Math. Z. 214, 527–565 (1993)

Mathematische Zeitschrift

New surfaces of constant mean curvature

Karsten Große-Brauckmann

Mathematisches Institut, Universität Bonn, Beringstrasse 4, D-53115 Bonn 1, Germany

Received 27 January 1992; in final form 16 October 1992

0 Introduction

Immersed surfaces of constant mean curvature 1 in Euclidean three-space (henceforth: H-surfaces) have been found by many authors. We would like to mention the following. In 1841 Delaunay determined all surfaces of revolution [D], which are simply periodic except for the sphere. In 1970 Lawson described two doubly periodic surfaces [L]. In 1984 Wente discovered immersed H-tori, and others contributed to a further study of H-tori, namely Pinkall and Sterling achieved a classification [PS]. Karcher obtained some triply periodic H-surfaces in 1989 [Ka]. In 1990 Kapouleas proved existence of a wealth of H-surfaces, compact and non-compact [Kp1, Kp2]. Karcher's and Kapouleas' surfaces come in continuous families of the same topological type.

In this work we extend Karcher's method – which is in turn based on Lawson's original ideas – to a broader class of surfaces, namely to surfaces with ends. We get the full one-parameter family of certain symmetric surfaces. The limiting cases for these families are as follows: surfaces whose Delaunay-shaped building blocks have small necks and the centres are spherical, as described by Kapouleas on the one hand, and surfaces with again small necks but a new type of centre (which asymptotically is n-noid-shaped) on the other hand. In between we find surfaces with maximal neck-size. For example we can embed Lawson's surfaces into a continuous one-parameter family of surfaces with the same symmetry and we are able to prove that they are exactly those in the family having maximal neck-size (Theorem 3.3). We obtain a similar result for the symmetric surfaces with n onduloid ends, see Theorem 6.1.

The work is organized as follows. The conjugate surface construction for H-surfaces is given in Sect. 1. Basically this is a geometric transformation which reduces the free bundary value problem for desired fundamental patches of H-surfaces to a Plateau problem for a geodesic polygon in the three-sphere S^3. If this polygon is embedded in the boundary of an H-convex set, Morrey's solution to the Plateau problem in S^3 followed by conjugation yields the desired Euclidean H-surface patch. Since the Morrey solution minimizes we get H-surfaches provided their fundamental patch is small enough, i.e. we have to suppose sufficiently high symmetry.

In Sect. 2 we study particular surfaces in $\mathbb{S}^3$. First we describe the surfaces associated to the Delaunay surfaces, the spherical helicoids. Their particular significance is that any embedded end of an H-surface is asymptotic to a Delaunay surface [KKS]. A foliation of $\mathbb{S}^3$ without two great circles leads to a similar result for spherical ends in Subsect. 4.4. Then we consider tori and Clifford tori. The solid Clifford torus serves as an H-convex set in this work and replaces the intersection of hemispheres used by Lawson and Karcher. In order to embed arbitrarily long boundary contours we use the universal covering of solid Clifford tori instead of subsets of $\mathbb{S}^3$.

The periodic H-surfaces we obtain are described in Sect. 3. For each surface we specify the associated boundary contour and the H-convex set containing it.

In the remaining sections we prove existence of surfaces with ends. Their spherical boundary polygons are infinitely long. In Sect. 4 we approximate them by a sequence of bounded polygons and prove convergence of their Plateau solutions, using curvature estimates by R. Schoen and establishing local area bounds. In Sect. 5 we apply this scheme to obtain H-surfaces with cylinder ends, for which the H-convex barriers are easy to describe. In terms of the parameter these surfaces are most distant to those of Kapouleas. In Sect. 6 we give examples of H-surfaces with ends of general Delaunay type.

1 Constant mean curvature surfaces in $\mathbb{R}^3$ and associated minimal surfaces in $\mathbb{S}^3$

1.1 Associated surfaces

Locally a *surface* in an oriented three-dimensional Riemannian manifold N with metric g is an immersion $f: \Omega^2 \to N^3$ of class C^2, where Ω is a domain in $\mathbb{R}^2$. f induces a metric $\langle v, w\rangle := g(df(v), df(w))$ on Ω. By the orientation of Ω and the induced metric the rotation by 90 degrees, $R^{90}: T\Omega \to T\Omega$, is given by $R^{90} e_1 = e_2$ and $R^{90} e_2 = -e_1$ for a positively oriented orthonormal base e_1, e_2 of the tangent space. We use the same notation R^{90} for the induced 90° rotation in $df(T\Omega) \subset TN$, $R^{90} df(v) = df(R^{90} v)$.

The second fundamental tensor $S: T\Omega \to T\Omega$ with respect to a continuous choice of normal $\nu: \Omega \to TN$ is defined by

$$\langle S v, w\rangle = g(-\nabla_{df(v)} \nu, df(w)),$$

where ∇ denotes the covariant derivative in N. The *mean curvature* of the immersion is

$$H = \tfrac{1}{2} \operatorname{trace} S = \tfrac{1}{2}\langle S e_1, e_1\rangle + \tfrac{1}{2}\langle S e_2, e_2\rangle,$$

for an orthonormal base e_1, e_2 of the surface. f is an immersion of constant mean curvature or *H-surface* if $H \equiv 1$ on Ω, and a *minimal surface* if $H \equiv 0$. Under these conventions the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ together with its inner normal becomes an H-surface.

Basic for this work is Lawson's

Theorem 1.1 [L, p. 364] *Let $f: \Omega^2 \to N^3(1)$ be a minimal immersion of a simply connected domain into a manifold of constant sectional curvature* 1 *with metric* g *and second fundamental tensor* S. *Then there exists an associated* H-*surface* $\tilde{f}: \Omega^2 \to \mathbb{R}^3$ *with metric* $\tilde{\langle}.,.\tilde{\rangle} = \langle .,. \rangle$ *and second fundamental tensor*

$$\tilde{S} = R^{90} S + \mathrm{id}. \tag{1.1}$$

On the other hand for each H-*surface in* $\mathbb{R}^3$ *there is an associated minimal surface in the* 3-*sphere* $\mathbb{S}^3$ *with* $S = R^{-90}(\tilde{S} - \mathrm{id})$.

Proof [Ka]. We verify the equations of Gauß and Codazzi for the data of $\tilde{f}$. The existence of the surfaces is then a consequence of the fundamental theorem for hypersurfaces [doC, p. 236].

By the symmetry of S we have $\mathrm{trace}(R^{90} S) = 0$, and therefore

$$\begin{aligned} \det \tilde{S} &= \det(R^{90} S + \mathrm{id}) \\ &= \det(R^{90} S) + \mathrm{trace}(R^{90} S) + 1 \\ &= \det S + 1. \end{aligned} \tag{1.2}$$

On the other hand f satisfies the Gauß equation $K = \det S + R = \det S + 1$, where K is Gauß curvature of the surface, and R sectional curvature of the space. It follows $K = \det \tilde{S}$. But the two surface metrics coincide and they determine the Gauß curvature, hence we have $K = \tilde{K}$. This establishes the Gauß equation $\tilde{K} = \det \tilde{S} + 0$ for $\tilde{f}$. For Codazzi's equations we verify $\nabla_X \tilde{S} Y = \nabla_Y \tilde{S} X$ (wlog. $[X, Y] = 0$). Since $\nabla_X SY = \nabla_Y SX$ and R^{90} commutes with ∇, we have $\nabla_X R^{90} SY = \nabla_Y R^{90} SX$. Adding $\nabla_X Y = \nabla_Y X$, the claim follows.

Finally $\mathrm{trace}\, \tilde{S} = \mathrm{trace}(R^{90} S + \mathrm{id}) = 2$ by the symmetry of S and thus $\tilde{f}$ is an H-surface. By the same calculations the converse of the theorem holds using the fundamental theorem for hypersurfaces in $\mathbb{S}^3$. □

Remarks. 1. The same method yields H-surfaces in $N^3(c-1)$ from minimal surfaces in $N^3(c)$, e.g. hyperbolic H-surfaces from Euclidean minimal surfaces.

2. Every rotation of the second fundamental tensor, $\tilde{S}^\phi = R^\phi S + \mathrm{id}$, describes an H-surface since (1.2) holds. In analogy to minimal surfaces this family of H-surfaces is called the *associated family*. However $\phi = \pm 90°$ leads to two different H-surfaces, which are obtained as well by the two choices of ν in (1.1) and likewise by $R^{90} S \pm \mathrm{id}$.

3. $\mathbb{S}^2 \subset \mathbb{R}^3$ with $\tilde{S} = \mathrm{id}$ is the associated surface of the great sphere $\mathbb{S}^2 \subset \mathbb{S}^3$ with $S = 0$. The cylinder in $\mathbb{R}^3$ with $\tilde{S} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is associated to the Clifford torus with $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The Delaunay surfaces are associated to the spherical helicoids, see Subsect. 2.1.

1.2 *Associated boundary curves*

We denote by $\tilde{M}^2 \subset \mathbb{R}^3$ an H-surface and by $M^2 \subset \mathbb{S}^3$ its associated minimal surface. Let c be a curve in Ω.

Lemma 1.2 *$\tilde{\gamma}(s)=\tilde{f}(c(s))$ is a curvature line and geodesic of the surface $\tilde{M}\subset\mathbb{R}^3$ iff the associated curve $\gamma(s)=f(c(s))$ in M describes a geodesic of S^3, that is a great circle arc.*

Proof. We choose a normal v resp. $\tilde{v}$ in M resp. $\tilde{M}$. Let us use $\cdot$ for the scalar product in $\mathbb{R}^3$ and $S^3\subset\mathbb{R}^4$ and denote the covariant derivative $\nabla_{\dot{\gamma}}$ along a geodesic by d/ds in $\mathbb{R}^3$ and by D/ds in S^3. Since $d\tilde{v}/ds$ is parallel to $\dot{\tilde{\gamma}}$ we have

$$(1.3)\qquad \begin{aligned} 0&=\frac{d}{ds}\tilde{v}\cdot R^{90}\dot{\tilde{\gamma}}\\ &=-\langle\tilde{S}\dot{c},R^{90}\dot{c}\rangle=-\langle(R^{90}S+\mathrm{id})\dot{c},R^{90}\dot{c}\rangle=-\langle S\dot{c},\dot{c}\rangle\\ &=\frac{D}{ds}v\cdot\dot{\gamma}=-v\cdot\frac{D}{ds}\dot{\gamma}.\end{aligned}$$

As $g=\tilde{g}$, γ is a geodesic of the surface iff $\tilde{\gamma}$ is. Hence $D\dot{\gamma}/ds$ is parallel to v, and by (1.3) $D\dot{\gamma}/ds=0$, that is γ is a geodesic of S^3, and vice versa. □

$\tau := g(\nabla_{\dot{\gamma}}v, R^{90}\dot{\gamma})$ is called *torsion* and $\kappa := g(\nabla_{\dot{\gamma}}\dot{\gamma}, v)$ *curvature* of the curve γ; Eq. (1.3) then reads

$$\tilde{\tau}=-\kappa.$$

The torsion of a geodesic in N^3 is the rotation speed of the normal and thus depends on the surface. We compute this speed in terms of the associated curve:

$$(1.4)\qquad \begin{aligned} \tilde{\kappa}&=\tilde{v}\cdot\frac{d}{ds}\dot{\tilde{\gamma}}\\ &=\langle\tilde{S}\dot{c},\dot{c}\rangle=\langle(R^{90}S+\mathrm{id})\dot{c},\dot{c}\rangle=-\langle S\dot{c},R^{90}\dot{c}\rangle+1\\ &=\frac{D}{ds}v\cdot R^{90}\dot{\gamma}+1=\tau+1.\end{aligned}$$

Since the sign of curvature and torsion depends on the normal let us fix orientations.

Definition 1.3 (i) (*Orientation of* S^3) r, v, $w\in T_pS^3$ are positively oriented, if r, v, w, $p\in\mathbb{R}^4$ are positively oriented.
(ii) (*Choice of normal*) The normal v of a surface $f:\Omega\to N^3$ orients $df(v)$, $df(w)$, v positively if v, $w\in T\Omega$ are positively ortiented. In particular $df(v)$, $R^{90}df(v)$, v are positively oriented.
(iii) Let γ be a geodesic in N^3 and v a vector field along γ. Then v *right* (*left*) *rotates* with respect to the axis $\dot{\gamma}$ if $\dot{\gamma}$, v, $\nabla_{\dot{\gamma}}v$ are positively (negatively) oriented.

1.3 Geometric data of H-surfaces

We construct H-surfaces invariant under a group of planar symmetries. A *fundamental patch* generates the comlete surface by the group of reflections. We assume the patch is simply connected. Its boundary consists of planar curves $\tilde{\gamma}_i$ which must be symmetry lines ($\tilde{S}\dot{\tilde{\gamma}}_i \parallel \dot{\tilde{\gamma}}_i$). See images in Subsect. 3.1 for an example.

A fundamental patch, bounded by n curvature lines $\tilde{\gamma}_i$ (parameterized with unity speed), defines the following *geometric data:*

(i) The *length* $\tilde{l}_i$ of the curvature line $\tilde{\gamma}_i$, also denoted by $|\tilde{\gamma}_i|$.

(ii) The *vertex angle* $\tilde{e}_i=\pi/(m_i+1)$, $m_i\in\mathbb{N}$, of two edges $\tilde{\gamma}_i$, $\tilde{\gamma}_{i+1}$ satisfies

$$\cos\tilde{e}_i=\cos\angle(\tilde{\gamma}_i,\tilde{\gamma}_{i+1})=-\dot{\tilde{\gamma}}_i(\tilde{l}_i)\cdot\dot{\tilde{\gamma}}_{i+1}(0).$$

(iii) The *tilting angle* $\tilde{t}_i\in\mathbb{R}$ of the normal $\tilde{v}$,

$$\tilde{t}_i=\int_0^{\tilde{l}_i}\frac{d}{ds}\tilde{v}\cdot\dot{\tilde{\gamma}}_i\,d\sigma.$$

This angle measures the total turn of the normal, $\cos\tilde{t}_i=\tilde{v}(\tilde{\gamma}_i(0))\cdot\tilde{v}(\tilde{\gamma}_i(l_i))$.

Vertex and tilting angle modulo π are determined by the symmetry type, that is by the boundary polygon. However the lengths are unknown. If in addition the position of the normal at the vertices is prescribed then the tilting angle can only be chosen modulo 2π.

By Lemma 1.2 the associated contour is a geodesic polygon in S^3 with the same lengths $l_i=\tilde{l}_i$ and the same vertex angles

$$e_i=\tilde{e}_i \tag{1.5}$$

because the metrics coincide. An integration of (1.4) along a geodesic arc γ_i yields:

$$\begin{aligned}\tilde{t}_i&=\int_0^{l_i}\frac{d}{ds}\tilde{v}\cdot\dot{\tilde{\gamma}}_i\,d\sigma\\&=-\int_{\tilde{\gamma}_i\subset\mathbb{R}^3}\tilde{\kappa}\,d\sigma\\&=-\int_{\gamma_i\subset S^3}(r+1)\,d\sigma\\&=-\int_0^{l_i}\frac{D}{ds}v\cdot R^{90}\dot{\gamma}_i\,d\sigma-l_i\\&=r_i-l_i.\end{aligned}\tag{1.6}$$

Here we used the notation

$$r:=-\int_0^{l}\frac{D}{ds}v(\gamma)\cdot R^{90}\dot{\gamma}\,d\sigma$$

for the *rotation angle* of the tangent plane along a geodesic arc $\gamma\subset S^3$. Indeed $r\in\mathbb{R}$ measures the total rotation of the normal with respect to parallel transport. r is positive for right rotations ($\dot{\gamma}$, v, $Dv/ds=-R^{90}\dot{\gamma}$ positively oriented). Given $\tilde{l}_i$, $\tilde{e}_i$, $\tilde{t}_i$ the associated contour in S^3 can be found arcwise: Prescribing a length $l_i=\tilde{l}_i$ to γ_i, the position of the normal $v(\gamma_i(l_i))$ at the endpoint is determined by (1.6). Then (1.5) gives the correct vector $\dot{\gamma}_{i+1}(0)$ in the tangent plane $v^{\perp}(\gamma_i(l_i))$. In Subsect. 1.5 we prove there exist lengths, such that the boundary contour closes.

