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Product formulas for resultants and Chow forms

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1 Introduction

Sparse elimination theory is concerned with the algebraic and combinatorial study of multivariate polynomial systems with prescribed Newton polytopes. Its starting point lies in the large body of work by Gel'fand, Kapranov and Zelevinsky (see [8, 9]) on generalized hypergeometric functions, $\mathscr{A}$-discriminants and regular triangulations of Newton polytopes. In the subsequent work [11] the $\mathscr{A}$-resultant was introduced and shown to be equivalent to the Chow form of the associated toric variety. In the present paper we extend the results in [8] and [11] by establishing product formulas of Poisson type for general Chow forms, for $\mathscr{A}$-resultants, and for sparse mixed resultants.

We consider a system of $k+1$ generic Laurent polynomials

$$f_i := c_{i1}\,\mathbf{t}^{a_{i1}} + c_{i2}\,\mathbf{t}^{a_{i2}} + \ldots + c_{in_i}\,\mathbf{t}^{a_{in_i}} \in K[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}], \tag{1.1}$$

where $i=0, \ldots, k$, $a_{ij}=(a_{ij1}, \ldots, a_{ijr}) \in \mathbf{Z}^r$, and $\mathbf{t}^{a_{ij}} := t_1^{a_{ij1}} t_2^{a_{ij1}} \ldots t_r^{a_{ijr}}$. The coefficients c_{ij} in (1.1) are algebraically independent indeterminates over $\mathbf{Q}$, and the field K is the algebraic closure of the rational function field $\mathbf{Q}(c_{01}, \ldots, c_{kn_k})$. The *support* of f_i is the set of exponent vectors $\mathscr{A}_i = \{a_{ij}\}_{j=1,\ldots,n_i} \subset \mathbf{Z}^r$, and its *Newton polytope* is the convex hull $P_i = \mathrm{conv}(\mathscr{A}_i)$ in $\mathbf{R}^r$. We assume that $\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_k$ lie in a common k-dimensional affine sublattice of $\mathbf{Z}^r$. This sublattice is identified with $\mathbf{Z}^k$.

Each Laurent polynomial f_i can be viewed as a generic point $(c_{i1} : \ldots : c_{in_i})$ in a complex projective space $\mathbf{P}^{n_i-1}$. Let $Z_0 = Z_0(\mathscr{A}_0, \ldots, \mathscr{A}_k)$ denote the set of all points (c_{ij}) in the product of projective spaces $\mathbf{P}^{n_0-1} \times \ldots \times \mathbf{P}^{n_k-1}$ such that the system $f_0 = f_1 = \ldots = f_k = 0$ has a solution in the algebraic torus $(\mathbf{C}^*)^k$. Let $Z = Z(\mathscr{A}_0, \ldots, \mathscr{A}_k)$ denote the closure of Z_0 in $\mathbf{P}^{n_0-1} \times \ldots \times \mathbf{P}^{n_k-1}$. In Sect. 2 we will show that Z is a proper irreducible subvariety, defined over $\mathbf{Q}$.

The *mixed resultant* $R_{\mathscr{A}_0,\ldots,\mathscr{A}_k}$ of the system (1.1) is an irreducible polynomial in $\mathbf{Z}[c_{01}, \ldots, c_{kn_k}]$. It is uniquely defined up to a sign factor ± 1 as follows. If $\mathrm{codim}(Z)=1$ then $R_{\mathscr{A}_0,\ldots,\mathscr{A}_k}$ is the defining polynomial of the hypersurface

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Z. If $\operatorname{codim}(Z)>1$ then $R_{\mathscr{A}_0,\ldots,\mathscr{A}_k}$ is the constant 1. In [8, Proposition 1.3.1] the mixed resultant was called the $(\mathscr{A}_0, \ldots, \mathscr{A}_k)$-*resultant*. It is our objective to find an expression as a *Poisson product*

$$R_{\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_k}(f_0,f_1,\ldots,f_k)=R'\cdot\prod_{\gamma\in V(f_1,\ldots,f_k)} f_0(\gamma), \tag{1.2}$$

where the product is taken over the set of discrete solutions

$$V(f_1,\ldots,f_k):=\{\gamma\in(K^*)^k \mid f_1(\gamma)=\ldots=f_k(\gamma)=0\}. \tag{1.3}$$

The extraneous factor R' is a rational function in $\mathbf{Q}(c_{11},\ldots,c_{kn_k})$, which is to be determined. It is our goal to express R' as a product of lower dimensional resultants depending only on the distinct leading forms of $f_1,\ldots,f_k$. In order to state our formula, we need to introduce a number of definitions and conventions.

We fix an origin O in the lattice $\mathbf{Z}^k$, and we use the following notations throughout.

- We select a normalized volume form $\operatorname{vol}(\cdot)$ for each sublattice of interest, which assigns volume 1 to an elementary simplex of that lattice. The normalized volume $\operatorname{vol}(\cdot)$ for $\mathbf{Z}^k$ is related to the standard volume $\operatorname{Vol}(\cdot)$ on $\mathbf{R}^k$ by $\operatorname{vol}(\cdot)=k!\cdot\operatorname{Vol}(\cdot)$.
- Let S^{k-1} denote the unit $(k-1)$-sphere in $\mathbf{R}^k$. For $w\in S^{k-1}$ let P_i^w denote the face of P_i on which the linear functional $(w,\cdot)$ is maximized. We also refer to $\mathscr{A}_i^w=\mathscr{A}_i\cap P_i^w$.
- Given $w\in S^{k-1}$, then $f_i^w:=\sum_{a\in\mathscr{A}_i^w} c_a\,\mathbf{t}^a$ denotes the *leading form* of the Laurent polynomial f_i with respect to w. Note that f_i^w is a monomial for generic w.
- We abbreviate the pointwise sums

$$\mathscr{A}:=\mathscr{A}_0+\mathscr{A}_1+\ldots+\mathscr{A}_k,$$
$$\bar{\mathscr{A}}:=\quad\ \mathscr{A}_1+\ldots+\mathscr{A}_k.$$

The corresponding *Minkowski sums* of Newton polytopes are $\bar{P}:=P_1+\ldots+P_k=\operatorname{conv}(\bar{\mathscr{A}})$ and $P:=P_0+\bar{P}=\operatorname{conv}(\mathscr{A})$. Let $L(\mathscr{A})$, $L(\bar{\mathscr{A}})$, etc., denote the sublattice of Z^k affinely spanned by the points of $\mathscr{A}$, $\bar{\mathscr{A}}$, etc.

