}$$

where Ψ is increasing in both its arguments since $(\cdot)^{q/p}$ is convex for $q/p > 1$. With

$$\Phi(t) = \Psi([g(t)]^p, [t^{-\theta}\xi \min(1,t)]^p)$$

we see that (3) follows from (5).

It remains to show that E_0 is an interval. We have to show that if

$$\{t; g(t) > y\} \supset \{t; t^{-\theta}\xi \min(1,t) > y\} \tag{6}$$

holds for $y = y_0$ then it holds for all $y \leqq y_0$.

From (1) it follows that f is continuous and thus $\{t;\ t^{-\theta} f(t) > y_0\}$ is a union of open intervals $(t_{1,i}, t_{2,i})$, $i \in I$.

Using (1) again, we see that the graph of $t^{-\theta} f(t)$ does not lie above the graph of the function

$$h_i(t) = y_0 \min((t/t_{1,i})^{1-\theta}, (t/t_{2,i})^{-\theta}), \qquad i \in I$$

for $t_{1,i} \leqq t \leqq t_{2,i}$, and does not lie below otherwise. (The reader should draw a picture at this point.)

Now follows

$$|\{t; h_i(t) > y\}| = \log(t_{2,i}/t_{1,i}) + (\theta(1-\theta))^{-1} \log(y_0/y), \qquad y \leqq y_0. \tag{7}$$

But $g(t)$ is equi-measurable with $t^{-\theta} f(t)$ and so

$$|\{t; g(t) > y_0\}| = \sum_{i \in I} \log(t_{2,i}/t_{1,i}) = \log(t_2/t_1) \tag{8}$$

where $t_1 < 1 < t_2$ and $t_1^{1-\theta} = t_2^{-\theta} = y_0/\xi$.

Furthermore

$$|\{t, t^{-\theta}\xi \min(1,t) > y\}| = \log(t_2/t_1) + (\theta(1-\theta))^{-1} \log(y_0/y), \qquad y \leqq y_0. \tag{9}$$

Since the sum in (8) has at least one term, it follows from (7) and (9) that (6) holds for $y \leqq y_0$. This completes the proof of the theorem. □

2 Some comments

1. The idea to use rearrangements in the context of converse inequalities of Hölder type is independently conceived and used in [Bo]. However, the results in [Bo] do not seem to be immediately transferable to our case; see Remark 1 above.
2. Is there a connection with converse Hardy inequalities?
There is a lemma in this connection, which may be used as a key for the proofs of such inequalities [B-B-P]:

$$\left(\int_0^\infty f(t)\,dt/t\right)^p \leqq p \int_0^\infty [f(t)]^p\,dt/t$$

where $0<p<1$ and $0 \leqq f(t)/t$ is decreasing.
This lemma gives evidently a weaker version of our theorem. Note that in our present Theorem 1 we have two growth conditions on f.
3. There is an obvious question of other measures (groups) not covered in [Bo] or by us. We intend to treat this in a future note.

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