1.4 Hopf vector fields

Karcher [Ka] introduced the idea to measure the angles of rotation in S^3 with respect to rotating vector fields. In terms of such fields the right hand side of (1.6) does not contain the unknown length any more. In S^3 such vector fields exist:

Definition 1.4 A vector field $\tau: S^3 \to TS^3$ is called *Hopf vector field* if $\tau(x) = T\cdot x$ with a skew symmetric orthogonal 4×4-matrix $T = -{}^tT = -T^{-1}$.

According to Definition 1.3(iii) T right rotates along the great circle $\cos sx + \sin sy$, if y, Tx, Ty, x is positively oriented for $x \perp y$. By a homotopy argument this orientation agrees for all x, y it is defined for. Hence *right* and *left rotating Hopf vector fields* are well defined.

Lemma 1.5 (i) *Hopf vector fields are Killing fields whose integral curves are great circles.*

(ii) *The angle of a Hopf field with a great circle is constant.*

(iii) *If a Hopf field is not tangent to a great circle it rotates with constant speed and once around the great circle.*

(iv) *Furthermore two right rotating Hopf fields make a constant angle with each other and are thus determined by their values at one point $x \in S^3$.*

Proof. (i) Since $Tx \perp x$ the integral curves of T are the circles $s \mapsto (\cos s)\,x + \sin s)\,Tx$.

(ii) Let $\gamma(s) = -\ddot{\gamma}(s)$ be the great circle. Then we have

$$\frac{d}{ds}(T\gamma\cdot\dot{\gamma}) = T\dot{\gamma}\cdot\dot{\gamma} + T\gamma\cdot\ddot{\gamma} = 0.$$

(iii) By (ii) $T\gamma$ rotates around $\dot{\gamma}$ provided $\dot{\gamma} \neq \pm T\gamma$. The velocity vector $\frac{d}{ds}T\gamma = T\dot{\gamma}$ projected to the tangent space of S^3, $\frac{D}{ds}T\gamma = T\dot{\gamma} - (T\dot{\gamma}\cdot\gamma)\,\gamma$, is of constant length

$$\left|\frac{D}{ds}T\gamma\right|^2 = |T\dot{\gamma}|^2 - 2(T\dot{\gamma}\cdot\gamma)^2 + (T\dot{\gamma}\cdot\gamma)^2|\gamma|^2. \tag{1.7}$$

On the other hand $T\gamma$ has a component orthogonal to $\dot{\gamma}$ with length

$$|T\gamma - (T\gamma\cdot\dot{\gamma})\,\dot{\gamma}|^2 = 1 - 2(T\gamma\cdot\dot{\gamma})^2 + (T\gamma\cdot\dot{\gamma})^2|\dot{\gamma}|^2. \tag{1.8}$$

By the skew symmetry of T (1.8) and (1.7) coincide. Therefore the rotation speed of the orthogonal component of $T\gamma$ is ± 1.

(iv) is a consequence of (ii) and (iii). □

We can find an orthogonal basis of three right rotating fields

$$(1.9)\quad A=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\quad C=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},$$

where Ax, Bx, Cx, x are positively orientated. For $\alpha^2+\beta^2+\gamma^2=1$ a linear combination $\alpha A+\beta B+\gamma C$ again is a right rotating Hopf field. In the following all Hopf vector fields are right rotating unless stated otherwise.

Using property (iv) of the preceding lemma we define angles between Hopf fields pointwise:

$$\angle(A, B):=\alpha,$$

if $A\cdot B=\cos\alpha$ for $0\leqq\alpha\leqq 180°$. Let the oriented angle of two fields B, C with respect to a third field $A\perp(B, C)$ be $\angle_A(B, C):=\pm\angle(B, C)$, depending on Ax, Bx, Cx, x being positively resp. negatively oriented in $\mathbb{R}^4$. Finally let the oriented angle of two fields B, C with respect to a linear independent field A be defined by the projection of B, C to $A^\perp$:

$$\angle_A(B, C):=\angle_A(B-(A\cdot B)A, C-(A\cdot C)A).$$

Thus $\angle_A(B, C)$ is the dihedral angle of the planes $\operatorname{span}\{A, B\}$ and $\operatorname{span}\{A, C\}$. If a right rotation by $+90°$ leads from B to C with respect to the axis A in the sense of Definition 1.3(iii) then $\angle_A(B, C)=+90°$, provided A, B, C are positively oriented. All computations with respect to oriented angles are modulo 2π so that we have identities like $\angle_A(B, C)=\angle_A(B, D)+\angle_A(D, C)$.

1.5 Existence of closed associated boundary polygons

Theorem 1.6 *Let $n\geqq 4$ angles $0<e_i<180°$ and n angles $t_i\in\mathbb{R}$ be given and suppose $e_{n-2}=e_{n-1}=90°$ and $t_{n-1}\neq 0 \bmod \pi$. Then there is a closed polygon Γ consisting of n geodesic arcs γ_i in S^3, with n normals in the vertices, having*

(i) *vertex angle $\angle(\gamma_i, \gamma_{i+1})=e_i$ (indices* mod *n) and*

(ii) *rotation angle of the tangent plane along an arc γ_i satisfying $r_i-|\gamma_i|=t_i \bmod 2\pi$.*

(iii) *We can prescribe any lengths $0<l_1, \ldots, l_{n-3}<\pi$ except for isolated values to have Γ embedded.*

Proof. We claim that if we take γ_i to be an integral curve of A_i with $\gamma_i(0)=\gamma_{i-1}(l_{i-1})$, where A_i satisfies

$$(1.10)\qquad \angle(-A_i, A_{i+1})=e_i \quad\text{and}\quad \angle_{A_i}(-A_{i-1}, A_{i+1})=t_i \bmod 2\pi,$$

then (i) and (ii) hold, no matter which lengths $0<l_i=|\gamma_i|<\pi$ we choose.

Clearly the first equation of (1.10) implies (i). To compute r_i we split the rotation into two parts: A rotation of A_{i-1} to A_{i+1} at $\gamma_i(0)$ and the rotation of $A_{i+1}(\gamma_i(s))$ along $\gamma_i(s)$. For the latter we let $A^\perp$ be the Hopf field orthogonal to $A_i=\dot\gamma_i$ and A_{i+1}. We obtain

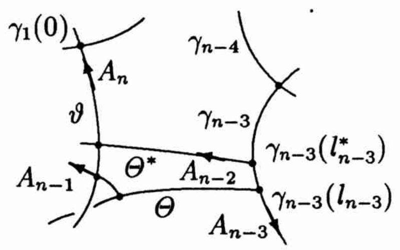
$$r_i = \angle_{A_i}(-A_{i-1}, A_{i+1}) - \int_0^{l_i} \frac{D}{ds} A^\perp \cdot R^{90}\dot\gamma_i \, d\sigma \mod 2\pi.$$

Since both r_i and $\angle_\bullet(.,.)$ are positive for right rotations with respect to the axis A_i, the sign of the first term is correct. But $\frac{D}{ds}(A^\perp\gamma_i)=A^\perp\dot\gamma_i$ and since $A^\perp\dot\gamma_i$ is orthogonal to $\dot\gamma_i, A^\perp\gamma_i, \gamma_i$ we must have $A^\perp\dot\gamma_i=\pm R^{90}\dot\gamma_i$. In fact $A^\perp\dot\gamma_i=-R^{90}\dot\gamma_i$ since $A^\perp$ is right rotating, that is $\dot\gamma_i, A^\perp\gamma_i, A^\perp\dot\gamma_i, \gamma_i$ is positively orientated as is $\dot\gamma_i, R^{90}\dot\gamma_i, v=A^\perp\gamma_i, \gamma_i$ by Definition 1.3(ii). This yields (ii) for the claim:

$$r_i = \angle_{A_i}(-A_{i-1}, A_{i+1}) + \int_0^{l_i} A^\perp\dot\gamma_i \cdot A^\perp\dot\gamma_i \, d\sigma = t_i + l_i \mod 2\pi.$$

We now prove we can satisfy (1.10) and find lengths such that the polygon closes. To start we choose A_n and A_1 such that $\angle(-A_n, A_1)=e_n$. Then we determine fields A_{i+1} for $2\leqq i+1\leqq n-1$ from (1.10). We fix a point $\gamma_1(0)$ and choose lengths $0<l_1, l_2<\pi$. Furthermore we choose lengths $l_3, \ldots, l_{n-5}$ and l^*_{n-4}, l^*_{n-3} small enough so that no self-intersections occur. Let ϑ be the great circle passing through $\gamma_1(0)$ with tangent vector A_n and Θ^* be the great circle through $\gamma_{n-3}(l^*_{n-3})$ tangent to A_{n-2}. We look for an integral of A_{n-1} meeting both ϑ and Θ^* in an angle of $e_{n-2}=e_{n-1}=90°$ and defining thereby lengths l_{n-2}, l_{n-1}, l_n.

First we rule out that ϑ and Θ^* intersect eachover. By assumption on t_{n-1} they do not coincide. By Lemma 2.6 either we can find a length l_{n-3} (close to l^*_{n-3}), such that Θ, defined as the great circle along A_{n-2} passing through $\gamma_{n-3}(l_{n-3})$, does not intersect ϑ. Or $\gamma_{n-3}, \Theta^*, \vartheta$ are contained in a torus ∂T_r. However this can occur only for $n\geqq 5$, since γ_{n-3} is then Clifford-parallel to ϑ. Thus there is an edge γ_{n-4}. If γ_{n-4} is contained in ∂T_r then γ_{n-4} left rotates with respect to ∂T_r and changing l^*_{n-4} to l_{n-4} gives $\gamma_{n-3}\not\subset\partial T_r$. The same conclusion holds if $\gamma_{n-4}\not\subset\partial T_r$. Hence we can assume $\vartheta\cap\Theta=\varnothing$ for lengths l_{n-3} and l_{n-4}. The same argument shows we obtain an embedded polygon for $l_1, \ldots, l_{n-3}$ any numbers less than π except for isolated values.



Two disjoint great circles ϑ, Θ have two perpendicular great circles ϕ and Φ, which meet the former in a distance of $\pi/2$. Namely let $\vartheta(s)$ and $\Theta(t)$ be two points in minimal distance and let ϕ be the great circle containing these two points. ϕ meets ϑ orthogonally in $\vartheta(s)$ and therefore $\phi \subset \{\vartheta(s+\pi/2)\}^{\perp}$. In particular $\vartheta(s+\pi/2) \perp \Theta(t)$ and likewise $\Theta(t+\pi/2) \perp \vartheta(s)$. That is the great circle $\Phi := \{\vartheta(s), \Theta(t)\}^{\perp}$ is also perpendicular.

ϕ and Φ are orthogonal to the linear independent fields A_{n-2} and A_n. By assumption on e_{n-2}, e_{n-1} they are necessarily integrals of A_{n-1} – this is the main idea of the proof. Four great circle arcs, two of ϕ and two of Φ, join the great circles ϑ and Θ in correct orientation $+A_{n-1}$. For the respective lengths l_{n-2}, l_{n-1}, l_n the polygon closes. Possibly readjusting the lengths $l_1, \ldots, l_{n-3}$ again, this polygon is seen to be embedded. □

1.6 Adding handles

We can easily enlarge the spherical polygon to obtain H-surfaces with an additional handle.

Theorem 1.7 *Let Γ be a closed polygon consisting of $n \geqq 3$ geodesics $\gamma_1, \ldots, \gamma_n$ with tangent Hopf vector fields $A_1, \ldots, A_n$. Suppose the three edges γ_n, γ_1, γ_2 are not contained in a torus* (*see* 2.3), *that is*

$$(1.11) \qquad |\gamma_1| \neq \tfrac{1}{2} \angle_{A_1}(-A_n, \pm A_2) \quad \text{or} \quad \angle(-A_n, A_1) \neq \angle(-A_1, \pm A_2).$$

Then there is an enlarged closed polygon $\Gamma^* = \{\gamma_0^*, \gamma_1^*, \gamma_2^*, \gamma_3, \ldots, \gamma_{n-1}, \gamma_n^*\}$ along the Hopf fields $A_0, A_1, \ldots, A_n$, where $A_0 = \{A_1, A_n\}^{\perp}$, with the following properties:

(i) *The vertex angles of the new edge are $\angle(\gamma_n^*, \gamma_0^*) = \angle(\gamma_0^*, \gamma_1^*) = 90°$, the other vertex angles remain the same.*

(ii) *The rotation angle along the new edge γ_0^* is the vertex angle of γ_n, γ_1: $\angle_{A_0}(-A_n, A_1) = \pm \angle(\gamma_n, \gamma_1)$. The rotation angles of the two adjacent edges change by* 90°:

$$\angle_{A_n}(-A_{n-1}, A_0) = \angle_{A_n}(-A_{n-1}, A_1) \mp 90°$$

and

$$\angle_{A_1}(-A_0, A_2) = \angle_{A_1}(-A_n, A_2) \mp 90°.$$

All other rotation angles remain unchanged.

(iii) *Γ^* can be chosen such that the new edge is arbitrarily short and the lengths of the adjacent edges change by a small amount.*

Proof. We choose a length $l_2^* < l_2$. Then the integral of A_1 passing through $\gamma_2(l_2^*)$ does not intersect the great circle which contains γ_n by assumption (1.11). We take γ_0^* to be the common perpendicular of the great circles γ_1^* and γ_n. Now we proceed as in the proof of Theorem 1.6. The other sign in (ii) relates to the choice $l_2^* > l_2$. □

Depending on the sign chosen in (ii) the associated H-surface has a handle to the inside or outside which closes after $2m-1$ reflections if $\angle_{A_0}(A_n, A_1) = \pm \pi/m$. Theorem 1.7 allows to construct polygons with many right angles by the successive insertion of handles.

1.7 Solution of Plateau's problem in H-convex sets

We want to solve Plateau's problem for the boundary polygon found in Subsect. 1.5. By the results of [H2] the Plateau problem is solvable in C^2-*H-convex* Riemannian manifolds, that is in manifolds whose boundary ∂N is of class C^2 and has non-negative mean curvature with respect to the inner normal. In order to control the boundary behaviour of the surface we need to enclose the boundary polygon in the H-convex boundary and therefore we want to allow for edges of N.

Let M be a compact minimal surface in $\mathbb{R}^3$ with boundary Γ. By the maximum principle a complete embedded minimal surface S, moved from ∞ towards M, cannot touch the interior of M before S intersects the boundary Γ. It is therefore convenient to extend the notion of C^2-H-convexity to the following:

Definition 1.8 A closed manifold N with boundary is *H-convex*, if there are closed submanifolds N_i, $0 \leqq i \leqq I$, and embedded C^2-surfaces $S_i \subset N_{i-1}$ with boundary satisfying:

(i) N_0 is C^2-H-convex and $N_1 = N$.

(ii) N_i is (the closure of) a connected component of $N_{i-1} - S_i$ for $i \geqq 1$.

(iii) $\partial S_i \subset \partial N_{i-1}$ and S_i has non-negative mean curvature w.r.t. the inner normal of N_i.

(iv) For each S_i there exists a continuous family (w.r.t. distance of surfaces) of H-convex embedded C^2-surfaces $S_i^\sigma \subset N_{i-1} - \operatorname{int} N_i$, with boundary $\partial S_i^\sigma \subset N_{i-1}$, such that $S_i^0 = S_i$ and either
(a) $0 \leqq \sigma \leqq 1$ and $S_i^1 \subset \partial N_{i-1}$
(b) or (applicable to non-compact N_{i-1} only) $0 \leqq \sigma < \infty$ and $\operatorname{dist}(S_i^\sigma, S_i) \geqq \sigma$.

We call the sets S_i *barriers*. Our definition of H-convexity is slightly tighter than the usual one [MY]. However the solvability of Plateau's problem in N is immediate, since by the maximum principle the solution is contained in each N_i and therefore in N:

Theorem 1.9 *Let N^3 be an H-convex manifold with boundary, and Γ a Jordan curve in N which bounds a topological disk of finite area. Then there exists a map of the disk $f: D \to N$, such that* (i) *in the interior of D, f is a smooth minimal immersion,* (ii) *$f(\partial D) = \Gamma$ is continuous and monotone, and* (iii) *f has minimal area among all maps of the same topological type.*

f is an immersion by the results of [O] and [G].