For each direction $w\in S^{k-1}$ we define a parallel vector $w[\mathscr{A}]=\lambda\cdot w$ as follows. If P^w is not a *facet* (=face of dimension 1) of P then $\lambda=0$ and $w[\mathscr{A}]$ is the zero vector. If P^w is a facet then we choose $w[\mathscr{A}]$ subject to the normalization condition:

$$\langle w[\mathscr{A}], x-a\rangle\cdot\operatorname{vol}_{\mathscr{A}^w}(P^w)=\operatorname{vol}(\operatorname{conv}(P^w\cup\{x\})) \quad \text{for all } a\in P^w, x\in\mathbf{Z}^k. \tag{1.4}$$

Here $\operatorname{vol}_{\mathscr{A}^w}(\cdot)$ denotes the $(k-1)$-dimensional volume form normalized with respect to the lattice $L(\mathscr{A}^w)$. Observe that $\langle w[\mathscr{A}], a\rangle$ does not depend on the choice of $a\in\mathscr{A}^w$, so that the abbrevation $\langle w[\mathscr{A}], \mathscr{A}^w\rangle$ is unambiguous. We similarly define $w[\bar{\mathscr{A}}]$, and we also use the notation $\langle w[\bar{\mathscr{A}}], \mathscr{A}_i^w\rangle$ for any other $\mathscr{A}_i$. Our main result is the following:

Theorem 1.1 *We have the following identity of rational functions in* $\mathbf{Q}(c_{01}, \ldots, c_{kn_k})$:

$$\prod_{\gamma \in V(f_1,\ldots,f_k)} f_0(\gamma) = R_{\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_k}(f_0, f_1, \ldots, f_k) \cdot \prod_{w \in S^{k-1}} R_{\mathscr{A}_1^w,\ldots,\mathscr{A}_k^w}(f_1^w, \ldots, f_k^w)^{\langle w[\mathscr{A}], \mathscr{A}^\vee \rangle}. \tag{1.5}$$

The product over w is finite; it includes only directions w which are normal to facets of the Minkowski sum $\bar{P} = P_1 + \ldots + P_k$. *For all other directions the face resultants equal* 1.

In order to state Theorem 1.1 in a more precise fashion we need a notion of "evaluating f_0 at the roots of $f_1, \ldots, f_k$" which takes into account the structure of the inputs as polynomials in $r \geqq k$ variables. We translate the affine lattices $L(\mathscr{A}_i)$ into a common lattice of dimension k, and then find the coarsest common refinement up to translations. Call this lattice $L(\mathscr{A}_0, \ldots, \mathscr{A}_k)$.

We consider the semigroup algebra

$$S := K[\mathbf{t}^a \mid a \in L(\mathscr{A}_0, \ldots, \mathscr{A}_k)], \tag{1.6}$$

and we define $V(f_1, \ldots, f_k)$ to be the spectrum of the quotient algebra $R = S/\langle f_1, \ldots, f_k \rangle$. This algebra is a finite-dimensional K-vector space, and we consider the K-linear map

$$\mathrm{Times}_{f_0} : R \to R, \quad g \mapsto f_0 \cdot g. \tag{1.7}$$

We now apply the methods in [14] to see that the endomorphism (1.7) may be diagonalized over the algebraically closed field K. It has the distinct eigenvalues $f_0(\gamma)$, where $\gamma \in V(f_1, \ldots, f_k)$. We obtain

$$\det(\mathrm{Times}_{f_0}) = \prod_{\gamma \in V(f_1,\ldots,f_k)} f_0(\gamma). \tag{1.8}$$

This determinant is what we mean by the left hand side of (1.5).

Remark 1.2 The expression (1.8) can be evaluated using only computations over the rational numbers. We compute a Gröbner basis for the ideal $\langle f_1, \ldots, f_k \rangle$, and we construct a matrix for Times_{f_0} using Gröbner basis normal forms. Then we compute the determinant of that matrix. For an introduction to Gröbner bases we refer to the book of Cox et al. [6]. See also [1, 14, 16] and the references given there.

Example 1.3 We illustrate the construction in (1.6) for two sparse binary forms:

$$f_0(t_1, t_2) = c_{02} t_1^4 + c_{01} t_2^4 \quad \text{and} \quad f_1(t_1, t_2) = c_{11} t_1^2 + c_{12} t_2^2. \tag{1.9}$$

We embed the affine lattices $\mathbf{Z}[(2,0),(0,2)]$ and $\mathbf{Z}[(4,0),(0,4)]$ in the one-dimensional lattice $\mathbf{Z}^1$ by sending $(2,0) \mapsto 1$, $(0,2) \mapsto 0$. Under this embedding we get $f_0 \mapsto c_{01} x^2 + c_{02}$, $f_1 \mapsto c_{11} x + c_{12}$, and $S/\langle f_1 \rangle \simeq K[x]/\langle c_{11} x + c_{12} \rangle$, where x is a new variable. The spectrum of $S/\langle f_1 \rangle$ can be identified with the singleton $\{-c_{12}/c_{11}\}$. In this example the expression (1.5) or (1.8) equals

$$c_{01}(-c_{12}/c_{11})^2 + c_{02} = (c_{01} c_{12}^2 + c_{02} c_{11}^2) \cdot (1/c_{11}^2). \tag{1.10}$$

We see that the mixed resultant equals $R_{\mathscr{A}_0,\mathscr{A}_1}(f_0,f_1)=c_{01}c_{12}^2+c_{02}c_{11}^2$. This is the square root of the usual Sylvester resultant of the binary quartic f_0 and the binary quadric f_1.

2 Basic facts on mixed volumes and mixed resultants

Let $f_1, \ldots, f_k$ be Laurent polynomials as in the introduction. After an affine transformation we may assume that their supports $\mathscr{A}_1, \ldots, \mathscr{A}_k$ affinely span a k-dimensional lattice $\mathbf{Z}^k$. We recall the following fundamental result of Bernstein [2].

Theorem 2.1 (Bernstein) *The set of roots $V(f_1, \ldots, f_k)$ is finite, and its cardinality equals the mixed volume $\mathscr{M}(P_1, \ldots, P_k)$ of the given Newton polytopes $P_i=\mathrm{conv}(\mathscr{A}_i)$.*

See [7, Chap. 5] for a nice intersection-theoretic proof of Theorem 2.1. We now recall several equivalent definitions and basic properties of the *mixed volume*. For an introduction to mixed volumes and convexity in general we refer to Bonnesen and Fenchel [4].

The mixed volume $\mathscr{M}(\cdot, \cdot, \ldots, \cdot)$ is an operator on k polytopes in a k-dimensional space $\mathbf{R}^k$, which attains non-negative real values. It is symmetric, it is multilinear, and it is $\mathrm{SL}_k(\mathbf{R})$-invariant and translation invariant. These three requirements uniquely define the mixed volume, subject to the normalization condition

$$\mathscr{M}(P, P, \ldots, P)=\mathrm{Vol}(P)\cdot k!=\mathrm{vol}(P). \tag{2.1}$$

This implies that $\mathscr{M}(P_1, P_2, \ldots, P_k)$ is an integer whenever $P_1, \ldots, P_k$ are lattice polytopes. The mixed volume can be computed by any of the following four methods:

(a) For $\lambda_1, \ldots, \lambda_k>0$ consider the polytope $\lambda_1 P_1+\ldots+\lambda_k P_k$. Its volume is a homogeneous polynomial of degree k in $\lambda_1, \lambda_2, \ldots, \lambda_k$. The mixed volume $\mathscr{M}(P_1, P_2, \ldots, P_k)$ equals the coefficient of the monomial $\lambda_1 \lambda_2 \ldots \lambda_k$ in the expansion of the polynomial $\mathrm{Vol}(\lambda_1 P_1+\ldots+\lambda_k P_k)$.
(b) We have the inclusion-exclusion formula

$$\mathscr{M}(P_1, \ldots, P_k)=(-1)^k\cdot \sum_{I\subset\{1,\ldots,k\}} (-1)^{|I|}\cdot \mathrm{Vol}\Big(\sum_{i\in I} P_i\Big). \tag{2.2}$$

(c) Consider the linear map of polytopes

$$\phi: P_1\times\ldots\times P_k\to P_1+\ldots+P_k, (p_1, \ldots, p_k)\mapsto p_1+\ldots+p_k. \tag{2.3}$$

A *mixed subdivision* is a polyhedral subdivision Δ of $P_1+\ldots+P_k$ which is *induced* from the linear map ϕ as in [3]. Let Δ be any mixed subdivision. The mixed volume $\mathscr{M}(P_1, P_2, \ldots, P_k)$ is the sum of the volumes of all parallelotopes $E_1\times E_2 \times\ldots\times E_k$ in Δ, where E_i is an edge of P_i. This representation appears to be most useful for practical computations. It also proves that $\mathscr{M}(P_1, P_2, \ldots, P_k)$ is non-negative.
(d) The *lattice distance* d_i^w from the face P_i^w to the origin O is defined to be the number of steps from O to the image of the face P_i^w in the one-dimensional

factor lattice $\mathbf{Z}^k/\langle w\rangle^\perp$. We have the following recursive formula for the mixed volume:

$$\mathcal{M}(P_1, P_2, \ldots, P_k) = \sum_{w\in S^{k-1}} d_1^w \cdot \mathcal{M}(P_2^w, \ldots, P_k^w). \tag{2.4}$$

In this formula it is understood that the faces P_i^w are projected perpendicularly into a common $(k-1)$-dimensional lattice, with its normalized mixed volume. The sum over w is finite; it includes only directions w which are normal to facets of $P_2+\ldots+P_k$. For all other directions the mixed volume on the right hand side is zero.

Lemma 2.2 *Using the notations introduced in Sect.* 1, *we have the following identities*:

$$\sum_{w\in S^{k-1}} \langle w[\mathcal{A}], \mathcal{A}^w\rangle \cdot \mathrm{vol}_{\mathcal{A}^w}(P^w) = -\mathrm{vol}(P) \tag{2.5}$$

$$\sum_{w\in S^{k-1}} \langle w[\bar{\mathcal{A}}], \mathcal{A}_0^w\rangle \cdot \mathcal{M}_{\bar{\mathcal{A}}^w}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) = -\mathcal{M}(P_0, \ldots, \hat{P}_i, \ldots, P_k). \tag{2.6}$$

Proof. Suppose that $x=0$ lies inside of P. Then summing (1.4) over $w\in S^{k-1}$ gives

$$\sum_w \langle w[\mathcal{A}], \mathcal{A}^w\rangle \cdot \mathrm{vol}_{\mathcal{A}^w}(P^w) = -\sum_w \mathrm{vol}(\mathrm{conv}(P^w \cup \{O\})) = -\mathrm{vol}(P).$$

Let $\mathrm{span}_{\mathbf{R}}(\bar{\mathcal{A}}^w)$ denote the real affine space spanned by the point configuration $\bar{\mathcal{A}}^w$. We abbreviate the restricted lattice $L(\bar{\mathcal{A}}, w) := \mathrm{span}_{\mathbf{R}}(\bar{\mathcal{A}}^w) \cap L(\bar{\mathcal{A}})$. To prove (2.6) we first observe that each $a\in\mathcal{A}_0^w$ satisfies the relation

$$\langle w[\bar{\mathcal{A}}], -a\rangle = \frac{\mathrm{vol}_{\bar{\mathcal{A}}}(\mathrm{conv}(\bar{P}^w \cup \{O\}))}{\mathrm{vol}_{\bar{\mathcal{A}}^w}(\bar{P}^w)} = [L(\bar{\mathcal{A}}, w): L(\bar{\mathcal{A}}^w)] \cdot d_0^w. \tag{2.7}$$

To see this, note that the index of the sublattice $L(\bar{\mathcal{A}}^w)$ in $L(\bar{\mathcal{A}}, w)$ measures precisely the ratio between the normalized volume forms in these two lattices. The mixed volume $\mathcal{M}(\cdot, \ldots, \cdot)$ computed with respect to the possibly finer lattice $L(\bar{\mathcal{A}}, w)$ is related to $\mathcal{M}_{\bar{\mathcal{A}}^w}$ by the same index. Using formulas (2.7) and (2.4), we find

$$\begin{aligned}
&\sum_w \langle w[\bar{\mathcal{A}}], \mathcal{A}_0^w\rangle \cdot \mathcal{M}_{\bar{\mathcal{A}}^w}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) \\
&\quad = -\sum_w d_0^w \cdot [L(\bar{\mathcal{A}}, w): L(\bar{\mathcal{A}}^w)] \cdot \mathcal{M}_{\bar{\mathcal{A}}^w}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) \\
&\quad = -\sum_w d_0^w \cdot \mathcal{M}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) = -\mathcal{M}(P_0, P_1, \ldots, \widehat{P_i}, \ldots, P_k).
\end{aligned}$$

This completes the proof of Lemma 2.2. □

We will now show that the mixed resultant $R_{\mathcal{A}_0,\ldots,\mathcal{A}_k}$ is well-defined and irreducible. As in the introduction, let Z be the closure in $\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}$ of the set Z_0 of all systems (1.1) which have a root in the torus $(\mathbf{C}^*)^k$.

Proposition 2.3 *The variety Z is a proper subvariety of $\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}$. It is irreducible and defined over the rationals* $\mathbf{Q}$.

Proof. Consider the incidence variety

$$W=\{(\mathbf{t},(c_{ij}))|\sum_{j=1}^{n_i} c_{ij}\,\mathbf{t}^{a_{ij}}=0 \text{ for } i=0,1,\ldots,k\} \tag{2.8}$$
$$\subset(\mathbf{C}^*)^k\times(\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}).$$

It has two projections $\pi_1: W\to(\mathbf{C}^*)^k$ and $\pi_2: W\to\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}$.

The first projection π_1 is surjective. The fiber $\pi_1^{-1}(\mathbf{t})$ over a torus element $\mathbf{t}\in(\mathbf{C}^*)^k$ consists of all systems (c_{ij}) such that $\mathbf{t}$ is a common root of (1.1). This fiber is the product of the perpendicular hyperplanes to $(\mathbf{t}^{a_{i1}},\ldots,\mathbf{t}^{a_{in_i}})$, for $i=0,1,\ldots\ldots,k$. Therefore the fibers of π_1 form a family of irreducible varieties of constant dimension. Clearly, the base space $(\mathbf{C}^*)^k$ is an irreducible quasiprojective variety. We can thus apply the proof of Theorem I.1.6.8 in [15] to conclude that W is an irreducible variety.

The image of the incidence variety W under the second projection π_2 is the quasi-projective variety Z_0. Since W is irreducible, $\pi_2(W)=Z_0$ is also irreducible, hence its closure Z is irreducible.

The fact that Z is a proper subvariety follows from Theorem 2.1 because the system $f_1(\gamma)=\ldots=f_k(\gamma)=0$ has only finitely many roots γ, and we can choose coefficients $c_{01},\ldots,c_{0n_0}$ such that $f_0(\gamma)\neq 0$ for all roots γ. Standard elimination theory implies that Z is defined over $\mathbf{Q}$. □

Bernstein's Theorem 2.1 implies that Z has codimension greater than 1 if and only if $\mathscr{M}(P_0,\ldots,\hat{P}_i,\ldots,P_k)=0$ for $i=0,\ldots,k$. It can be shown that this is equivalent to the proposition, $(\forall i)(\exists I\subseteq\{0,\ldots,\hat{i},\ldots,k\})$: $\dim(\sum_{j\in I}P_j)<|I|$. We therefore impose the following condition on our input polynomials:

$$\forall I\subset\{1,\ldots,n\},\quad |I|\leqq\dim(\sum_{j\in I}P_i), \tag{2.9}$$

where I is a proper subset of $\{1,\ldots,n\}$. With condition (2.9), the sparse mixed resultant $R_{\mathscr{A}_0,\ldots,\mathscr{A}_k}$ is a non-constant irreducible polynominal in $\mathbf{Z}[c_{01},\ldots,c_{kn}]$, separately homogeneous in the coefficients of each f_i, and we have the following corollary to Theorem 2.1.