Theorem 1.10 (Boundary regularity [H1]) *If $f: D \to N$ is a minimal surface of class $C^0(\bar{D}, N) \cap C^2(D, N)$ which maps a boundary* arc *$\delta \subset \partial D$ to an analytic Jordan* arc *$\gamma \subset N$, then f is analytic on $D \cup \delta$.*

Lemma 1.11 (Reflection principle [L, Proposition 3.1]) *Let f be a spherical minimal surface of class $C^2(D^+ \cup I, N^3(c))$, where $D^+ = \{(x, y) \in D \mid y > 0\}$ and $I = (-1, 1) \times \{0\} \subset \partial D^+$. If $f(I)$ is either contained in a geodesic γ, or perpendicular to a totally geodesic plane σ, then f can be extended to D by reflection to a minimal surface f^* of class $C^2(D, N)$.*

This reflection is either 180°-rotation around γ or plane reflection in σ. For H-surfaces a similar theorem holds [DHKW, 3.4 Theorem 2].

If $f: D^+ \to N$ is a minimum of area with geodesic boundary $f(I)$ the extension by reflection f^* is free of *branch points* on I, i.e. of points with $|\nabla f^*|=0$; this is a result of [GL]. If two adjacent geodesic arcs of the boundary curve make an angle of $e_i=\pi/(m_i+1)$, $m_i \in \mathbb{N}$, we can extend the surface to a complete neighbourhood of the vertex by Lemma 1.11. This neighbourhood can be reparameterized regularly according to Theorem 1.10, however (true) branch points may occur.

If $f: D \to S^3$ parameterizes a spherical minimal surface without branch points, then the associated H-surface $\tilde{f}: D \to \mathbb{R}^3$ is regular, since the second fundamental tensor $\tilde{S}$ is bounded. Higher regularity follows from the H-surface equation.

In the next theorem we embed the boundary curve of the H-surface into the boundary of an H-convex set in order to exclude vertex branch points and to prescribe the rotation angle t_i as a real number instead of mod 2π.

Theorem 1.12 *Let Γ be a closed polygon, embedded in an H-convex manifold $N^3(1)$. Let the arcs be integrals of the Hopf fields $A_1, \ldots, A_n$, and e_i, t_i be angles such that* (1.10) *is satisfied. Suppose Γ bounds a disk of finite area in N and satisfies:*

(i) *Every vertex of the polygon is contained in two different H-convex barriers, i.e. in an edge of ∂N.*

(ii) *There is a vector field v_i along γ_i contained in the boundary tangent space $T_{\gamma_i} N - \operatorname{span}\{\dot{\gamma}_i\}$, coinciding with $-\dot{\gamma}_{i-1}(l_{i-1})$ and $\dot{\gamma}_{i+1}(0)$ at the vertices, with rotation angle $r_i(v_i)=t_i+l_i$.*

Then there exists a fundamental H-surface patch $\tilde{M} \subset \mathbb{R}^3$ with the geometric data e_i and t_i, whose reflection in $\mathbb{R}^3$ extends to a complete immersed H-surface.

Proof. We take the Plateau solution M given by Theorem 1.9. With the help of the maximum principle we see from assumption (i) that the surface extended by reflection is free of branch points. □

This theorem reduces the existence proof for H-surfaces to the construction of suitable barriers for the associated boundary curve. By the positive curvature of S^3 this becomes more difficult the bigger the boundary polygon is.

2 Some surfaces in S^3

2.1 Spherical helicoids and Delaunay surfaces

The helicoids are the associated surfaces of the Delaunay surfaces. We choose a great circle $(\cos u, \sin u, 0, 0)$ and an orthogonal great circle $(0, 0, \cos au, \sin au)$ parameterized with speed $a \in \mathbb{R}$. Joining corresponding points with great circles we obtain a ruled surface $f^a = f: \mathbb{R}^2 \to S^3 \subset \mathbb{R}^4$,

$$f(u,v)=\begin{pmatrix} \cos v \cos u \\ \cos v \sin u \\ \sin v \cos au \\ \sin v \sin au \end{pmatrix}. \tag{2.1}$$

f^0 parameterizes a great sphere, f^1 is a Clifford torus (see 2.3). The tangent space is spanned by

$$f_u(u,v)=\begin{pmatrix}-\cos v\sin u\\ \cos v\cos u\\ -a\sin v\sin au\\ a\sin v\cos au\end{pmatrix},\qquad f_v(u,v)=\begin{pmatrix}-\sin v\cos u\\ -\sin v\sin u\\ \cos v\cos au\\ \cos v\sin au\end{pmatrix}.$$

f^a is an immersion called (*spherical*) *helicoid* provided $a\neq 0$, with metric

$$\begin{pmatrix}f_u^2 & f_u\cdot f_v\\ f_u\cdot f_v & f_v^2\end{pmatrix}=\begin{pmatrix}\cos^2 v+a^2\sin^2 v & 0\\ 0 & 1\end{pmatrix}.$$

In the following we suppose $a\neq 0$ and write $\gamma(v):=|f_u|>0$. The normal

$$n(u,v)=\frac{1}{\gamma(v)}\begin{pmatrix}-a\sin v\sin u\\ a\sin v\cos u\\ \cos v\sin au\\ -\cos v\cos au\end{pmatrix}$$

orients $\frac{1}{\gamma}f_u, f_v, n, f\in\mathbb{R}^4$ positively. The second fundamental tensor is

$$Sf_u=-n_u=-\frac{a}{\gamma}f_v \quad\text{and}\quad Sf_v=-n_v=-\frac{a}{\gamma^3}f_u.$$

f_u/γ and f_v are orthonormal and thus $2H=\operatorname{trace}S=S\frac{1}{\gamma}f_u\cdot\frac{1}{\gamma}f_u+Sf_v\cdot f_v=0$, i.e. the helicoids are minimal surfaces.

By Lemma 1.2 all v-lines are curvature lines on the associated H-surfaces. Thus there is a continuous group of reflections and the H-surfaces are either rotationally symmetric (if $a\neq -1$ we have $\tilde{S}f_u=(1+a/\gamma^2)f_u\not\equiv 0$) or translation invariant and cylinders ($\tilde{S}f_u\equiv 0$ in case $a=-1$). The Delaunay surfaces are characterized as the only non-compact rotationally symmetric H-surfaces [D]. We want to show that all Delaunay surfaces can be derived this way.

Suppose $a>-1$. The boundary of the helicoid $f^a:[0,\pi/(2+2a)]\times[0,\pi/2]\to\mathbb{S}^3$, $\pm a>0$, consists of integrals of the Hopf fields $\mp A$, C, A, $-B$ where A, B, C are as in (1.9). Indeed we have

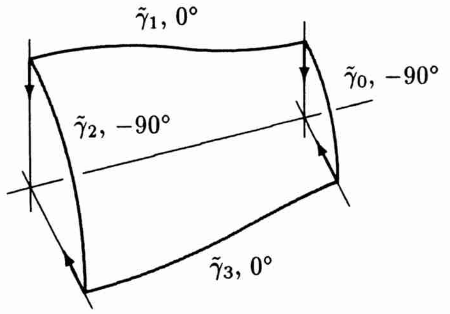
$$-af=\frac{d}{du}f\left(-u,\frac{\pi}{2}\right),\qquad Cf=\frac{d}{dv}f(0,-v),\qquad Af=\frac{d}{du}f(u,0),$$

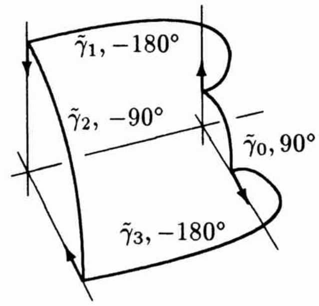
and using $\cos\pi/(2+2a)=\sin a\pi/(2+2a)$

$$-Bf=\frac{d}{dv}f\left(\frac{\pi}{2+2a},v\right). \tag{2.2}$$

We label the arcs in this order by $\gamma_0,\gamma_1,\gamma_2,\gamma_3$. We have four rectangular vertex angles and four tilting angles $-90°$, $0°$, $-90°$, $0°$ in case $a>0$ resp. $90°$, $-180°$, $-90°$, $-180°$ if $-1<a<0$, e.g. the first tilting angle is $\angle_{\mp A}(B,C)$

$=\mp 90°$ since A, B, C is positively oriented. These are exactly the geometric data of half a period of a 90° *segment* of the Delaunay surfaces; either the embedded *onduloid*, or the immersed *nodoid* (for $a>$ resp. <0):





The u-lines are circles in $\mathbb{R}^3$ of the length $4\gamma(v)\,\pi/(2+2a)$, which is extremal for $v=0 \bmod \pi$, where $\gamma=1$, and $v=\pi/2 \bmod \pi$, where $\gamma=|a|$. Thus the ratio of the radii is

$$\frac{r_{\min}}{r_{\max}}=|a| \qquad \text{for } -1<a\leqq 1$$

resp. $1/|a|$ otherwise. It therefore suffices to consider $0<a\leqq 1$ for the onduloids and $-1<a<0$ for the nodoids to obtain all Delaunay surfaces. Note $a=\pm 1$ is associated to the cylinder of radius $1/2$ and $a=0$ to S^2.

Theorem 2.1 *Each Delaunay surface is the associated H-surface of a helicoid $f^a(u,v)$ with $-1<a\leqq 1$ and $a\neq 0$ in S^3. A segment of angle ϕ of the Delaunay surface, containing k periods is associated to a piece of the helicoid with $0\leqq u<\phi/(1+a)$ and $0\leqq v\leqq k\pi$.*

2.2 *Foliation of* S^3 *with helicoids*

Let $a\neq 0$ and set $\alpha:=\dfrac{\pi}{2}\,\dfrac{1}{1+|a|}$,

$$K:=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \operatorname{sign} a & 0 \\ 0 & -\operatorname{sign} a & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

For $a>0$ the field K is the right rotating Hopf field B from (1.9) whereas K is left rotating in case $a<0$.

Lemma 2.2 *$Kf(u,v)\cdot n(u,v)>0$ for $|u|<\alpha$ and $=0$ for $u=\pm\alpha$.*

Proof.

$$Kf\cdot n=\begin{pmatrix}\sin v\sin au\\ \operatorname{sign} a\sin v\cos au\\ -\operatorname{sign} a\cos v\sin u\\ -\cos v\cos u\end{pmatrix}\cdot\frac{1}{\gamma(v)}\begin{pmatrix}-a\sin v\sin u\\ a\sin v\cos u\\ \cos v\sin au\\ -\cos v\cos au\end{pmatrix}$$

$$=\frac{1}{\gamma(v)}(a\sin^2 v(-\sin u\sin au+\operatorname{sign} a\cos u\cos au)$$

$$+\cos^2 v(-\operatorname{sign} a\sin u\sin au+\cos u\cos au))$$

$$=\frac{1}{\gamma(v)}(|a|\sin^2 v(-\sin u\sin|a|u+\cos u\cos au)$$

$$+\cos^2 v(-\sin u\sin|a|u+\cos u\cos au))$$

$$=\frac{\cos^2 v+|a|\sin^2 v}{\sqrt{\cos^2 v+a^2\sin^2 v}}\cos(u+|a|u)$$

$$>0,\quad \text{if } (1+|a|)|u|<\pi/2. \quad \square$$

Thus the flow of the helicoids $f^a((-\alpha,\alpha)\times[0,2\pi))$, by the field K gives rise to a foliation of S^3 without the two great circles $\eta^a_\pm(t):=f^a(\pm\alpha,t)$. Namely $\Psi^a=\Psi\colon \mathbb{R}^3\to S^3$, $\Psi(\phi,u,v):=\cos\phi f^a(u,v)+\sin\phi Kf^a(u,v)$ satisfies:

Lemma 2.3 (i) $\Psi(\phi,\pm\alpha,v)=\Psi(0,\pm\alpha,v\mp\phi)=\eta_\pm(v\mp\phi)$

(ii) $\Psi(\phi+\pi,u,v)=\Psi(\phi,u,v+\pi)$

(iii) $\Psi\colon D\to S^3$ *is injective on* $D:=[0,\pi)\times(-\alpha,\alpha)\times[0,2\pi)$ *and*

(iv) *the image of* D *is* $S^3-\eta_\pm=S^3-(\eta_+\cup\eta_-)$.

Proof. (i) For $a>0$ we have $\dot\eta_\pm=\mp B\eta_\pm$ by (2.2) and the result follows. For $a<0$ the calculation is similar.

(ii) Use $f(u,v+\pi)=-f(u,v)$.

(iii) $(u,v)\mapsto\Psi(\phi,u,v)$ with $|u|<\alpha$ is a minimal surface with boundary $\eta_\pm$ for each ϕ. We set

$$\phi_0:=\inf\{\phi>0\,|\,\Psi(\phi.\,U,V)=f(u,v)\text{ and }(u,v)\neq(U,V);|u|,|U|<\alpha;0\leqq v,V<2\pi\}$$

and claim $\phi_0=\pi$.

$\mu_0\leqq\pi$ by (ii). We show $\phi_0>0$. $f\colon[-\alpha,\alpha]\times[0,2\pi)\to S^3$ is injective. Namely if $f(u,v)=f(U,V)$ the quotient of the first two components is $\tan u=\tan U$ and since $\alpha<\pi/2$ it follows $u=U$ and $v=V$. Hence $\cos tf+\sin tn$ is injective for $0\leqq t<\varepsilon$ and $(u,v)\in[-\alpha,\alpha]\times[0,2\pi)$. Finally, by the preceding lemma Ψ is an injective reparameterization of the above on $[0,\varepsilon(\delta))\times[-\alpha+\delta,\alpha-\delta]\times[0,2\pi)$ for any $\delta>0$. Since f is locally minimizing $\varepsilon(\delta)$ is bounded away from 0. Either $\Psi(\phi_0,U,V)$ intersects $f(u,v)$ in the interior ($|u|,|U|<\alpha$). Because the touching is locally on one side the surfces coincide by the maximum principle. Or the two surfaces are tangential in a point of the common boundary without interior intersection. By the boundary maximum principle they coincide. In order to prove $\phi_0 \geqq \pi$ it is therefore sufficient to conclude from

$$\Psi(\phi, 0, 0) = \begin{pmatrix} \cos\phi \\ 0 \\ 0 \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \cos v \cos u \\ \cos v \sin u \\ \sin v \cos a u \\ \sin v \sin a u \end{pmatrix} = f(u, v)$$

with $0 \leqq \phi < \pi$ and $-\alpha < u < \alpha$, $0 \leqq v < 2\pi$ that ϕ, u, v are equal to 0. However, all other cases lead to a contradiction:

$$v \neq 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi \Rightarrow \cos v, \sin v \neq 0 \Rightarrow \sin u, \cos a u = 0,$$

$$v \in \left\{\frac{\pi}{2}, \frac{3}{2}\pi\right\} \Rightarrow \phi = \pi/2 \Rightarrow \sin a u = 1,$$

$$v = \pi \Rightarrow u = 0 \Rightarrow \cos\phi = -1.$$

(iv) The image of D under Ψ is open and closed in $S^3 - \eta_\pm$. □

By the maximum principle a uniqueness theorem follows:

Theorem 2.4 *Let $M \subset S^3 - \eta_\pm$ be a compact minimal surface with boundary satisfying $\partial M \subset \Psi(0,.,.)$ and $\Psi(\phi_0,.,.) \cap M = \varnothing$ for a ϕ_0. Then M is contained in the sheet $\Psi(0,.,.)$.*

Corollary 2.5 *The helicoids $W_\xi^a = \{f^a(u, v) \mid -\xi \leqq u \leqq \xi, 0 \leqq v < 2\pi\}$ are stable for $\xi < \alpha$.*

By Theorem 2.1 W_α^a belongs to a 180° segment of the onduloid if $a > 0$.