Corollary 2.4 *The degree of the sparse mixed resultant in the coefficients $(c_{i1},\ldots,c_{in_i})$ of the i-th input polynomial f_i equals the mixed volume*

$$\deg_{f_i}(R_{\mathscr{A}_0,\ldots,\mathscr{A}_k})=\mathscr{M}(P_0,\ldots,\widehat{P_i},\ldots,P_k). \tag{2.10}$$

3 Product formula for the $\mathscr{A}$-resultant

We begin by restricting to the unmixed case $\mathscr{A}=\mathscr{A}_0=\mathscr{A}_1=\ldots=\mathscr{A}_k$. There is only one set of lattice points $\mathscr{A}=\{a_1,a_2,\ldots,a_n\}\subset\mathbf{Z}^k$, shared by the exponent vectors of all $k+1$ input polynomials. Here the resultant is called the *$\mathscr{A}$-resultant*

(or *unmixed resultant*) and denoted $R_{\mathscr{A}}$. Basic properties of $\mathscr{A}$-resultants were studied in [11] and [17].

We assume that $\mathscr{A}$ affinely spans the lattice $\mathbf{Z}^k$. Let $P=\text{conv}(\mathscr{A})$ be the Newton polytope, and abbreviate the normalized volume of P by $d:=\text{vol}(P)$ $=\text{Vol}(P)\cdot k!$. Our input polynomials in (1.1) have the following form:

$$
\begin{aligned}
f_0(\mathbf{t}) &= c_{01}\,\mathbf{t}^{a_1} + c_{02}\,\mathbf{t}^{a_2} + \ldots + c_{0n}\,\mathbf{t}^{a_n},\\
f_1(\mathbf{t}) &= c_{11}\,\mathbf{t}^{a_1} + c_{12}\,\mathbf{t}^{a_2} + \ldots + c_{1n}\,\mathbf{t}^{a_n},\\
&\ldots \quad \ldots\ldots \quad \ldots \quad \ldots \quad \ldots\\
f_k(\mathbf{t}) &= c_{k1}\,\mathbf{t}^{a_1} + c_{k2}\,\mathbf{t}^{a_2} + \ldots + c_{kn}\,\mathbf{t}^{a_n}.
\end{aligned} \tag{3.1}
$$

As before, the coefficients c_{ij} are taken to be indeterminates over $\mathbf{Q}$, and we let

$$
V = V(f_1, \ldots, f_k) = \{\gamma \in (K^*)^k \mid f_1(\gamma) = \ldots = f_k(\gamma) = 0\}. \tag{3.2}
$$

The set V consists of precisely d distinct points by a result of Kushnirenko [12].

We have the following product formula for the $\mathscr{A}$-resultant. Theorem 3.1 will be proved in Sect. 6. In Sect. 7 we will derive the more general Theorem 1.1 from Theorem 3.1.

Theorem 3.1 *We have the following identity of rational functions in* $\mathbf{C}(c_{01}, \ldots, c_{kn})$:

$$
\prod_{\gamma \in V} f_0(\gamma) = R_{\mathscr{A}}(f_0, f_1, \ldots, f_k) \cdot \prod_{w \in S^{k-1}} R_{\mathscr{A}^w}(f_1^w, \ldots, f_k^w)^{\langle w[\mathscr{A}], \mathscr{A}^w \rangle}. \tag{3.3}
$$

The right hand product is over all facets of the Newton polytope $P=\text{conv}(\mathscr{A})$. If w selects a facet P^w then the degree of $R_{\mathscr{A}^w}$ is equal to the volume of that facet, normalized with respect to $L(\mathscr{A}^w)$. The corresponding exponent $\langle w[\mathscr{A}], \mathscr{A}^w \rangle$ can be computed as the ratio of $\text{vol}(\text{conv}(\mathscr{A}^w \cup \{O\}))$ and this normalized facet volume.

The left hand side of (3.3) is a rational function of degree zero in the coefficients of $f_1, \ldots, f_n$. So is the right hand side, since the degree of $R_{\mathscr{A}}(f_0, f_1, \ldots, f_k)$ is d by Corollary 2.4, and the product over $w \in S^{k-1}$ has degree $-d$ by Eq. (2.5) in Lemma 2.2.

Example 3.2 (*A quadrangle in the plane*) Let $\mathscr{A}=\{(1,2), (3,2), (2,3), (1,3)\}$ and consider the system

$$
\begin{aligned}
f_0 &= c_{01}\,xy^2 + c_{02}\,x^3y^2 + c_{03}\,x^2y^3 + c_{04}\,xy^3,\\
f_1 &= c_{11}\,xy^2 + c_{12}\,x^3y^2 + c_{13}\,x^2y^3 + c_{14}\,xy^3,\\
f_2 &= c_{21}\,xy^2 + c_{22}\,x^3y^2 + c_{23}\,x^2y^3 + c_{24}\,xy^3.
\end{aligned} \tag{3.4}
$$

The Newton polytope $\text{conv}(\mathscr{A})$ is a quadrangle with normalized area 3. The $\mathscr{A}$-resultant equals

$$
R_{\mathscr{A}}(f_0, f_1, f_2) = [1\,2\,3]^2[1\,3\,4] + [1\,2\,4]^2[2\,3\,4], \tag{3.5}
$$

where $[ijk]$ abbreviates the determinant

$$\begin{vmatrix} c_{0i} & c_{0j} & c_{0k} \\ c_{1i} & c_{1j} & c_{1k} \\ c_{2i} & c_{2j} & c_{2k} \end{vmatrix}.$$

The two bracket monomials in (3.5) express the two triangulations of $\mathrm{conv}(\mathscr{A})$, as predicted by [11, Theorem 5.3]; see also [17, Theorem 3.1]. The extraneous factor on the right hand side of (3.3) is found to be

$$\prod_{w \in S^1} R_{\mathscr{A}^w}(f_1^w, f_2^w)^{\langle w[\mathscr{A}], \mathscr{A}^w \rangle} = [1\,2]^4 [2\,3]^{-5} [3\,4]^{-3} [1\,4], \tag{3.6}$$

where $[ij]$ abbreviates the determinant $\begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix}$. The factors in the product (3.6) express the boundary of the quadrangle $\mathrm{conv}(\mathscr{A})$, joined to the origin. (See Fig. 1). □

Remark 3.3 As an immediate application of Theorem 3.1 we obtain a new and elementary proof of the product formula for the *principal $\mathscr{A}$-determinant* given in the earlier work of Gel'fand, Kapranov and Zelevinsky [8, Theorem 2F.2]. To this end we apply the *toric specialization* of [11, Theorem 5.10] to both the right hand side and the left hand side of (3.3). The result of this specialization is the formula in [8, Theorem 2F.2].

4 Projections and intersections of projective varieties in terms of Chow forms

Our strategy in proving Theorem 3.1 is to utilize the known interpretation of the $\mathscr{A}$-resultant as the Chow form of the toric variety defined by $\mathscr{A}$ [11, §5]. In fact, we will give a completely general product formula (Theorem 5.1) for arbitrary projective varieties. Theorem 3.1 will then be derived as a special case in Sect. 6. In this section we show how to use toric deformations of Chow forms for computing coordinate projections and intersections of projective varieties.

Throughout Sects. 4 and 5 we let X be any irreducible, complex projective variety of dimension k and degree d in $\mathbf{P}^n$. We represent X by its *Chow form* R_X [11; 15, I.1.6.5]. The Chow form is an irreducible polynomial function of the coefficients c_{ij} of $k+1$ generic linear forms $\ell_i = c_{i0}x_0 + c_{i1}x_1 + \ldots + c_{in}x_n$; $i = 0, \ldots, k$. The Chow form vanishes under specialization of the c_{ij} if and only if the accordingly specialized linear forms ℓ_i have a common zero on X. In other words, R_X is the defining polynomial of the hypersurface of all $(n-k-1)$-planes in $\mathbf{P}^n$ which meet X. Note that R_X can also be expressed as a polynomial in *brackets*, i.e., in the $(k+1)\times(k+1)$-minors of the coefficient matrix (c_{ij}).

We fix the point $e_0 = (1:0:\ldots:0)$, the affine space $\mathbf{A}^n = \{x \in \mathbf{P}^n \mid x_0 \neq 0\}$, the hyperplane "at infinity" $\mathbf{P}^{n-1} = \{x \in \mathbf{P}^n \mid x_0 = 0\}$, and the projection $\pi: \mathbf{P}^n \backslash e_0 \to \mathbf{P}^{n-1}$ onto $\mathbf{P}^{n-1}$ with center at e_0. The projection π determines a mapping π_* of algebraic cycles. We are interested in computing the Chow forms of the image cycle $Y = \pi_*(X)$ and the intersection cycle $X \cap \mathbf{P}^{n-1}$ directly from the Chow form R_X. We assume that $X \cap \mathbf{P}^{n-1}$ has dimension $k-1$ and that Y

has dimension k. Then the degree of $X \cap \mathbf{P}^{n-1}$ equals d, while the degree $\tilde{d}$ of Y satisfies the inequality $\tilde{d} \leqq d$. Let $(X \cap \mathbf{P}^{n-1}) * e_0$ denote the cone over $X \cap \mathbf{P}^{n-1}$ with apex e_0. Thus both $(X \cap \mathbf{P}^{n-1}) * e_0$ and Y are algebraic cycles of the same dimension k in $\mathbf{P}^n$.

Let $w: \mathbf{C}^* \to (\mathbf{C}^*)^{n+1}$, $t \mapsto \operatorname{diag}(t^{w_0}, \ldots, t^{w_n})$ be the one-parameter subgroup of $(\mathbf{C}^*)^{n+1}$ defined by the weight vector $w = (1, 0, 0, \ldots, 0)$. The natural action of $w(t)$ on $\mathbf{P}^n$ extends to a covariant action on Chow forms via $c_{i0} \mapsto t \cdot c_{i0}$ (i.e., by scaling the first column). It extends to a contravariant action on the polynomial ring $\mathbf{C}[x_0, x_1, \ldots, x_n]$ via $x_0 \mapsto t^{-1} \cdot x_0$.

Let $I \subset \mathbf{C}[x_0, x_1, \ldots, x_n]$ denote the homogeneous prime ideal defining the variety X. For each parameter value $t \in \mathbf{C}^*$ we get an isomorphic ideal $w(t) \cdot I$ after the substitution $x_0 \mapsto t^{-1} \cdot x_0$. The *initial ideal* $\operatorname{init}_w(I)$ of I is the ideal generated by all leading forms f^w where f runs over I. Computing the initial ideal $\operatorname{init}_w(I)$ amounts to a *Gröbner basis* computation for I with respect to a term order which refines the partial ordering

$$\mathbf{x}^\alpha \leqq \mathbf{x}^\beta :\Leftrightarrow \langle w, \alpha \rangle \leqq \langle w, \beta \rangle . \tag{4.1}$$

By the results of Bayer et al. (see [1]), the initial ideal $\operatorname{init}_w(I)$ equals the limit $\lim_{t \to 0} w(t) \cdot I$ in the *Hilbert scheme*. Similary, we obtain the initial terms of the Chow form $\operatorname{init}_w(R_X)$ as the limit $\lim_{t \to 0} w(t) \cdot R_X$ in the *Chow variety*. Algebraically, the initial Chow form $\operatorname{init}_w(R_X)$ is the sum of all terms in R_X whose weight has maximum inner product with w. For details see [11, § 3; 17, Corollary 3.5].

Our main result in this section is the following geometric description of the two toric deformations of X defined by w and $-w$ respectively.

Proposition 4.1 *We have the following identities in the polynomial ring* $\mathbf{C}[c_{00}, \ldots, c_{kn}]$:

$$\operatorname{init}_{-w}(R_X) = R_{(X \cap \mathbf{P}^n) * e_0} \quad \text{and} \quad \operatorname{init}_w(R_X) = R_Y \cdot R_{Z'}, \tag{4.2}$$

where Z' is an algebraic cycle of the form $Z * e_0$ of degree $d - \tilde{d}$ and dimension k in $\mathbf{P}^n$.

Proof. We use the general fact that $\operatorname{init}_w(R_X) = R_{\operatorname{init}_w(I)}$, where $R_{\operatorname{init}_w(I)}$ denotes the Chow form of the algebraic cycle underlying the subscheme of $\mathbf{P}^n$ defined by the initial ideal $\operatorname{init}_w(I)$. This relation holds because the action of the one-parameter group $w(t)$ is equivariant with respect to the natural morphism from the Hilbert scheme to the Chow variety.

The initial ideal $\operatorname{init}_{-w}(I)$ is obtained from I by the substitution $x_0 \mapsto 0$, since in this instance the action is $x_0 \mapsto t \cdot x_0$, and we are giving x_0 arbitrarily small weight as $t \to 0$. The algebraic cycle underlying $\operatorname{init}_{-w}(I)$ equals $(X \cap \mathbf{P}^{n-1}) * e_0$. Note that we need to cone the intersection cycle with e_0 because the ideal $\operatorname{init}_{-w}(I)$ is generated by polynomials which depend only on $\bar{x} = (x_1, \ldots, x_n)$, yet we are looking at its cycle in $\mathbf{P}^n$. We get

$$\operatorname{init}_{-w}(R_X) = R_{\operatorname{init}_{-w}(I)} = R_{(X \cap \mathbf{P}^{n-1}) * e_0}. \tag{4.3}$$

The initial ideal $\text{init}_w(I)$ may be determined by computing a Gröbner basis G for I with respect to any elimination order which is lexicographic between $x_0 > \bar{x} = (x_1, \ldots, x_n)$. Such a Gröbner basis has the form

$$G = \{g_1(\bar{x}), \ldots, g_r(\bar{x}), h_1(x_0, \bar{x}), \ldots, h_s(x_0, \bar{x})\}. \tag{4.4}$$

Forgetting the ordering imposed on the $\bar{x}$ variables in the elimination order, we find that the desired initial ideal $\text{init}_w(I)$ is generated by

$$\text{init}_w(G) = \{g_1(\bar{x}), \ldots, g_r(\bar{x}), x_0^{m_1} q_1(\bar{x}), \ldots, x_0^{m_s} q_s(\bar{x})\}, \tag{4.5}$$

where $q_i(\bar{x})$ is the coefficient of the highest power of x_0 appearing in h_i. The variety defined by the polynomials in (4.5) decomposes as

$$\{x_0 = g_1(\bar{x}) = \ldots = g_r(\bar{x}) = 0\} \cup \{g_1(\bar{x}) = \ldots = g_r(\bar{x}) = q_1(\bar{x}) = \ldots = q_s(\bar{x}) = 0\}. \tag{4.6}$$

Using (4.6), we may express the cycle underlying $\text{init}_w(I)$ as the sum of two algebraic cycles. The first cycle in this decomposition is precisely the projection cycle $Y = \pi_*(X)$. Let Z' denote the second cycle. We see in (4.6) that the scheme underlying Z' is defined by a set of equations not involving the variable x_0 at all. Therefore Z' is the cone with apex e_0 over some cycle in $\mathbf{P}^{n-1}$; in symbols $Z' = Z * e_0$.