2.3 Tori and Clifford tori

Let us define $F: \mathbb{R}^3 \to S^3$ by

$$F(r, x, y) := \begin{pmatrix} \cos r \cos(x+y) \\ \cos r \sin(x+y) \\ \sin r \cos(y-x) \\ \sin r \sin(y-x) \end{pmatrix}. \tag{2.3}$$

The image of $0 \leqq r \leqq \pi/2$, $0 \leqq x < 2\pi$, $0 \leqq y < \pi$ parameterizes the whole sphere since

$$F(r, x + 2\pi, y) = F(r, x, y) \quad \text{and} \quad F(r, x, y + \pi) = F(r, x + \pi, y). \tag{2.4}$$

The set $T_r := \{F(\rho, x, y) \mid 0 \leqq \rho \leqq r\}$ is called a *solid torus* resp.

$$T := T_{45} = \{x \in S^3 \mid x_3^2 + x_4^2 \leqq 1/2\} \tag{2.5}$$

the *solid Clifford torus.* These are tubular neighbourhoods of the *soul,* i.e. the great circle $F(0, x, 0) = F(0, 0, x)$. We call the boundary ∂T_r for $0 < r < 90°$ a *torus,* resp. $\partial T = \partial T_{45}$ the *Clifford torus.*

$x \mapsto F(r, x, y)$ and $y \mapsto F(r, x, y)$ are great circles. They have a constant distance r to the soul and are called *r-Clifford parallels* to the soul. Their tangent vectors are

$$\frac{\partial}{\partial x} F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} F \quad \text{and} \quad \frac{\partial}{\partial y} F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} F.$$

Thus

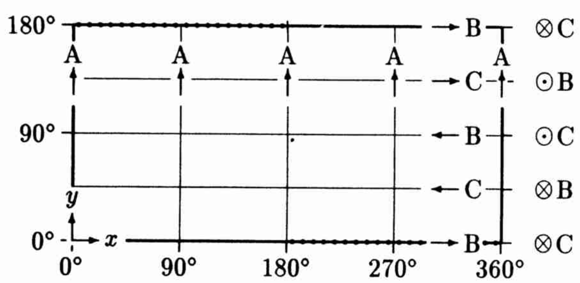
$$(2.6) \qquad \frac{\partial}{\partial x} F(r, x, y) \cdot \frac{\partial}{\partial y} F(r, x, y) = \cos^2 r - \sin^2 r = \cos 2r$$

and two intersecting great circles on ∂T_r make an angle of $2r$. In particular the Clifford torus has orthogonal asymptotic lines and therefore is a minimal surface.

By Definition 1.4 $\partial F/\partial y$ is a right rotating Hopf vector field, whereas $\partial F/\partial x$ is a left rotating Hopf field. Indeed using (1.9) we have $\partial F/\partial y = A \cdot F$. Along an A-line the field $\partial F/\partial x$ once left rotates w.r.t. parallel fields resp. twice left rotates w.r.t. right rotating fields. We describe this left rotating field on the Clifford torus with the right rotating fields from (1.9):

$$(2.7) \qquad \frac{\partial}{\partial x} F(45°, x, y) = (\cos 2y\, B - \sin 2y\, C)\, F(45°, x, y).$$

Drawing the Hopf fields into the domain we get (identification of the boundary as indicated by the dots, see (2.4)).



The inner normal of the solid Clifford torus T_r

$$(2.8) \qquad n(x, y) = -\frac{\partial}{\partial r} F(r, x, y)\Big|_{r=45°} = (\cos 2y\, C + \sin 2y\, B)\, F(45°, x, y)$$

is indicated on the right: Fields with ⊗ point in the normal direction, those with ⊙ in the opposite direction.

The following fact for tori was used in 1.5 and 1.6:

Lemma 2.6 *Let γ and ϑ be two great circles to Hopf fields A and C. Let Θ_s be the great circle along B($\mp \pm A$ and $\mp \pm C$) with $\Theta_s(0) = \gamma(s)$. Suppose that* *for* $s=0$ *we have* $\Theta_0(l)=(0)$, *that is the* arc Θ_0 *of length* $0<l<2\pi$ *ends in* ϑ. *Then*

(i) *either* $\Theta_s(l)$ *is a point of* ϑ *for all* $0\leqq s<2\pi$ *and* γ, Θ_s, ϑ *are contained in the same torus* ∂T_r. *This case is equivalent to either*

$$l=\tfrac{1}{2}\angle_B(A,-C) \quad \text{and} \quad \angle(A,B)=\angle(B,C),$$

or

$$l=\tfrac{1}{2}\angle_B(A,C) \quad \text{and} \quad \angle(A,B)=180°-\angle(B,C).$$

(ii) *Or* $\vartheta \cap \Theta_s$ *for small* $s \neq 0$.

Proof. There is a torus ∂T_r which contains γ and Θ_s for all s. Namely we take A to be one of the left rotating fields, B right rotating, and $\cos 2r=A\cdot B$, see (2.6). Now suppose for (i) that ϑ is also contained in ∂T_r. Since $B\neq \pm C$ then $\pm C$ belongs to the left rotating fields. Let us first assume that $+C$ arises from the right rotation of A along the arc Θ_0. Because the rotation speed is 2 w.r.t. right rotating fields we have $2l=\angle_B(A,-C)$ and $\angle(A,B)=\angle(B,C)$. Otherwise the same equations hold with $-C$.

The intersection of ϑ with ∂T_r is either transverse or tangent. In the latter case and if (i) does not hold the intersection is in a non-asymptotic direction of ∂T_r. So either a first or a second order comparison proves that ϑ intersects the torus (and therefore Θ_s) in a neighbourhood of Θ_0 in the points $\Theta_0(l)$ and $\Theta_0(l+\pi)$ only. $\square$

2.4 *Intersection of solid Clifford torus and great sphere*

Lemma 2.7 *Let* T *be a solid Clifford torus and* c *a geodesic with* $c(0)\in\partial T$, *whose tangent vector* $\dot c(0)$ *is contained in the plane spanned by the inner normal* $n(c(0))$ *of the torus and one asymptotic direction of the torus in the point* $c(0)$. *If* $\dot c(0)\cdot n>0$ *then* $c(t)\in T$ *for* $0\leqq t\leqq 90°$.

Proof. By a rotation of S^3 we assume that $p:=c(0)=F(45°,0,0)$ with normal $n(p)=n(0,0)$ as in (2.3) and (2.8) and we take the great circle $c^*(t)=F(45°,0,t)\subset \partial T$ to be the asymptotic direction. The great circle which makes an angle $\beta(0<\beta<180°)$ with the asymptotic direction is

$$\begin{aligned} c(t)&=\cos t\,p+\sin t(\cos\beta\, c^*(90°)+\sin\beta\, n(0,0))\\ &=\cos t\begin{pmatrix}\frac{1}{2}\sqrt{2}\\ 0\\ \frac{1}{2}\sqrt{2}\\ 0\end{pmatrix}+\sin t\,\tfrac{1}{2}\sqrt{2}\begin{pmatrix}\sin\beta\\ \cos\beta\\ -\sin\beta\\ \cos\beta\end{pmatrix}.\end{aligned}$$

By (2.5) the claim follows from the inequality

$$\text{(2.9)}\qquad \begin{aligned}\tfrac{1}{2}\geqq c_3^2(t)+c_4^2(t)&=\tfrac{1}{2}(\cos t-\sin t\sin\beta)^2+\tfrac{1}{2}\sin^2 t\cos^2\beta\\ &=\tfrac{1}{2}-\cos t\sin t\sin\beta. \quad \square\end{aligned}$$

Corollary 2.8 *Let $G \subset S^3$ be a great sphere containing an asymptotic line c^* of a Clifford torus ∂T. Then $G \cap \partial T = \{c^*\} \cup \{c^{**}\}$, where $c^{**} \subset \partial T$ is another asymptotic line, which intersects c^* in two antipodal points.*

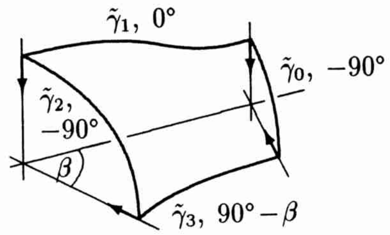
Proof. In (2.9) equality holds for $\beta = 0$ and $t = 90°$, that is the great circles $c^*(t)$ and $c^{**}(\beta) = \cos\beta\, c^*(90°) + \sin\beta\, n(p)$ are contained in ∂T. □

3 Periodic *H*-surfaces

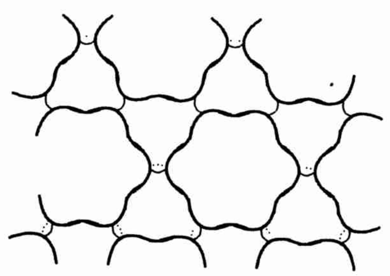
In this section we prove existence of various compact H-surface patches; by reflection we obtain doubly and triply periodic surfaces.

3.1 The simplest doubly periodic surfaces

We consider an H-surface patch $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ with vertex angles $90°$, $0 < \beta < 90°$, $90°$, $90°$ and tilting angles $-90°$, $0°$, $-90°$, $90° - \beta$. Note that with these tilting angles we restrict attention to the onduloid case.



The surfaces extended by reflection are doubly periodic with triangular, quadratic, or hexagonal lattice, if $\beta = 30°$, $45°$, or $60°$. E.g. in the hexagonal case the surface will look like:



We can choose the Hopf fields of $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ to be $-A$, C, $\sin\beta A - \cos\beta C$, $-B$ (with A, B, C orthogonal). It is easy either to check (1.10) or to determine the fields from the degenerate situation $|\gamma_0| = 0$, which is a spherical triangle, using Theorem 1.7.

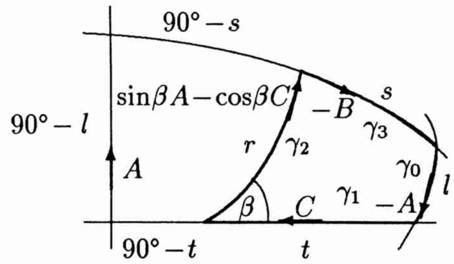
Lemma 3.1 (i) *Let $0<l, t, r, s<90°$ denote the lengths of a quadrilateral along the Hopf fields $-A$, C, $\sin\beta A-\cos\beta C$, $-B$ and $0<\beta<90°$. Then*

$$0<l\leqq\frac{\beta}{2}.$$

(ii) *There exists a continuous one-parameter family Γ_t, $0<t<90°$, of quadrilaterals with the Hopf fields considered. The family consists of* small quadrilaterals *whose lengths l, t, r, s increase monotonously from all lengths 0 to $\beta/2$, 45°, 45°, $\beta/2$ and continues with* large quadrilaterals *where l decreases monotonously to 0 again, but t, r, s increase monotonously to $t=90°$, $r=90°$, $s=\beta$. The two degenerate cases are a point for Γ_0 and a spherical triangle for Γ_{90}.*

Proof. (i) We need the cosine law of spherical geometry $\cos c=\cos a\cos b+\cos\gamma\sin a\sin b$, which holds in a triangle with an angle $0<\gamma<180°$ enclosed by the sides a and b.

At the ends on γ_2 we extend the edges γ_1 and γ_3 to 90°. The resulting endpoints can be joined by an A-perpendicular. We recognize the Hopf fields and the lengths of the extended arcs γ_0, γ_1, γ_2 from the onduloid case of the spherical helicoids, i.e. $a>0$ (see Subsect. 2.1) and therefore the length of the perpendicular arc is $90°-l$. In stereographic projection:



In the given quadrilateral we consider the diagonal passing through the angle β. The cosine law yields for the two rectangular triangles

$$\cos s\cos r=\cos l\cos t. \tag{3.1}$$

Similarly in the other quadrilateral:

$$\sin s\cos r=\sin l\sin t. \tag{3.2}$$

Dividing (3.2) by (3.1) we get

$$\tan s=\tan l\tan t. \tag{3.3}$$

For the other two diagonals we compute:

$$\cos s\cos l=\cos r\cos t+\cos\beta\sin r\sin t \tag{3.4}$$

$$\sin s\sin l=\cos r\sin t-\cos\beta\sin r\cos t. \tag{3.5}$$

Since $\cos l \cdot (3.1) = \cos r \cdot (3.4)$ we can eliminate s

$$(3.6) \qquad \cos^2 l \cos t = \cos^2 r \cos t + \cos\beta \sin r \cos r \sin t.$$

In the same way from $\sin l \cdot (3.2) = \cos r \cdot (3.5)$,

$$(3.7) \qquad \sin^2 l \sin t = \cos^2 r \sin t - \cos\beta \sin r \cos r \cos t.$$

Adding $\sin t \cdot (3.6) + \cos t \cdot (3.7)$ we get

$$\tfrac{1}{2} \sin 2t = \sin 2t \cos^2 r - \cos\beta \sin r \cos r \cos 2t$$

or $\sin 2t \cos 2r = \cos\beta \sin 2r \cos 2t$. So $t = 45°$ is equivalent to $r = 45°$ and otherwise

$$(3.8) \qquad \tan 2t = \cos\beta \tan 2r.$$

Subtracting $\sin t \cdot (3.6) - \cos t \cdot (3.7)$ we obtain

$$\cos 2l \sin t \cos t = \cos\beta \sin r \cos r \Rightarrow \cos 2l = \cos\beta \sin 2r \frac{1}{\sin 2t}$$

and finally by (3.8)

$$(3.9) \quad \cos 2l = \cos\beta \sin 2r \sqrt{1 + \frac{1}{\cos^2\beta \tan^2 2r}} = \sqrt{\cos^2\beta \sin^2 2r + \cos^2 2r},$$

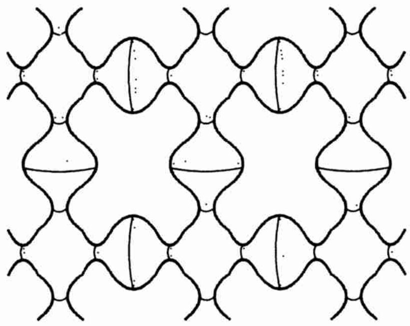
that is $\cos 2l \geqq \cos\beta$.

(ii) We can reverse (i) instead of applying Theorem 1.6: For each $0 < r < 90°$ we get a length $0 < l < \beta/2$ by (3.9), $0 < t < 90°$ by (3.8) and $0 < s < 90°$ by (3.3), and obtain a quadrilateral for which (3.1) (3.2) (3.4) (3.5) hold. Therefore it has the desired Hopf fields. □

A geodesic quadrilateral whose edges are shorter than 180° is contained in the boundary of an H-convex set: Any three vertices of Γ_t are contained in a great sphere and the fourth vertex is in one of the two hemi-S^3. Thereby we get four hemi-S^3 whose intersection is H-convex. From Theorem 1.12 we conclude the existence of the periodic surfaces.

We can lengthen γ_1 and γ_3 by integer multiples of 90° to obtain *long quadrilaterals* with the same Hopf fields: We fix γ_2 and let $\Gamma^{k,\beta} := \{\gamma_0^k, \gamma_1^k, \gamma_2, \gamma_3^k\}$ be the quadrilateral with edge lengths $k\pi/2 < |\gamma_1^k|$, $|\gamma_3^k| \leqq (k+1)\pi/2$, $k \in \mathbb{N}_0$, and with angle $0 < \angle(\gamma_1^k, \gamma_2) = \beta \leqq 90°$. Since short and long perpendiculars alternate, we then have $0 < |\gamma_0^k| \leqq \beta/2$ for $k/2 \in \mathbb{N}_0$, and $90° - \beta/2 \leqq |\gamma_0^k| < 90°$ for $(k+1)/2 \in \mathbb{N}$.

However, only for $k=1$ the lengths do not exceed 180° and the same existence proof works. We obtain surfaces with an additional bubble. $\tilde{M}^{1,45^\circ}$ might look like this:



3.2 H-convex sets for long helicoids

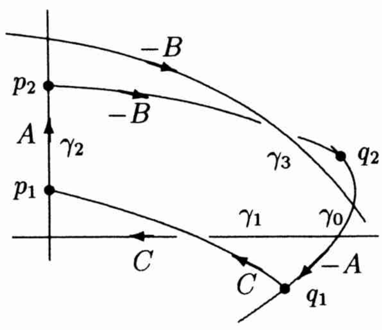
In the onduloid case, 90°-segments $W^a_{\xi/2}$ (see Theorem 2.4) can be embedded in solid Clifford tori.

Lemma 3.2 *Let* $a \neq 0$ *and* $\xi \leqq \alpha = \frac{\pi}{2}\frac{1}{1+|a|}$.