The Chow form of a sum of cycles is the product of the Chow forms of the summands:

$$\text{init}_w(R_X) = R_{\text{init}_w(I)} = R_{Y + Z * e_0} = R_Y \cdot R_{Z * e_0}. \tag{4.7}$$

This completes the proof of Proposition 4.1. □

Example 3.2 (*continued*) We illustrate Proposition 4.1 for the toric surface $X = X_{\mathscr{A} \cup \{(0,0)\}}$, where $\mathscr{A} = \{(1, 2), (3, 2), (2, 3), (1, 3)\}$. Here $n = 4$, $\tilde{d} = 3$ and $d = 8$.

Let $I \subset \mathbf{C}[x_0, x_1, x_2, x_3, x_4]$ denote the ideal generated by the set

$$G = \{\underline{x_0 x_2 x_4^3} - x_1^4 x_3, \underline{x_0 x_3^3} - x_1^2 x_2 x_4, \underline{x_0 x_3 x_4} - x_1^3, \underline{x_1 x_3^2} - x_2 x_4^2\}. \tag{4.8}$$

Let X denote the projective variety in $\mathbf{P}^4$ defined by I. The set G is a Gröbner basis for I with respect to the purely lexicogrpahic term order induced from the variable ordering $x_0 > x_1 > x_2 > x_3 > x_4$. From this it can be deduced that X has dimension 2 and degree 8, using standard methods of Gröbner basis theory. To be more precise, using the notation of [18, §2], we find that the underlined initial monomials give rise to precisely eight *standard pairs*:

$$\begin{aligned}&(1, \{x_2, x_3, x_4\}), (1, \{x_1, x_2, x_4\}), (x_3, \{x_1, x_2, x_4\}), (1, \{x_0, x_1, x_4\}),\\ &(1, \{x_0, x_1, x_2\}), (x_3, \{x_0, x_1, x_2\}), (x_4, \{x_0, x_1, x_2\}), (x_4^2, \{x_0, x_1, x_2\}).\end{aligned} \tag{4.9}$$

These eight standard pairs will appear as the bracket factors in the second term of $\text{init}_{-w}(R_X) = R_{X,5}$ below.

The two initial ideals considered in the proof of Proposition 4.2 are

$$\begin{aligned}\text{init}_w(I) &= \langle x_0 x_2 x_4^3, x_0 x_3^3, x_0 x_3 x_4, x_1 x_3^2 - x_2 x_4^2 \rangle \quad \text{and}\\ \text{init}_{-w}(I) &= \langle x_1^4 x_3, x_1^2 x_2 x_4, x_1^3, x_1 x_3^2 - x_2 x_4^2 \rangle.\end{aligned} \tag{4.10}$$

We now determine the Chow form R_X. Consider three generic linear forms

$$\begin{aligned}\ell_0 &= c_{00}x_0 + c_{01}x_1 + c_{02}x_2 + c_{03}x_3 + c_{04}x_4 \\ \ell_1 &= c_{10}x_0 + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 \\ \ell_2 &= c_{20}x_0 + c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + c_{24}x_4.\end{aligned} \tag{4.11}$$

By definition, the Chow form is the unique irreducible polynomial $R_X = R_X(c_{00}, \ldots, c_{24})$ in the coefficients of (4.11), which vanishes if and only if the system of linear equations $\ell_0(x) = \ell_1(x) = \ell_0(x) = 0$ has solution x in X.

For $0 \leqq i < j < k \leqq 4$ we abbreviate $[ijk] :=$

$$\det\begin{pmatrix} c_{0i} & c_{0j} & c_{0k} \\ c_{1i} & c_{1j} & c_{1k} \\ c_{2i} & c_{2j} & c_{2k} \end{pmatrix}.$$

The Chow form equals $R_X = R_{X,5} + R_{X,6} + R_{X,7} + R_{X,8}$, where

$$\begin{aligned}
R_{X,5} &= (-[123]^2[134] - [124]^2[234])[012]^4[014], \\
R_{X,6} &= [012]^2(-[012]^2[013][034][234]^2 + 3[012]^3[023][034][134][234] \\
&\quad - [012][013][023][024][134][234] + 7[012][014][024]^2[234]^2 \\
&\quad - 5[012][023]^2[024][134]^2 - 5[012][024]^3[134][234] \\
&\quad + 4[013][023]^2[024][124][134] + 4[013][024]^3[124][234]) \\
R_{X,7} &= -7[012]^2[023]^2[024][034]^2[234] + [012][013][023]^4[034][134] \\
&\quad + 3[012][013][023]^2[024]^2[034][234] \\
&\quad + 5[012][023]^3[024]^2[034][134] + 7[012][023][024]^4[034][234] \\
&\quad - [013]^2[023]^4[024][134] - [013]^2[023]^2[024]^3[234] \\
&\quad - 2[013][023]^3[024]^3[134] - 2[013][023][024]^5[234] \\
&\quad - [023]^2[024]^5[134] - [024]^7[234] \\
R_{X,8} &= [023]^5[034]^3.
\end{aligned}$$

These bracket polynomials have been computed with the algorithm described in [17, Sect. 4.2]. The two initial Chow forms of the algebraic cycles defined by (4.10) are

$$\mathrm{init}_w(R_X) = R_{X,8} \quad \text{and} \quad \mathrm{init}_{-w}(R_X) = R_{X,5}. \tag{4.12}$$

The projection $Y = \pi_*(X)$ is a surface of degree three in $\mathbf{P}^3$. Its defining polynomial is $x_1 x_3^2 - x_2 x_4^2$. Therefore its Chow form equals

$$R_Y = [234][124]^2 - (-[134])(-[123])^2.$$

The intersection cycle $X \cap \mathbf{P}^3$ is supported on the union of the two lines $\{x_1 = x_2 = 0\}$ and $\{x_1 = x_4 = 0\}$ in $\mathbf{P}^3$. The cone over this reducible curve is a reducible surface in $\mathbf{P}^4$. It has the Chow form

$$R_{(X \cap \mathbf{P}^3) * e_0} = [023]^5[034]^3.$$

Finally, the extraneous cycle $Z' = Z * e_0$ has degree five and is given by the Chow form

$$R_{Z'} = [012]^4 [014].$$

Returning to our general discussion, we now state a lemma, to be used in the subsequent section, for computing the Chow form of the cone over a projective variety (or cycle). Let Z be any algebraic cycle of dimension $k-1$ in $\mathbf{P}^{n-1}$. Its Chow form R_Z is a polynomial in brackets $[i_1 \dots i_k]$, $1 \leqq i_1 < \dots < i_k \leqq n$. The cone $Z * e_0$ over Z is a cycle of dimension k in $\mathbf{P}^n$, and its Chow form $R_{Z * e_0}$ is a polynomial in brackets $[j_0 j_1 \dots j_k]$ for $0 \leqq j_0 < j_1 < \dots < j_k \leqq n$.