(i) *The helicoid* $W^a_{\xi/2} = \{f^a(u,v) | 0 \leqq u \leqq \xi, 0 \leqq v < 2\pi\}$ *is contained in a solid Clifford torus* T_1.

(ii) *There exists an H-convex set T with* $W^a_{\xi/2} \subset T$ *and* $\partial W^a_{\xi/2} \subset \partial T$.

Proof. (i) It suffices to take $\xi = \alpha$. We consider the helicoid $f^a([0,\alpha] \times [0, \pi/2])$ with its boundary Hopf fields given in 2.1. Let p_1 and q_1 be two points on the A-lines, being 45° away of the $-B$-line, towards γ_0 resp. γ_2. The great circle passing through p_1 and q_1 is again a C-line, since it belongs to the boundary of a quadrilateral with $l=45°$ (i.e. $a = \pm 1$). *Let* T_1 be the Clifford torus with this C-soul.



Being a 45°-Clifford parallel the $-B$-line is contained in ∂T_1. The three other boundary curves of the quadrilateral are also contained in T_1: the C-line is

$|l-45°|$-Clifford-parallel to the soul; the two A-lines are perpendicular to the soul and do not contain points in a distance greater than 45° since $\xi \leqq \alpha < \pi/2$. In fact T_1 encloses the complete half period of the helicoids: every great circle arc $v \mapsto f(u, v)$ is at most in a distance of the soul given by the length of the longer common perpendicular. The perpendiculars are the A-lines and their length is at most 45°.

(ii) We define a second torus T_2 w.r.t. two points p_2 and q_2 on the A-lines in 45°-distance of the C-line. Its soul is a $-B$-line and the C-line is contained in ∂T_2. $T = T_1 \cap T_2$ is H-convex: The solid Clifford torus T_1 is C^2-H-convex and the family of surfaces needed for property (iv. a) of Definition 1.8 consists of Clifford tori ∂T_σ, whose soul interpolates between the one of T_1 and the one of T_2. □

Topologically T is again a solid torus. Let T^c be the universal covering of this torus. Topologically T^c is a solid cylinder and again H-convex. We shall use T^c to embed the long quadrilaterals. Let us describe the remaining barriers in the easiest case of $\Gamma^{k,90}$, which leads to an existence proof for long 90° segments of the onduloids ($a>0$) with the help of Theorem 1.12.

To embed γ_0^k and γ_2 in an H-convex boundary we use great spheres. We call a rectangular geodesic 2-gon bounded by two 180° arcs a *quarter great sphere*. By Corollary 2.8 two asymptotic lines of a Clifford torus bound a quarter great sphere.

Let G_1 be a quarter great sphere with boundary in T_1^c defined as follows. Let one boundary curve be a 180°-arc of the $-B$-line and let the A-line γ_2 be contained in the interior of G_1. The boundary curve $\Gamma^{k,90}$ is seen to be completely contained in one of the two components of $T^c - G_1$ by the following argument: Each Clifford parallel to the soul only intersects a quarter sphere once, since in general a great circle only intersects a great sphere in two antipodal points. However int G_1 does not contain any of its antipodal points. The great circle γ_3 bounding the quarter sphere and the Clifford parallel to the soul γ_1 have the same sense of rotation and therefore the long edges of the quadrilateral lie on the same side of the quarter great sphere. Condition (iv. b) of Definition 1.8 can be satisfied by moving G_1 from ∞ to its position.

In the same way we obtain a quarter sphere $G_2^k \subset T_1^c$ at the other end of the quadrilateral. Then the compact connected component $N^{k,90}$ of $T^c - G_1 - G_2^k$ is an H-convex set, satisfying the requirements of Theorem 1.12.

3.3 Doubly periodic surfaces with many bubbles on the edges

We modify the H-convex set $N^{k,90}$ presented in the last section for the boundary curve $\Gamma^{k,\beta}$. By Lemma 2.7 the $(\sin\beta A - \cos\beta C)$-line is contained in T_1. Indeed it lies in the plane of the torus normal $-A$ and the asymptotic line C (since T_1 has a C-soul, C is asymptotic direction on the whole of ∂T_1); finally its length is $r \leqq 90°$. Similarly this line is contained in T_2, where A is normal and C asymptotic line.

We can take over G_2^k from 3.2. We intersect T_1^c on the side of γ_2 with a quarter great sphere G_1^β, bounded by a 180° arc of the $-B$ asymptotic line and with the $(\sin\beta A - \cos\beta C)$-line in its interior. Again we can move G_1^β from

∞ to satisfy Definition 1.8(iv. b). The argument given in 3.2 implies that the boundary curve is completely contained in a connected component $N^{k,\beta}$ of $T^c - G_1^\beta - G_2^k$. Theorem 1.12 provides a solution $M^{k,\beta}$ of Plateau's problem for the long quadrilaterals. The solution $M^{k,\beta}$ has k *bubbles* on an edge, that is the curvature lines along the edges have length in between $k\pi$ and $(k+1)\pi$.

Theorem 3.3 *Given an angle* $\beta \in \{\pi/3, \pi/4, \pi/6\}$ *there is a family* $\tilde{M}_t^{k,\beta}$ *of* H-*surfaces, with any number of bubbles for the edges* $k \in \mathbb{N}$. *The family is continuous with respect to the parameter* $0<t<\pi/2$. *The surfaces are contained in a slab of* $\mathbb{R}^3$ *and their fundamental domain is contained in a triangle with angles* β, $90°$, $90°-\beta$ *times the ray* $[0, \infty)$.

Proof. It remains to prove continuity of the family. From Lemma 3.1 we know that $t \to \Gamma_t$ is continuous. Suppose the spherical Plateau solutions M_s do not converge to M_t for $s \to t$ w.r.t. distance of surfaces. By the technique of Theorem 4.4 we get another minimal surface M bounded by the same curve $\Gamma = \Gamma_t^{k,\beta}$ as M_t. Therefore continuity for M is reduced to uniqueness. But a continuous convergence is also in C^2 (or C^∞) in the interior and hence the associated surfaces $\tilde{M}_s$ converge w.r.t. distance (or in C^∞) in the interior; applying this argument to the surfaces extended by reflection yields the convergence in the desired sense up to the boundary.

To prove uniqueness we specify a family of isometries Ψ_σ of S^3 with $\Psi_0 = \mathrm{id}$ and $-\sigma_0 < \sigma < \sigma_0$, and find an H-convex set $N \supset \Gamma$ such that for every $0 < |\sigma| < \sigma_0$ there are neighbourhoods of Γ and $\Psi_\sigma(\Gamma)$ disjoint to $\Psi_\sigma(N) \cap N$. Then the two Plateau solutions M and $\Psi_\sigma(M)$ do not intersect each other in a neighbourhood of their boundaries. We claim they cannot intersect in the interior. Both surfaces are minimizing, and a replacement of any piece of one surface cut out by a piece of the other surface does not reduce area. But the intersection is transveral at all but finitely many points in the intersection set, so that smoothing out the edge made by a piece of one and a piece of the other surface does reduce area. Thus M is enclosed in between $\Psi_\sigma(M)$ and $\Psi_{-\sigma}(M)$ for any $\sigma \neq 0$.

For Ψ_σ we take the flow to the Killing field $B-C$. The Clifford torus ∂T_1^c contains the arc γ_3, being a $-B$ integral and it contains C-lines as asymptotic lines, because the T_1-soul is a C-line. Thus the field $B-C$ is tangent along γ_3. No other bundary arc is contained in ∂T_1^c and hence sufficiently small rotations of T_1^c with axis γ_3 also contain the quadrilateral Γ. We take N_1 to be the intersection of two such tori rotated in a different sense. Then N_1 does not contain (a neighbourhood of) the $\pm(B-C)$-direction along the arc γ_3. In the same way we define N_2 for the arc γ_1 along which the direction $B-C$ is again tangent to T_2^c.

We now describe two pairs of great spheres for the remaining arcs. This is more technical because we have to convert Hopf fields into parallel fields. We give the barriers for $k=0$ so that we can use hemi-S^3; for the general case they have to be replaced by the appropriate component in the covering space. We start with the arc γ_0. The great sphere G_2 from 3.2 has a normal $\nu(\gamma_0(0)) = -B$. On a $90°$ arc γ along A the field B rotates to C (see Definition 1.3). That is a parallel translation of $-B$ from $\gamma(0)$ to $\gamma(90°)$ yields C. Viewing the $-A$-line γ_0 in decreasing parameterization we conclude that the normal of G_2 along γ_0 is $\nu(\gamma_0(s)) = -\cos sB + \sin sC$. But γ_0 is shorter than $90°$, and so $\nu(\gamma_0(l_0)) \cdot \dot{\gamma}_1$ is positive for the C-line γ_1. γ_1 and γ_2 are no longer than $90°$ and hence contained in the hemi-$S^3 = \{x \in S^3 \mid x \cdot v \geqq 0\}$. Thus Γ is contained in that hemi-S^3.

We consider a great sphere G_3 containing γ_0 and γ_1. We claim that Γ is contained in the hemi-S^3 defined by $v(\gamma_0(l_0)) = B$. Indeed, $v(\gamma_0(s)) = -\sin(l_0 - s)\, C + \cos(l_0 - s)\, B$ has positive scalar product with B in 0. Again $v(\gamma_0(0)) \cdot (-\dot\gamma_3(l_3)) > 0$ as the latter direction is B and $l_0 < 90°$.

Now we have to show that the intersection N_3 of the two hemispheres does not contain the vector $\pm(B - C)$ along the arc γ_0. That is the product of $\pm(B - C)$ with v has to be negative for at least one normal for each sign of $B - C$. For the normal of G_2 we compute $v(\gamma_0(s)) \cdot (B - C) = -\cos s - \sin s$ and $v(\gamma_0(s)) \cdot (B - C) = \cos(l_0 - s) + \sin(l_0 - s)$ for the normal of G_3. Clearly the first product is negative and the second product is positive for $0 \leqq s \leqq l_0 < 90°$, so the second one is negative taken with $-(B - C)$.

We need similar calculations along γ_2. A normal of G_1^β at $\gamma_2(l_2)$ is $v(\gamma_2(l_2)) = -\cos\beta A - \sin\beta C$. As $\angle_{\sin\beta A - \cos\beta C}(-B, -\cos\beta A - \sin\beta C) = +90°$ the parallel translation of v to the point $\gamma_2(l_2 - 90°)$ is $+B$. Now $v(\gamma_2(s)) = \sin(l_2 - s)\, B + \cos(l_2 - s)(-\cos\beta A - \sin\beta C)$ and the product with $-C$ is positive so that Γ lies in the hemi-S^3 defined by v.

For a great sphere G_4^β defined by the normal $-B$ in the point $\gamma_2(0)$ the normal is $v(\gamma_2(s)) = -\cos s B + \sin s(\cos\beta A + \sin\beta C)$. Again $v(\gamma_2(l_2)) \cdot (-B) > 0$ and Γ is on the side of v. Finally we compute the products $\pm(B - C) \cdot v$ along γ_2. For the normal of G_1^β we have $(B - C) \cdot v(\gamma_2(s)) = \sin(l_2 - s) + \cos(l_2 - s) \sin\beta$ and $(B - C) \cdot v(\gamma_2(s)) = -\cos s - \sin s \sin\beta$ for G_4^β. The second product is negative for the given sign of $(B - C)$ as the first one is for the other sign. Hence the intersection of these two hemi-S^3, N_4, with N_1, N_2, and N_3 is an H-convex set with the property claimed.

When we compare the surface M with itself moved along the flow $\Psi_\sigma(M)$ this proof recovers M as a graph w.r.t. the $B - C$ direction. □

Remark. The H-surfaces have a different asymptotic behaviour for large and small quadrilaterals.

(i) The large quadrilaterals belong to H-surfaces with *spherical centres.* In fact, taking the limit $t \to 90°$ the large quadrilaterals $\Gamma_t^{0,\beta}$ converge to a triangle contained in a great sphere $S^2 \subset S^3$ and the associated H-surfaces converge to punctured spheres which touch each other with the given symmetry. The same applies to the centre spheres of the surfaces $\tilde M_t^{k,\beta}$. For a small neck perimeter $4|\gamma_0|$ the existence of the surfaces with a spherical centre was proved by Kapouleas [Kp1].

However, for the small quadrilaterals the cases $k = 0$ and $k \in \mathbb{N}$ are different.

(ii) Computing the lengths of the small quadrilaterals $\Gamma_t^{0,\beta}$ in terms of t, we see from (3.8) and (3.9) that r and l shrink of order $O(t)$, whereas s shrinks of order $O(t^2)$ by (3.3). Asymptotically the patches are therefore triangles. Hence when we blow up the associated surfaces in a way that t and the lattice is of constant size $(H \to 0)$, the patches tend to flat triangles and the complete H-surfaces $\dfrac{1}{O(t)} \tilde M_t^{0,\beta}$ tend to punctured double planes. That is $\tilde M_t^{0,\beta}$ itself lies in a slab of height $O(t^2)$ and converges the more to a plane of multiplicity 2, but the singular points become dense.

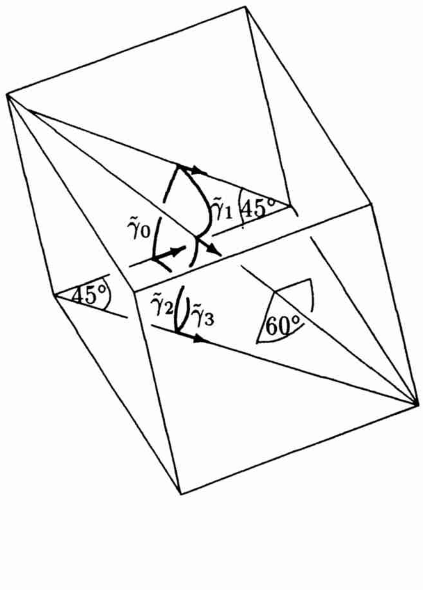
(iii) The associated H-surfaces $\tilde M_t^{k,\beta}$ for $k > 0$ have k additional bubbles on the edges of the lattice. The blow up of the surfaces belonging to small quadrilaterals can be shown to converge to the Jorge-Meeks n-noids. Indeed, a blow up by $1/O(t)$ in the space $S^3_{1/O(t)}$ has two lines γ_1 and γ_3 of unbounded length $1/O(t)$ and one line γ_2 of bounded length $O(1)$. The limit boundary contour coincides with the associated boundary contour of the Euclidean n-noids. By an appropriate maximum principle at infinity (for minimal surfaces with boundary going to infinity) it can be proved that minimal surfaces spanned by the limit contour are unique. A proof will appear elsewhere. We say these surfaces have *n-noidal centres.*

The hexajonal surface $\tilde{M}^{0,\pi/3}_{45}$ and the quadratic surface $\tilde{M}^{0,\pi/4}_{45}$ with maximal neck size in their family were found by Lawson [L, Theorem 9]. They join the surfaces to large and small quadrilaterals in the family.

Centre-spheres and centre n-noids within the same surface. A rotation by 180° with axis γ^k_0 of the quadrilaterals $\Gamma^{k,\beta}_t$ is contained in the same topological torus T^c from 3.2 which contains $\Gamma^{k,\beta}_t$ itself. The Plateau solutions are associated to doubled fundamental H-patches and yield the same surfaces $\tilde{M}^{k,\beta}_t$ as before. By Lemma 3.1 the quadrilaterals $\Gamma^{k,\beta}_t$ and $\Gamma^{k,\beta}_{90-t}$ have the same length $|\gamma_0|$. We can glue them together along γ_0 and obtain H-surfaces each k-bubble edge of which joins a spherical with an n-noidal centre in case $\beta=45°$ and $\beta=60°$. Of course the same barriers work. This yields another one parameter family, being continuous w.r.t. to t with the same proof as in Theorem 3.3. One of the symmetries of $\tilde{M}^{k,\beta}_t$ is removed and the family under consideration coincides with the former family in the surface with the parameter $t=45°$.