Lemma 4.2 *The Chow form $R_{Z * e_0}$ is gotten from the Chow form R_Z by the substitutions*

$$[i_1 \dots i_k] := [0 i_1 \dots i_k] \quad \text{for } 1 \leqq i_1 < \dots < i_k \leqq n. \tag{4.13}$$

Proof. Let L be a generic flat of codimension $k+1$ in $\mathbf{P}^n$, having Plücker coordinates $[j_0 j_1 \dots j_k]$, $0 \leqq j_0 < \dots < j_k \leqq n$. Consider its projection $\pi(L)$ into the coordinate hyperplane $\mathbb{P}^{n-1}$. The flat $\pi(L)$ has codimension k in $\mathbf{P}^{n-1}$. The formula (4.13) expresses the Plücker coordinates of $\pi(L)$ directly in terms of the Plücker coordinates of L.

It suffices to prove Lemma 4.2 for the case when Z is an irreducible variety. The cone $Z * e_0$ is the unique irreducible variety in $\mathbf{P}^n$ defined by the property that a point $x = (x_0 : x_1 : \dots : x_n) \in \mathbf{P}^n$ lies in $Z * e_0$ if and only if $\pi(x) = (x_1 : \dots : x_n) \in \mathbf{P}^n$ lies in Z. Therefore a flat L of codimension $k+1$ in $\mathbf{P}^n$ lies in the hypersurface defined by $R_{Z * e_0}$ if and only if its projection $\pi(L)$ lies in the hypersurface defined by R_Z. □

5 Product formula for the Chow form

We retain the notations of Sect. 4. We now concentrate on the special case where the projection $\pi: X \to Y$ is a birational map. This means that the ideal I of X contains a homogeneous polynomial of the form $x_0 f(x_1, \dots, x_n) - g(x_1, \dots, x_n)$. The rational function $\dfrac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)}$, which inverts π, is homogeneous of degree 1.

For each generic form $\ell_i = c_{i0} x_0 + c_{i1} x_1 + \dots + c_{in} x_n$ we let $\tilde{\ell}_i = c_{i1} x_1 + \dots + c_{in} x_n$ denote its restriction to $\mathbf{P}^{n-1}$. The intersection cycle $X \cap \mathbf{P}^{n-1}$ is a cycle of dimension $k-1$ and degree d on $\mathbf{P}^{n-1}$. Its Chow form is a polynomial function on the space of k linear forms in $n-1$ variables, so $R_{X \cap \mathbf{P}^{n-1}}(\tilde{\ell}_1, \dots, \tilde{\ell}_k)$ is well-defined.

We consider the generic section $X' = \{\alpha \in X \mid \ell_1(\alpha) = \dots = \ell_k(\alpha) = 0\}$. This linear section consists of d distinct points, with coordinates in the field K. The Chow form of X can be factored as

$$R_X(\ell_0, \ell_1, \dots, \ell_k) = \prod_{l=1}^{d} (\alpha_{l0} c_{00} + \alpha_{l1} c_{01} + \dots + \alpha_{ln} c_{0n}). \tag{5.1}$$

(See [15, I.1.6.5 Eq. (2)], where we have absorbed the constant α into the α_{ij}.) The coefficients α_{lj} are the homogeneous coordinates of the points in the generic section:

$$X' = \{(\alpha_{l0} : \alpha_{l1} : \ldots : \alpha_{ln}) \mid l = 1, 2, \ldots, d\}. \tag{5.2}$$

The generic $(n-k-1)$-plane defined by $\ell_1 = \ldots = \ell_k = 0$ in $\mathbf{P}^{n-1} = \{x_0 = 0\}$ does not intersect the $(k-1)$-dimensional variety $X \cap \mathbf{P}^{n-1}$. Therefore the point configuration X' lies in the affine space $\mathbf{A}^n$ and we have $\alpha_{l0} \neq 0$ for $l = 1, 2, \ldots, d$. We will use local coordinates $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$ for all points in the affine space $\mathbf{A}^n$.

Let $Y = \pi_*(X)$ be the projection of Y into $\mathbf{P}^{n-1}$ as before. We consider its generic section $Y' = \{\beta \in Y \mid \tilde{\ell}_1(\beta) = \ldots = \tilde{\ell}_k(\beta) = 0\}$. This section is a zero-dimensional variety in $\mathbf{P}^{n-1}$, consisting of $\tilde{d}$ distinct points. By genericity, the form g does not vanish at any point in the linear section Y'. Therefore we can represent each element in Y' uniquely by a vector $\beta \in \mathbf{C}^n$ such that $x_0 = f(\beta)/g(\beta) = 1$. Our main result in this section is the following product formula. It relates the Chow form of the projection of X into $\mathbf{P}^{n-1}$ to the Chow form of the intersection of X with $\mathbf{P}^{n-1}$.

Theorem 5.1 *We have the following identity of rational functions in* $\mathbf{C}(c_{01}, c_{02}, \ldots, c_{kn})$:

$$\prod_{\beta \in Y', \frac{f(\beta)}{g(\beta)} = 1} \tilde{\ell}_0(\beta) = \frac{R_Y(\tilde{\ell}_0, \tilde{\ell}_1, \ldots, \tilde{\ell}_k) \cdot R_Z(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)}{R_{X \cap \mathbf{P}^{n-1}}(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)}. \tag{5.3}$$

Proof. Consider the following rational function on $\mathbf{P}^n \times \mathbf{P}^n \times \ldots \times \mathbf{P}^n (k+1 \text{ times})$:

$$\frac{\mathrm{init}_w(R_X)}{\mathrm{init}_{-w}(R_X)}(\ell_0, \ell_1, \ldots, \ell_k) = \frac{R_Y \cdot R_{Z * e_0}}{R_{(X \cap \mathbf{P}^{n-1}) * e_0}}(\ell_0, \ell_1, \ldots, \ell_k). \tag{5.4}$$

The identity in (5.4) is derived from Proposition 4.1. We claim that the right hand side of Eq. (5.3) is obtaind from the right hand side of Eq. (5.4) by taking the limit

$$\lim_{c_{00} \to \infty} (c_{00}^{\tilde{d}} \cdot (\text{r.h.s. (5.4)})). \tag{5.5}$$

This is a consequence of the following three observations.

(a) The factor $R_Y = R_Y(\ell_0, \ldots, \ell_k)$ does not contain any of the first column variables $c_{00}, c_{10}, \ldots, c_{k0}$ because Y lies in $\mathbf{P}^{n-1}$. Therefore we can write $R_Y = R_Y(\tilde{\ell}_0, \ldots, \tilde{\ell}_k)$.

(b) The factor $R_{Z * e_0}$ is a homogeneous polynomial of degree $d - \tilde{d}$ in brackets of the form $[0 i_1 \ldots i_k]$, by Lemma 4.2. Each occurring bracket is linear in the variable c_{00}, with leading coefficient equal to the complementary k-bracket $[i_1 \ldots i_k]$. Thus $R_{Z * e_0}$ is a polynomial of degree $d - \tilde{d}$ in c_{00}, and its leading coefficient equals $R_Z(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)$.