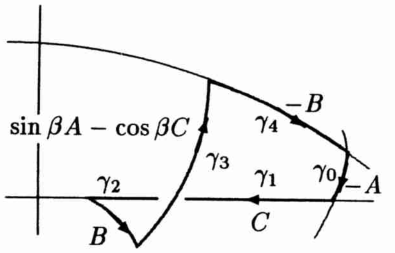
3.4 *Triply periodic surfaces*

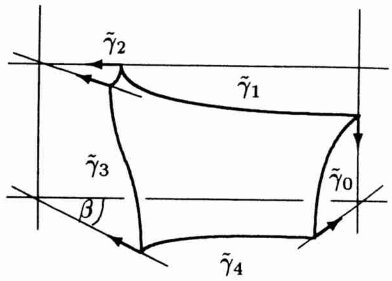
(a) Cubic lattice. The simplest triply periodic surface has the symmetry of a cubical lattice and is the H-surface analogue of the Schwarz P-surface. Again



there may be k bubbles on the edges. The boundary curve $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ has Hopf fields $-A$, $\frac{1}{2}\sqrt{2}(C+B)$, $\frac{1}{2}\sqrt{2}(A-C)$, $-B$. By the Handle-Theorem 1.7 the boundary curve closes for small $|\gamma_0|$. We are in the case of $\xi=\alpha/2$ of Lemma 3.2 and the respective tori as well as great spheres similar to those defined in the last section serve as barriers.

(b) Layers of doubly periodic surfaces joined by handles. We replace the vertex at the angle β in the quadrilateral from 3.3, $0<\beta\leqq 90°$, by a fifth edge along the field B to obtain an additional outward handle of the H-surface patch ($\angle_B(-C, \sin\beta A-\cos\beta C)=-\beta$):

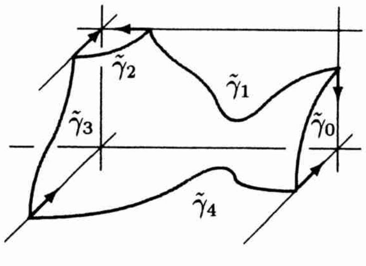


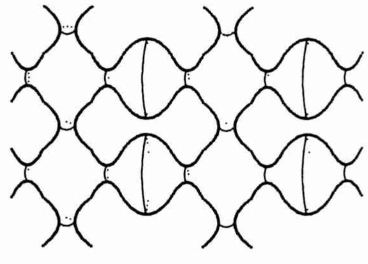


The Handle Theorem 1.7 ensures existence of the boundary curve with a small handle length $|\gamma_2|$ for any length of γ_0 less than $\beta/2$; only in case $l=\beta/2$ assumption (1.11) is not satisfied since $\gamma_1, \gamma_3, \gamma_4$ are then contained in a Clifford torus. In fact the handle length can be any value in $(0, \beta/2-l]$.

We can use the barriers from 3.2 and 3.3 to embed the boundary curve in an H-convex boundary. For $|\gamma_2|$ small the pentagon is certainly contained in T_1^c. The Clifford torus ∂T_2^c contains γ_2 and thus the pentagon is contained in T_2^c. To embed γ_3 we choose a great sphere containing this curve and the handle γ_2. Since the latter is an asymptotic line for T_2^c this great sphere cuts out a quarter sphere of T_2^c and γ_1 and γ_4 are on the same side of this great sphere by the argument given in 3.2. For γ_0 we use the quarter great sphere G_2^k defined in 3.2. Every vertex of the polygon is contained in two of these four barriers.

The complete H-surface consists of layers of doubly periodic surfaces, similar to those described before, whose vertices are vertically joined by handles. In case $\beta=45°$ the Schwarz P-surface is generalized in a less symmetric fashion than in (a). For $\beta=30°$ and $\beta=60°$ we obtain an H-surface analogue to A. Schoen's H'-T-surface. For $\beta=90°$ we get a merely doubly periodic H-surface: The lattice consists of rectangles the long edge of which takes k additional bubbles. E.g. for $k=1$ the fundamental domain and a portion of surface can be sketched as follows:





(c) Every second vertex joined by a handle. In the same way as in 3.3 we can consider boundary curves of double length. A handle on one side only yields a surface every second vertex of which is joined to the next layer. This makes sense for $\beta = 45°$ and $\beta = 60°$.

4 Convergence of ends

In Subsect. 3.3 we found Plateau solutions $M^{k,\beta}$ to quadrilaterals Γ^k of length approximately $k\pi$. For instance in case $\beta = 60°$ these are associated to H-surfaces with a triangular lattice and have k bubbles on the edges of the lattice (see second figure in Subsect. 3.1). To obtain a "Delaunay-surface with 3 ends" we let k tend to ∞. Convergence is proved for the spherical patches. In general these are not graphs and we use a local description. Interior convergence of the sequence is implied by curvature estimates and locally uniform area bounds. No regularity problems arise at the piecewise geodesic boundary. However to know the limiting position of the normal at the boundary of the end we have to prove uniqueness of the end in Subsect. 4.4.

4.1 Curvature estimate and graph lemma

A minimal surface is *stable*, if the second variation of the area is non-negative. R. Schoen proved that stable surfaces, in particular solutions to Plateau's problem, have uniformly bounded curvature outside a neighbourhood of the boundary curve:

Theorem 4.1 [S, Theorem 3] *Let M be a stable minimal surface immersed in a 3-manifold N and $x \in M$ be a point, such that the geodesic ball $B_r^2(x)$ is contained in M $(0 < r \leqq 1)$. Then the norm of the second fundamental form is bounded,*

$$(4.1) \qquad |A(x)|^2 := \langle Se_1, e_1\rangle^2 + 2\langle Se_1, e_2\rangle^2 + \langle Se_2, e_2\rangle^2 \leqq C(N, r),$$

where C depends on bounds for the sectional curvature of N and its derivative.

Locally hypersurfaces can be described as graphs. In manifolds we can take normal coordinates in a geodesic ball $B_r^3(p) \cap M$ over the tangent plane $T_p M$. The radius $r > 0$ is uniform in p, if the curvature of the hypersurface is bounded:

Lemma 4.2 [Wh, Lemma 1; KKS; S] *Suppose N^3 is a manifold with injectivity radius bounded away from 0, whose metric has bounded C^2-norm in normal coordinates. Let M be a minimal surface in N whose second fundamental form is bounded on a domain $\Omega \subset M$,*

$$|A(x)| \leqq A_0 \quad \text{for all } x \in \Omega.$$

Then there exists a radius $\rho(A_0, N)$ such that for a point p with $B_{2\rho}^2(p) \subset \Omega$ (i) *the connected component of $M \cap B_\rho^3(p)$ which contains p is a disk and* (ii) *is graph w.r.t. the tangent plane $T_p M$.* (iii) *Furthermore choosing $\rho(A_0, N, \alpha)$ the coordinates u of the graph have bounded Hölder norm on $B_\rho^2(0) \subset T_p M$:*

$$\|u\|_{C^{2,\alpha}} \leqq C(A_0, N, \alpha).$$

Note this lemma also allows to describe a surface as graph over slightly tilted planes close to the tangent plane.

4.2 Convergence of compact minimal surfaces

Definition 4.3 A sequence M_k of C^1-surfaces satisfies *uniform local area bounds* if there is a radius r such that for each ball with $B^3_\rho(x)\cap \partial M_k=\varnothing$ and all $\rho<r$ holds

$$|M_k\cap B_\rho(x)|<C.$$

In Subsect. 4.3 we give uniform local area bounds for helicoid ends. Assuming area bounds we solve improper Plateau problems, that is problems with non-closed boundary curve, using an exhaustion:

Theorem 4.4 *Let N^3 be a non-compact closed manifold with boundary (whose curvature tensor is C^1-bounded) and Γ a connected non-closed curve in N, being piecewise C^1. Assume there exist compact sets N_n* with $N_{n-1}\subset N_n$ and $\bigcup\limits_{n\in\mathbb{N}} N_n=N$, *and closed Jordan curves $\Gamma_n\subset N_n$* with $\Gamma_k\cap N_n=\Gamma\cap N_n$ for all $k>n$. *Finally let the Plateau problems for Γ_n be solvable with a sequence of minimal disks $M_n\subset N_n$ satisfying uniform local area bounds. Then there exists a minimal surface $M\subset N$ of the type of the disk which is regular in the interior and whose boundary is Γ.*

Proof. Let $N^r_n:=N_n-U_r(\bigcup\limits_{k>n}\Gamma_k)$ where U_r denotes an open r-neighbourhood of a set. Because Γ is piecewise C^1 we have $\bigcup\limits_{r>0} N^r_n=N_n-\Gamma$ and N^r_n is non-empty for small r. The Plateau solutions M_k accumulate in int N^r_n, for sufficiently small r; indeed, if there were accumulation points in ∂N_n only (for all but finitely many n), the maximum principle would yield a minimal surface in ∂N bounded by Γ and the proof finished. Thus we find a point $p\in\operatorname{int} N^r_n$, such that a subsequence of $k\mapsto M_k$ converges in distance to p. Taking a further subsequence we assume the normals to converge as well.

For each $x\in N^{2r}_n\cap M_k$, $k>n$, we have $\operatorname{dist}(x,\partial M_k)\geqq 2r$, and thus Theorem 4.1 provides a curvature bound on N^r_n, $|A(x)|^2\leqq C(N,r)$ for all $x\in N^r_n\cap M_k$. By the graph Lemma 4.2 we find a radius $\rho(\leqq r/2$ wlog.), such that $M_k\cap B^3_{2\rho}(p)$ is graph over the tangent plane $T_p M_k$ provided $p\in M_k\cap N^{2r}_n$. Therefore a connected component of a subsequence of M_k converges in $C^{2,\alpha}(B_\rho(p))$ as graph w.r.t. the limit tangent plane $T_p N$.

We cover N^{2r}_n with balls $B^3_\rho(y_i)\subset N^r_n$, $i=1,\dots,I$. For an i_1 we have proved convergence of a connected component of M_k in $B_\rho(y_{i_1})\subset B_{2\rho}(p)$. Since a subsequence converges in the whole ball $B_{2\rho}(p)$ there is a point $q\in B_{2\rho}(p)-B_\rho(y_{i_1})$ of convergence. Let q be in $B_\rho(y_{i_2})$, $i_1\neq i_2$. By another application of the graph lemma we obtain convergence of a connected component of a further subsequence in $B_{2\rho}(q)$ and thus in $B_\rho(y_{i_2})$. We continue to consider further balls $B_\rho(y_{i_l})$ such that either $i_l\neq i_1,\dots,i_{l-1}$ or for $i_l=i_j(l>j)$ another connected component of the surface is considered. By the uniform area bound the procedure ends after finitely many steps. Therefore we obtain $C^{2,\alpha}$-convergence of a subsequence of M_k, again denoted by $M^{(n)}_k$ to a surface $M^{(n)}_\infty$, such that $\partial M^{(n)}_\infty\cap N^r_n=\varnothing$.

Now we consider a sequence $N_n^{r_n}$ with $r_n \searrow 0$. With finitely many balls of smaller radius $\rho_n/2 < r_n$, given by curvature estimate and graph lemma, we extend the covering of $N_{n-1}^{r_{n-1}}$ to $N_n^{r_n}$. From the sequence $M_k^{(n-1)}$ we choose a subsequence $M_k^{(n)}$ converging on $N_n^{r_n}$. The diagonal sequence $M_n^{(n)}$ converges on each compact set $K \subset N - \Gamma = \bigcup_{n\in\mathbb{N}} N_n^{r_n}$ to $M = \bigcup_n M_\infty^{(n)}$ in $C^{2,\alpha}$. Since M has no boundary points in the interior of N and each subsequence accumulates in Γ we have $\partial M = \Gamma$.

Since the convergence is uniform in distance on each compact set we can compare M with a sufficiently close minimizing surface M_n. Hence M is a stable minimal surface and each curve in M is contractible. For connected Γ there is only one connected component M. □

If Γ consists of geodesic arcs making an angle of $\pi/(m+1), m\in\mathbb{N}$, then M is smooth at the boundary (see Subsect. 1.7). If N is H-convex and all vertices are contained in two barriers as in Theorem 1.12 M is free of boundary branch points.

4.3 *Uniform local area bounds for finite helicoid ends*

First we describe coordinates on the covering of a solid Clifford torus induced by the foliation of Subsect. 2.2.

Let a helicoid $W_\xi^a = f^a([-\xi, \xi], [0, 2\pi))$ for $\xi < \alpha$ and $-1 < a \leqq 1$, $a \neq 0$, be given together with its induced foliation $\Psi: \mathbb{R} \times (-\alpha, \alpha) \times \mathbb{R} \to S^3$. We consider closed sets $N \subset S^3$ satisfying three properties: (i) N contains the helicoids W_ξ^a, and $\zeta_\pm(v) := f^a(\xi, v)$ is contained in ∂N. (ii) The class of the curves $\zeta_\pm$ is the only non-trivial homotopy class in N and (iii) N contains no curve linked with $\eta_\pm = f^a(\pm a, .)$. Let X denote the connected component of $\Psi^{-1}(N - \eta_\pm)$ containing $(0,0,0)$. Because of (iii) and Lemma 2.3(ii) if $(\phi, u, v) \in X$ then $(\phi \pm \pi, u, v) \notin X$, so that $X \subset (-\pi, \pi) \times [-\xi, \xi] \times \mathbb{R}$. By (ii) and Lemma 2.3(iii), (iv) the parameterization Ψ lifts to a bijection $\Psi^c: X \to N^c - \eta_\pm^c$. By (i) X contains the set $0 \times [-\xi, \xi] \times \mathbb{R}$. We omit the suberscript c for N and $\eta_\pm$.

Let $N_x^y := \{\Psi^c(\phi, u, v) \in N - \eta_\pm \mid x \leqq v < y\}$ be the portion of N between x and y, $D_x := \{\Psi^c(\phi, u, x) \in N\}$ be the lateral boundary of N and $D_x^y := D_x \cup D_y$. We prove that k-times the area of the helicoid $w_\xi^a := |f^a([-\xi, \xi] \times [0, \pi))|$ is a lower bound for the area of any piece of length $2\pi k$ of the surfaces $M = M_t$ defined in Sect. 3:

Lemma 4.5 *Let $M \subset N_0^{\pi k}$ be an immersed C^1-surface with* finite helicoid end, *i.e.*

(i) $\partial M \subset \zeta_\pm^c \cup D_0^{\pi k}$ *with $\zeta_\pm$ as above.*
(ii) *Every closed curve in M is contractible in $M \cup D_0^{\pi k}$.*
(iii) *The two connected components of $(\partial N_x^y - D_x^y) - \zeta_\pm^c$ are contained in different components N_+ and N_- of $N - M$.*

Then we have the following area bound, where $c := 2|D_0^{\pi k}|$:

$$|M| \geqq k w_\xi^a - c. \tag{4.2}$$

Proof. We find an embedded comparison surface of the type of the disk which has the boundary of the helicoid. Let $M_0 := M \cap \partial N_+$. Rounding off corners and edges of M_0 yields an embedded surface M_1 with $|M_1| \leqq |M_0|$. We discard for all connected components of M_1 except for the one containing $\zeta^c_\pm$. $M_1 \cap D_0$ is a disjoint union of closed Jordan curves as well as another curve with end points $\zeta^c_\pm(0)$, being homotopic to a boundary curve of a helicoid $\gamma_0 \subset D_0 \cap W^a_\xi$. The area of this homotopy is at most $|D_0|$. The closed curves in $M_1 \cap D_0$ bound disjoint disks with total area less than $|D_0|$. Proceeding at the other end $D_{\pi k}$ in the same way, we obtain an embedded surface of the type of the disk by (ii) and with boundary of the helicoids. From Corollary 2.5, applied to k periods of the helicoid, we infer (4.2). $\square$

Lemma 4.6 *Let M_k be a sequence of minima of area fulfilling the hypothesis of Lemma* 4.5. *Then M_k satisfies uniform local area bounds.*

Proof (idea of E. Kuwert). Γ_k bounds a surface of area $k w^a_\xi + c_1$, where c_1 is the area of a disk spanned by the difference of the curve Γ_k to the helicoid boundary. This together with (4.2), applied to outer portions of M_k, yields an estimate for the area of a middle part $|M_k \cap N^{\pi(l+m)}_{\pi l}|$, where $l+m+n=k$:

$$\begin{aligned}|M_k \cap N^{\pi(l+m)}_{\pi l}| &= |M_k \cap N^{\pi(l+m+n)}_0| - |M_k \cap N^{\pi l}_0| - |M_k \cap N^{\pi(l+m+n)}_{\pi(l+m)}| \\ &\leqq (l+m+n)\, w^a_\xi + c_1 - l w^a_\xi + c - n w^a_\xi + c \\ &= m w^a_\xi + c_1 + 2c. \quad \square\end{aligned}$$

4.4 Uniqueness of ends and convergence of boundary normals

We call a non-compact minimal surface M contained in a half cylinder $N \cap \{v \geqq 0\}$ a *helicoid end* if ∂M is a Jordan curve consisting of two geodesic rays $\zeta^c_\pm(s)$, $s \in \mathbb{R}^+_0$, joined by a compact curve ζ_0. Here again $\zeta^c_\pm(s)$ denotes a covering of $f^a(\pm\xi, s)$ with nonzero $-1 < a \leqq 1$ and $\xi < \alpha$. We can pull back the end in its H-convex barriers to obtain a limit surface which is periodic. By Corollary 2.4 the limiting helicoid type is uniquely determined by $\zeta^c_\pm$. We say an end satisfies uniform local area bounds if this holds for the pull back sequence $M_k := \Xi_{-k} M$, where the pull back $\Xi: N \to N$ is defined by $\Xi_{-k}(p) = \Psi^c((\Psi^c)^{-1}(p) - (0, 0, k\pi))$.

Theorem 4.7 *Let M be a minimizing* (*w.r.t. compactly supported variations*) *helicoid end, satisfying uniform local area bounds and contained in an H-convex set N. Then M_n converges in distance to a helicoid W^c uniquely determined by $\zeta^c_\pm$.*

Proof. Similarly to Theorem 4.4 we get convergence of a subsequence of M_k to a minimal surface M_0, using area and curvature bounds. The convergence is in $C^{2,\alpha}$ in the interior.

Let us first note that we can assume the coordinates Ψ^c to be defined in a neighbourhood of any point $p \in \operatorname{int} M_0$. Otherwise p is contained in the boundary ∂N and by the maximum principle either $\operatorname{int} M_0 \subset \operatorname{int} N$ or M_0 coincides with a subset of ∂N bounded by $\zeta^c_\pm$, which must be W^c by minimality of M.

Suppose for a moment in addition to our assumptions that

$$\Xi_k M \text{ is a barrier for } M, \tag{4.3}$$

then $\Xi_{-k} M$ and $\Xi_{-k-1} M$ intersect in the boundary rays $\zeta^c_\pm$ only. Thus M_k converges monotonely to M_0 and therefore the whole sequence converges and M_0 is invariant under covering transformations, $\Xi M_0 = \lim \Xi_{-k} M = M_0$, that is M_0 projects to S^3. From Theorem 2.4 we obtain the uniqueness $\Pi M_0 = W^a_\xi$. (4.3) holds for all surfaces with cylinder ends we describe.

In general consider

$$\lim_{k\to\infty} \sup\{\phi \mid \Psi^c(\phi, u, v) \in M \cap N^\infty_{\pi k}\} =: \mu.$$

Since $\Psi^c(0, \pm\xi, v) \in \zeta^c_\pm \subset \partial M$ and Ψ^c is bounded by π we have $0 \leqq \mu \leqq \pi$. Suppose $\mu > 0$. Then there is a sequence of points $p_k := \Psi^c(\phi_k, u_k, v_k) \in M \subset N^\infty_{\pi k}$ with $\phi_k \to \mu$. Replacing $M_k := M \cap B^3_\varepsilon(p_k)$ by $\{\Psi^c(\phi - \varepsilon_k, u, v) \mid \Psi^c(\phi, u, v) \in M_k\}$, where $\varepsilon_k \searrow 0$, if necessary, we assume $\phi_k \leqq \mu$ for all $\Psi^c(\phi, u, v) \in M_k$. But for any subsequence M_{k_i} such that $(u_k, v_{k_i} \bmod 2\pi)$ converges to (u_0, v_0), ΠM_{k_i} is a minimal surface which converges in a uniform neighbourhood of $\Psi(\mu, u_0, v_0)$ to a M_0 by curvature estimate and graph lemma. In particular by the maximality of μ the normals converge. Since M_0 lies locally on one side of the minimal surface $\Psi(\mu, u, v)$ and touches in the interior point $\Psi(\mu, u_0, v_0)$ the surfaces locally agree by the maximum principle. Hence any subsequence M_{k_i} converges to $\Psi(\mu, u_0, v_0)$ in a uniform neighbourhood (depending on the distance to the boundary). Repeating these arguments for any accumulation point $\Psi(\mu, u, v)$ we infer that $\Xi_n M \to \Psi^c(\mu, u, v)$. By the boundary values we obtain $\mu = 0$. Applying the same arguments to $\lim\limits_{k\to\infty} \inf\{\dots\}$ proves the result. $\square$

Now we have proved that the spherical ends converge in distance a helicoid. Next we show C^1-convergence at the boundary: when moving towards infinity the normals at the boundary converge to those of the helicoids.

Since we know convergence in distance, we can use a local barrier argument. Take a point $p = f^a(u, v) \in \partial W^a$ in the boundary of a helicoid. Extend the helicoid to a full neighbourhood W^a_* of p and let $D := B^3_\varepsilon \cap W^a_*$ be a minimal disk. Now we rotate D first around the normal $n(p)$ by an angle $\pm\psi$. Since the principal curvatures of the helicoid do not vanish the rotated disks do not intersect the boundary of the helicoid ∂W^a except for the point p itself (for small ψ and ε). The tangent planes in p agree. Thus a further rotation of the disks around the boundary circle ∂W^a by the angle ψ yields two disks $D^\pm_\psi$ that do not intersect the helicoid, provided the rotations have the right sense.

Let M_k be a sequence of minimal surfaces converging in distance to the helicoid and with boundary of the helicoid. If M_k is close enough to the helicoid in a neighbourhood of a point p then ∂M_k is on one side of $D^\pm_\psi$. By the maximum principle the tangent halfplane at $T_p M_k$ is contained in the wedge between the tangent halfplanes of D^+_ψ and D^-_ψ. As ψ tends to 0, $T_p M_k$ converges to the tangent plane of $D^\pm_0 = W^a$ in p. A similar construction works for points with vanishing principal curvatures. Slightly generalizing we get

Lemma 4.8 *Let M_k be a sequence of minimal surfaces with the same C^2-boundary portion Γ which converges in distance to a minimal surface M of class C^2 including Γ. Then at each interior point of Γ the normals of the surfaces M_k converge to the normal of M.*

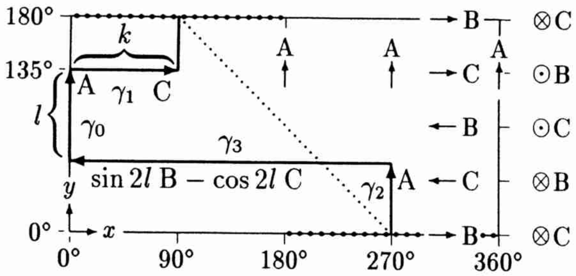
Thus we can read off the geometric data from the ends of the minimizing helicoid defined by $\zeta_\pm$ in case $\xi < \alpha$.

5 *H*-surfaces with cylinder ends and their periodic cousins

5.1 *Surfaces of Pinkall and Sterling*

The following surfaces are parameterized by a torus and described in [PS] where an explicit representation by solutions of the sinh-Gordon-equation is given. See also [We].

Simply periodic. The simply periodic cousins of these surfaces have immersed ends not converging to Delaunay surfaces. Take a solid Clifford torus T with Hopf fields as in Subsect. 2.3. Let $\Gamma^{k,l} \subset \partial T$ for $k>0$, $0<l<180°$ be a curve in ∂T, being the image of the following curve in parameter space (in the sketch we take $k=90°$):



$\Gamma^{k,l}$ is embedded in the Clifford cylinder T^c and has geometric data:

$$
\begin{array}{ll}
|\gamma_0^{k,l}|=l, & \angle_A(\cos 2l\, C-\sin 2l\, B, C)=-2l,\\
|\gamma_1^{k,l}|=k\pi, & \angle_C(-A, A)=180°,\\
|\gamma_2^{k,l}|=\pi-l, & \angle_A(-C, \sin 2l\, B-\cos 2l\, C)=2l-360°,\\
|\gamma_3^{k,l}|=\pi+k, & \angle_{\sin 2l\, B-\cos 2l\, C}(-A, A)=-180°.
\end{array}
$$

The given rotational Hopf angles are measured inside T in the sense of Theorem 1.12(ii). Namely along γ_1 the inner normal of the torus is $-B$ by (2.8). Therefore $-A$ rotates through $-B$ to A. Now $\angle_C(-A, -B)=\angle_C(-B, A)=90°$ and we get $t_1=90°+90°=180°$ as claimed. Similarly we get t_2 and t_3.

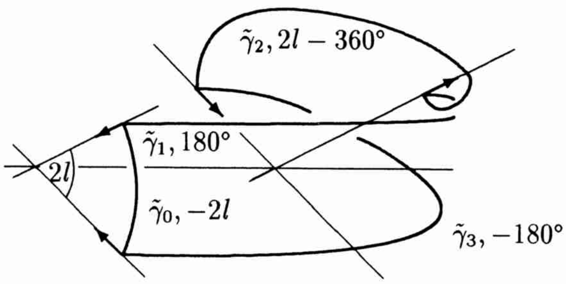
We have to verify that $\Gamma^{k,l}$ bounds a disk in T^c. We show that $\Gamma^{k,l}$ is contractible. $\Gamma^{k,l}$ is homotopic to the dotted curve $y+x=270°$. The image of the map $[0, 45°] \to T$, $\rho \mapsto F(\rho, x, x-\pi)$ is the dotted curve for $\rho=45°$ and shrinks to a point for $\rho=0$. This curve is also contractible in the covering cylinder T^c.

We need barriers in addition to ∂T^c. We define two great spheres: Extending γ_0 to 180° yields a new endpoint of γ_0 on γ_3. The arc $\gamma_3^k(t)$, $k \leqq t \leqq k+\pi$, closes γ_0 to a 2-gon. This 2-gon is contractible and thus defines a quarter sphere Q_1. In the same way the extended arc γ_2 and an arc of γ_3 bounds a quarter sphere Q_2. Then $\Gamma^{k,l}$ is contained in the compact component of $N^k := T^c - Q_1 - Q_2$ and all vertices of $\Gamma^{k,l}$ are contained in two different barriers.

By Theorem 1.12 we get existence of an H-surface patch with 4 edges and tilting angles $-2l$, 180°, $2l-360°$, $-180°$. The surfaces have different symmetry properties for l less or greater than 90°. The case $l=90°$ is exceptional: The boundary curve $\Gamma^{k,90}$ spans helicoids of negative a, whose associated surfaces are 180°-segments of the nodoids (see 2.1). Indeed, 90° arcs through $\gamma_3(t)$ in direction of $-\cos\frac{t}{1+k/\pi}A+\sin\frac{t}{1+k/\pi}B$ give the ruling which is contained in T by Lemma 2.7.

We claim that for $l \neq 90°$ the curves $\tilde{\gamma}_0^{k,l}$ and $\tilde{\gamma}_2^{k,l}$ are contained in different planes. Let $M^{k,l}$ be the Plateau solution of $\Gamma^{k,l}$. Any translation of $M^{k,l}$ in x-direction is a barrier for $M^{k,l}$ itself. The translation leaves the Hopf directions of γ_1 and γ_3 invariant and thus the rotational Hopf angle is monotone along γ_1 and γ_3. It follows that $\tilde{\gamma}_1$ and $\tilde{\gamma}_3$ are convex planar curves. On the other hand the nodoid $W=M^{k,90}$ with the same edge $\gamma_1^{k,90}$ is a barrier for $M^{k,l}$. But for the nodoid $\tilde{\gamma}_0^{k,90}$ and $\tilde{\gamma}_2^{k,90}$ are contained in the same plane. Thus the claim follows form the Comparison Lemma 5.1.

Reflection at the symmetry planes yields a simply periodic immersed H-surface $\tilde{M}^{k,l}$. If $2l\cdot m=\pi$ the surface closes through $2m-1$ reflections. The fundamental H-patch looks as follows:



The simply periodic H-surface $\tilde{M}^{k,l}$ is a cylinder with lobes of m spheres attached in a constant distance. $k>0$ is a parameter of this distance. If more generally $2l\cdot m/\mu=\pi$ (m, $\mu\in\mathbb{N}$ without common divisor), the cylindrical tubes are covered μ-times and lobes of $m\cdot\mu$ spheres are attached. For $90°<l<180°$ the roles of γ_0 and γ_2 swap and we obtain the same surfaces as for $90°-l$.

With ends. By Lemma 4.6 the sequence $M^{k,l}$ of minimal surfaces satisfies uniform area bounds and thus we obtain from Theorem 4.4 a minimal surface $M^{\infty,l}\subset T^c$, bounded by the two rays $\gamma_1^{\infty,l}$, $\gamma_3^{\infty,l}$ and by the arc γ_2^l. According to Subsect. 4.3 the solid Clifford torus T is foliated by Clifford tori, such that $\eta_\pm$ are the great circles $F(45°, x, 135°-l/2\pm45°)$. In case $l\neq90°$ we can apply Theorem 4.7 (note (4.3) holds) and obtain that the end is asymptotically the strip $S^l=\{F(45°, x, y)|135°-l<y<135°\}\subset\partial T$. For $0<l<90°$ the arc γ_0 is contained in this strip and the geometric data of the surface with end $M^{\infty,l}$ are those of the approximating surfaces $M^{k,l}$ by Lemma 4.8. Thus we obtain an H-surface $\tilde{M}^{\infty,l}$ with two cylinder ends in case $2l\cdot m=\pi$. To this cylinder a lobe of m spheres is attached in the middle. As in the periodic case the cylinder ends are covered μ-times, if $2l\cdot m/\mu=\pi$. For $90°<l<180°$ the attached spheres on $\tilde{M}^{k,l}$ disappear to infinity as $k\to\infty$ and $\tilde{M}^{\infty,l}$ is the ordinary cylinder.

The surface $\tilde{M}^{\infty,45}$ is pictured in the paper of Pinkall and Sterling [PS].

5.2 Comparison of planar curves

Lemma 5.1 *Let $\gamma, \Gamma\colon [0, l] \to \mathbb{R}^2$ be curves with $\gamma(0) = \Gamma(0) = 0$. Suppose for*

$$\dot\gamma(s) = \begin{pmatrix}\cos t(s)\\ \sin t(s)\end{pmatrix} \quad \text{and} \quad \dot\Gamma(s) = \begin{pmatrix}\cos T(s)\\ \sin T(s)\end{pmatrix}$$

and $s \in (0, l)$ holds

$$t(0) = T(0) = 0 \leqq t(s) < T(s) \leqq 180° = t(l) = T(l). \tag{5.1}$$

Then at the end points the first components of the curves satisfy

$$\Gamma_1(l) < \gamma_1(l).$$

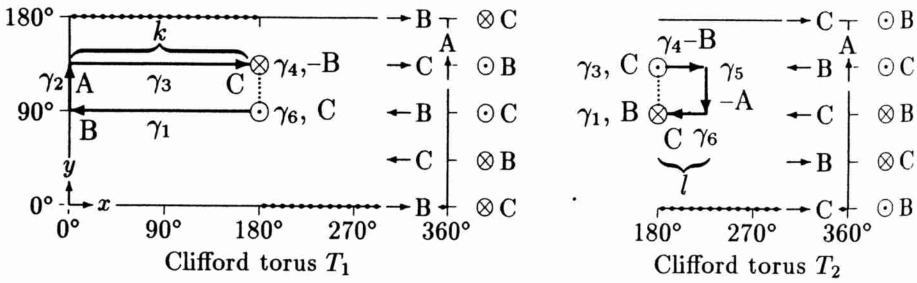
Proof.

$$\Gamma_1(l) = \int_0^l \dot\Gamma_1(s)\,ds = \int_0^l \cos T(s)\,ds < \int_0^l \cos t(s)\,ds = \int_0^l \dot\gamma_1(s)\,ds = \gamma_1(l). \quad \square$$

Let $\gamma = \Gamma$ be a geodesic arc in S^3 bounding two minimal surfaces M_1, M_2 which have the same tangent planes in $\gamma(0)$ and $\gamma(l)$ but do not intersect in the interior. By the boundary maximum principle the rotation angles r and R of M_1 and M_2 along γ satisfy $r(s) < R(s)$ for $0 < s < l$ and we get by (1.6) $t(s) < T(s)$. The other inequalities in (5.1) certainly hold for convex curves. We conclude that if the end points of $\tilde\gamma$ have the same height then the end points of $\tilde\Gamma$ don't.

5.3 Doubly periodic pile of cylinders

In the next two sections we construct H-surfaces with umbillic points, which are parameterized by surfaces of higher genus. Take two Clifford tori ∂T_1 and ∂T_2, such that ∂T_2 is the 90°-rotation of ∂T_1 about the great circle $x = 0°$, 180°.



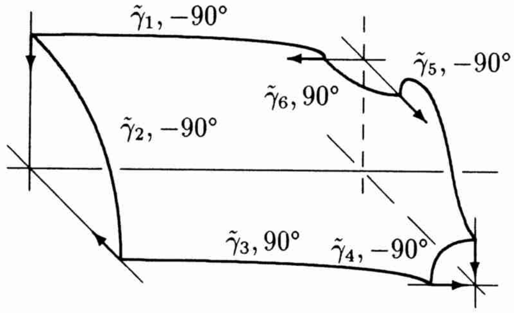
The curve $\Gamma^{k,l}$ has the geometric data ($k > 0$, $0 < l \leqq 45°$),

$$\begin{array}{llll}
|\gamma_1^{k,l}| = k, & \angle_B(-C, A) = -90°, & |\gamma_4^{k,l}| = l, & \angle_{-B}(-C, -A) = -90°,\\
|\gamma_2^{k,l}| = 45°, & \angle_A(-B, C) = -90°, & |\gamma_5^{k,l}| = 45°, & \angle_{-A}(B, C) = -90°,\\
|\gamma_3^{k,l}| = k, & \angle_C(-A, -B) = 90°, & |\gamma_6^{k,l}| = l, & \angle_C(A, B) = 90°.
\end{array}$$

$\Gamma^{k,l}$ is contained in the solid torus T_1. Indeed γ_5 is a $45° - l$ Clifford parallel to the A-soul of T_1, and γ_4 as well as γ_6 are no longer than 90°. In the covering torus T_1^c the curve $\Gamma^{k,l}$ is contractible.

Let T_3 be a solid Clifford torus with soul γ_3. γ_1 and γ_6 are contained in ∂T_3, and γ_2 and γ_4 are in T_3 since they are orthogonal to the soul and of length less than 45°. γ_5 is orthogonal to γ_6 and contained in T_3 by Lemma 2.7. In the same way the torus T_4 with soul γ_1 contains Γ. Let N be the H-convex component of $T_1^c \cap T_3^c \cap T_4^c$ which contains $\Gamma^{k,l}$.

By Theorem 1.12 we obtain an H-surface-patch with one interior handle $\tilde{\gamma}_6$ of width l and one exterior handle of width $\tilde{\gamma}_4$, looking as follows ($\tilde{\gamma}_5$ contains an umbilic point):



By reflection we get a doubly periodic surface $\tilde{M}^{k,l}$ with cylindrical tubes of length approximately k. By Theorem 4.4 we obtain the limiting surface for $k \to \infty$. This surface is doubly periodic and has cylinder ends according to Theorem 4.7 (note that (4.3) holds).

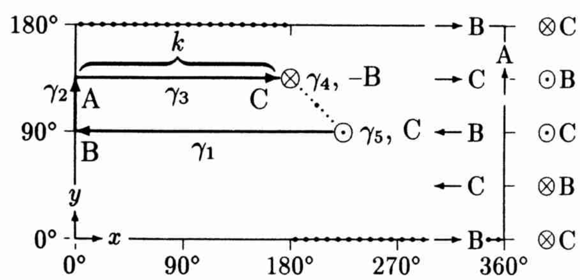
Dividing the complete surface by all translations yields a surface of genus 2. If we choose $|\gamma_2|$ and $|\gamma_5|$ to be 30°, we get a doubly periodic surface, each cylinder of which is joined with handles to six nearby cylinders (genus 3).

5.4 *Cylinder ends in a halfspace*

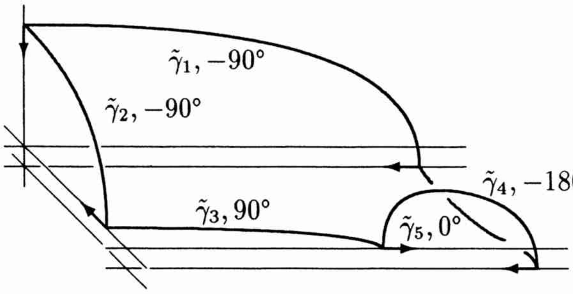
Let the curve Γ^k, $k > 0$, with the data

$$
\begin{array}{lll}
|\gamma_1^k| = k + 45°, & \angle_B(-C, A) = -90°, & \\
|\gamma_2^k| = 45°, & \angle_A(-B, C) = -90°, & \\
|\gamma_3^k| = k, & \angle_C(-A, -B) = 90°, & \\
|\gamma_4^k| = 45°, & \angle_{-B}(-C, C) = -180° & \text{(over } -A\text{)}, \\
|\gamma_5^k| = 45°, & \angle_C(B, B) = 0° &
\end{array}
$$

be given on a Clifford torus T as follows:



The arcs γ_4 and γ_5 have length 45° and meet each other exactly in a point of the soul of T. A disk G of radius 45° of a great sphere containing these curves (and orthogonal to the soul) furnishes a barrier. Using the cylinder barriers from the last section we obtain an H-convex set N^k, being the component of $T^c \cap T_3^c \cap T_4^c - G$ containing Γ, and the requirements of Theorem 1.12 are satisfied. The generating patch of the H-surface $\tilde{M}^k$ looks like:



The end points of $\tilde{\gamma}_4$ must sit in different planes. The periodic H-surface $\tilde{M}^k$ is contained in a slab between two parallel planes. The surface $\tilde{M}$ obtained by Theorem 4.4 as the limiting surface has cylinder ends in a halfspace by Theorem 4.7.

5.5 *Non-existence results*

Instead of listing more difficult examples we give non-existence proofs in certain symmetric classes. Suppose we had a cylinder as in 5.3 but with two handles only. A fundamental patch (a 90°-segment of one cylinder end) would be associated to a surface in S^3 with an approximating boundary curve $\Gamma = \gamma_1, \ldots, \gamma_5$ along, say, the Hopf fields B, A, C, $-A \pm C$. γ_5 is the handle and $|\gamma_2| = 45°$ is the curve approaching the cylinder end. Hence γ_1, γ_2, γ_3, γ_4 are contained in a cylinder ∂T and γ_5 is orthogonal. In conclusion the length of γ_5 must be a multiple of 90° in order to close the boundary curve. A shorter handle is not possible. (However with the lengths k, l, $k+90°$, $l+90°$, 90° the boundary curve closes such that for $l = 45°$ we have $\dot{\gamma}_5 = +C$.)

In the same way a cylinder as in 5.3 with two handles to the outside does not exist. Indeed we had to find lengths in 5.3 such that γ_6 is tangent to $-C$: as above we require $|\gamma_2| = 45°$ in order to obtain a 90°-segment of the cylinder end. But for $0 < |\gamma_4|, |\gamma_6| < 90°$ the end point of $\gamma_4(l_4)$ is contained in T_1 (independently of the lengths of γ_1 and γ_3) whereas $\gamma_6(0)$ is in $S^3 - T_1$. The A-great circles are either contained in T_1 or in $S^3 - T_1$ (see Subsect. 2.3) and therefore $\gamma_4(l_4)$ and $\gamma_6(0)$ can not be joined by an A-line γ_5.

6 *H*-surfaces with Delaunay-ends

In this section we describe H-surfaces with general ends of onduloid type $(0 < a < 1)$. The existence of these surfaces follows immediately from the existence of the periodic surfaces given in Sect. 3 by the limit methods of Sect. 4.

6.1 n-ends-surface

In Subsect. 3.3 we described barriers for doubly periodic surfaces with k bubbles on the edges. Using Lemma 4.6 and the H-convex sets from Subsect. 3.3 we see Theorem 4.4 applies and hence the surfaces $M^{k,\beta}$, bounded by $\Gamma^{k,\beta}$, converge to a surface M^{β} as $k\to\infty$. By Theorem 4.7 the ends converge to a helicoid f^a defined by γ_1 and γ_3. Since γ_0^k is in the helicoid f^a for all $k\in\mathbb{N}$ the geometric data of the end coincide with those of the finite quadrilaterals, see Lemma 4.8. Thus the H-surfaces have $n\geqq 3$ ends in a plane making the same angle with each other, provided $\beta=\pi/n$. The centres are spherical or n-noidal, depending on whether we start with a small or large quadrilateral. From Lemma 3.1 and Theorem 3.3 we conclude uniqueness and existence:

Theorem 6.1 (i) *Every n-ends-surface which lies between two parallel planes of* $\mathbb{R}^3$, *has maximal symmetry, and whose ends converge to an embedded Delaunay surface has a quotient of radii with*

$$0<a\leqq\frac{1}{n-1}, \tag{6.1}$$

if the boundary arc of finite length in the fundamental domain is no longer than 90°.

(ii) *There is a continuous one-parameter family of surfaces* $M_t^{\pi/n}$, $0<t<90°$, *with the properties mentioned above. In the family exactly the two different surfaces* $M_t^{\pi/n}$ *and* $M_{90-t}^{\pi/n}$ *for* $0<t<45°$ *have asymptotically the same Delaunay ends. Each value of a in (6.1) is taken twice except for the maximal value* $1/(n-1)$ *which is only taken by* $M_{45}^{\pi/n}$.

Remark. The surface $\tilde{M}_t^{\pi/n}$ for $0<t<45°$ has an n-noidal centre, in the terminology of the Remark in Subsect. 3.3. The k-th bubble is closer to the centre when compared with the surface with spherical centre $\tilde{M}_{90-t}^{\pi/n}$, because t is the length of the curvature line. Certainly these surfaces have the asymptotic behaviour for $t\to 0°$ and 90° claimed in Remark (i) and (iii) of Subsect. 3.3. It eases the imagination of the family to consider the two degenerate situations of $t\to 0$ and $t\to 90°$, when the ends tend to a chain of spheres.

Take a ray of spheres and mark the end point on the axis. To glue the rays with the n marked points onto a centre sphere S^2 in a regular fashion yields the limit for $t\to 90°$. To glue these points onto the origin of $\mathbb{R}^3$ gives the limit $t\to 0$; in this case the n spheres closest to the origin intersect each other, so that by continuity the surfaces $\tilde{M}_t^{\pi/n}$ cannot be embedded for small t.

Proof. (i) We consider the spherical fundamental path $M\subset S^3$ associated to a 90°-segment of an end. This is bounded by two great circle rays γ_1 and γ_3, and a great circle arc γ_2. By the symmetry of the surface we can take γ_1, γ_2, γ_3 to be integrals of the Hopf fields C, $\sin\beta A-\cos\beta C$, $-B$ as in Subsect. 3.1. By the assumption on the length $|\gamma_2|$ we are in the situation of Lemma 3.1 and the perpendiculars γ_1 and γ_3 have length $0<l\leqq\beta/2$. Thus the quotient of radii is

$$0<\frac{l}{90°-l}\leqq\frac{\beta/2}{90°-\beta/2}=\frac{1}{180°/\beta-1}=\frac{1}{n-1},$$

for $\beta=\pi/n$.

(ii) We get continuity with the double-barrier technique from the proof of Theorem 3.3. However, in this non-compact case we need a maximum principle at infinity to rule out a non-compact intersection of a minimal surface M with $\Psi_\sigma(M)$. But this is given through Theorem 4.7: Both surfaces converge to the same helicoid, so there is no intersection of M with $\Psi_\sigma(M)$ in a neighbourhood of infinity. □

6.2 *Ends having the symmetry of the regular polyhedra*

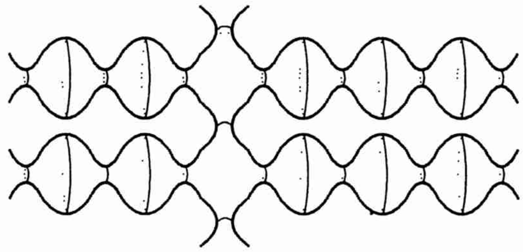
In 3.4(a) we found a surface patch with cubical symmetry. This can be generalized to the symmetry of any regular polyhedron. Since the segment needed is always less than 90° (in fact 36°, 45°, or 60°), it is easy to find the barriers.

6.3 *n-ends-surfaces joined with handles*

Taking the limit $k \to \infty$ of the triply periodic surface from Subsect. 3.4(b) we get a simply periodic surface with ends: It consists of (vertical) translations of n-ends-surfaces ($n \geqq 3$), where two adjacent translates are joined by a handle.

6.4 *A fence of Delaunay surfaces joined by handles*

Taking the doubly periodic surface of Subsect. 3.4(b) with $\beta = 90°$ the limit yields Delaunay surfaces, one of each bubbles is joined with a handle to the next surface. This can be considered the surface for $n = 2$ of the preceeding section. By the remark at the end of 3.4(b) each a-parameter for the end occurs except for $a = 1$, the cylinder end. We can similarly construct doubly periodic surfaces with Delaunay ends.



Acknowledgements. The present work originated from the author's PhD-thesis. It is a pleasure to thank my advisor Prof. Karcher who introduced me to this beautiful topic. The author was supported by Sonderforschungsbereich 256 of the Deutsche Forschungsgemeinschaft.

References

[doC] Do Carmo, M.: Differential geometry of curves and surfaces. New Jersey: Prentice Hall 1976

[D] Delaunay, C.: Sur la surface de révolution, dont la courbure moyenne est constante. J. Math. **6** (1841)

[DHKW] Dierkes, U., Hildebrandt, S., Küster, A., Wohlrab, O.: Minimal surfaces I. Berlin Heidelberg New York: Springer 1992

[G] Gulliver, R.: Regularity of minimizing surfaces of prescribed mean curvature. Ann. Math. **97**, 275–305 (1973)

[GL] Gulliver, R., Lesley, F.D.: On boundary branch points of minimizing surfaces. Arch. Ration. Mech. Anal. **52**, 20–25 (1973)

[H1] Hildebrandt, S.: Boundary behaviour of minimal surfaces. Arch. Ration. Mech. Anal. **35**, 47–82 (1969)

[H2] Hildebrandt, S.: Maximum principles for minimal surfaces and for surfaces of constant mean curvature. Math. Z. **128**, 253–259 (1972)

[Kpl] Kapouleas, N.: Complete constant mean curvature surfaces in Euclidean three-space. Ann. Math. **131**, 239–330 (1990)

[Kp2] Kapouleas, N.: Compact constant mean curvature surfaces in Euclidean three-space. J. Differ. Geom. **33**, 683–715 (1991)

[Ka] Karcher, H.: The triply periodic minimal surfaces of A. Schoen and their constant mean curvature companions. Manuscr. Math. **64**, 291–357 (1989)

[KKS] Korevaar, N., Kusner, R., Solomon, B.: The structure of complete embedded surfaces with constant mean curvature. J. Differ. Geom. **30**, 465–503 (1989)

[L] Lawson, H.B.: Complete minimal surfaces in S^3. Ann. Math. **92**, 335–374 (1970)

[MY] Meeks, W.H., Yau, S.T.: The existence of embedded minimal surfaces and the problem of uniqueness. Math. Z. **179**, 131–168 (1982)

[O] Osserman, R.: A proof of regularity everywhere of the classical solution to Plateau's problem. Ann. Math. **91**, 1092–1120 (1969)

[PS] Pinkall, U., Sterling, I.: On the classification of constant mean curvature tori. Ann. Math. **130**, 407–451 (1989)

[S] Schoen, R.: Estimates for stable minimal surfaces in three dimensional manifolds. In: Bombieri, E. (ed.) Seminar on Minimal Submanifolds. (Ann. Math. Stud., vol. 103, pp. 111–126) Princeton: Princeton University Press 1983

[We] Wente, H.: Constant mean curvature immersions of Enneper type. Memoirs of the AMS, vol. 478, 1992

[Wh] White, B.: Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals. Invent. Math. **88**, 243–256 (1987)