(c) The factor $R_{(X \cap \mathbf{P}^{n-1}) * e_0}$ is a polynomial of degree d in the variable c_{00}. By Lemma 4.2, its leading coefficient equals $R_{X \cap \mathbf{P}^{n-1}}(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)$.

For each parameter value $t \in \mathbf{C}^*$ the Chow form $w(t) \cdot R_X$ represents the deformed irreducible variety $w(t) \cdot X$. The intersection of $w(t) \cdot X$ with $\ell_1 = \ldots = \ell_k = 0$ consists of d points $(\alpha_{l0}(t) : \alpha_{l1}(t) : \ldots : \alpha_{ln}(t))$, $1 \leqq l \leqq d$. These points may be viewed as a vector-valued algebraic function with d branches. As t tends to zero, $\tilde{d}$ of these points approach the points of a generic section of $Y \subset \mathbf{P}^{n-1}$, while the remaining $d - \tilde{d}$ branches approach the points of a generic section of $Z * e_0$. The factorization (5.1) for the Chow form implies that the left hand side of (5.4) equals

$$\lim_{t \to 0} \frac{t^{\tilde{d}} \prod_{i=1}^{d} \left(c_{00} + \frac{\alpha_{i1}}{\alpha_{i0}}(t)\, c_{01} + \ldots + \frac{\alpha_{in}}{\alpha_{i0}}(t)\, c_{0n} \right)}{\prod_{i=1}^{d} \left(c_{00} + \frac{\alpha_{i1}}{\alpha_{i0}}(t^{-1})\, c_{01} + \ldots + \frac{\alpha_{in}}{\alpha_{i0}}(t^{-1})\, c_{0n} \right)}. \tag{5.6}$$

Here we need the factor $t^{\tilde{d}}$ because $\tilde{d}$ of the first coordinates $\alpha_{l0}(t)$ tend to zero as $t \to 0$. We now consider (5.6) as an algebraic function in c_{00}, and we determine its leading term. This is the ratio of the leading term in the denominator with the leading term in the numerator. The former is simply c_{00}^{d}, while the latter is found to equal

$$\lim_{t \to 0} t^{\tilde{d}} \prod_{i=1}^{d} \left(c_{00} + \frac{\alpha_{i1}}{\alpha_{i0}}(t)\, c_{01} + \ldots + \frac{\alpha_{in}}{\alpha_{i0}}(t)\, c_{0n} \right) = c_{00}^{d-\tilde{d}} \prod_{\beta \in Y'} \ell_0(\beta) + o(c_{00}^{d-\tilde{d}}). \tag{5.7}$$

The product on the right hand side is over representatives $\beta \in Y'$ such that $f(\beta)/g(\beta) = 1$. Therefore $\lim_{c_{00} \to \infty} (c_{00}^{\tilde{d}} \cdot (5.6))$ is equal to both the left and the right hand side in (5.3). This completes the proof of Theorem 5.1. □

Example 3.2 (continued) The limit in (5.5) equals the product of (3.5) and (3.6):

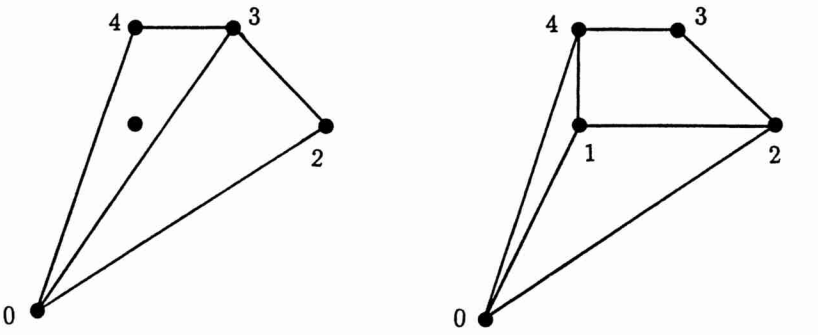
$$\lim_{c_{00} \to \infty} c_{00}^{3} \cdot \frac{([234][124]^2 + [134][123]^2)([012]^4[014])}{[023]^5[034]^3} = R_{\mathscr{A}} \cdot [12]^4 [23]^{-5} [34]^{-3} [14].$$

6 Toric varieties

Fix a configuration of lattice points $\mathscr{A} = \{a_0, a_1, \ldots, a_n\}$ which affinely spans the lattice $\mathbf{Z}^k$. This set defines a k-dimensional toric variety $X_{\mathscr{A}}$ in $\mathbf{P}^n$, namely, $X_{\mathscr{A}}$ is the projective spectrum of the monoid algebra $\mathbf{C}[s\mathbf{t}^{a_0}, s\mathbf{t}^{a_1}, \ldots, s\mathbf{t}^{a_n}]$ which is graded by a new variable s. It has been observed in [11] that the Chow form of $X_{\mathscr{A}}$ equals the $\mathscr{A}$-resultant $R_{\mathscr{A}}$. We recall the following result from [11, §5]:

Theorem 6.1 *The following objects are in one-to-one-to-one-to-one correspondence:*

- *the toric degenerations of $X_{\mathscr{A}}$ in the Chow variety,*
- *the distinct initial forms of the $\mathscr{A}$-resultant $R_{\mathscr{A}}$ (= Chow form of $X_{\mathscr{A}}$),*
- *the regular polyhedral subdivisions of the polytope $P = \mathrm{conv}(\mathscr{A})$ with vertices in $\mathscr{A}$,*
- *the faces of the secondary polytope $\Sigma(\mathscr{A})$ (= Chow polytope of $X_{\mathscr{A}}$).*



$R_{(X\cap \mathbf{P}^{n})*e_0} =$
$[023]^5[034]^3$

$R_Y \cdot R_{Z*e_0} = R_{Y+Z*e_0} =$
$([234][124]^2+[134][123]^2)\cdot[012]^4[014]$

Fig. 1. The two antipodal polyhedral subdivisions in Example 3.2

We apply the constructions from Sects. 5 and 6 to the toric variety $X=X_{\mathscr{A}}$ in $\mathbf{P}^n$. Its projection $Y=\pi_*(X)$ into $\mathbf{P}^{n-1}$ is again a toric variety. It corresponds to the point configuration $\tilde{\mathscr{A}}=\mathscr{A}\backslash\{a_0\}$ gotten by deleting one point; in symbols, $Y=X_{\tilde{\mathscr{A}}}$. We assume (without loss of generality) that the deleted point a_0 is the origin in the lattice $\mathbf{Z}^k$. Hence $\mathbf{t}^{a_0}=1$. It is our goal to derive Theorem 3.1 for the configuration $\tilde{\mathscr{A}}$ by applying Theorem 5.1 to the toric varieties $X=X_{\mathscr{A}}$ and $Y=X_{\tilde{\mathscr{A}}}$.

Our assumption in Theorem 3.1 that the lattice $\mathbf{Z}^k$ is already affinely spanned by the subset $\tilde{\mathscr{A}}$ implies that the map $X\to Y$ is birational. Hence there exists a representation $a_0=\sum_{i=1}^{n}\lambda_i a_i$ where $\lambda_1,\ldots,\lambda_n$ are integers satisfying $\sum_{i=1}^{n}\lambda_i=1$. Then the birational inverse of the projection can be chosen to be

$$x_0=\frac{g(x_1,\ldots,x_n)}{f(x_1,\ldots,x_n)}=x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_n^{\lambda_n}. \tag{6.1}$$

For instance, in Example 3.2 we can take any of the following representations by (4.8):

$$x_0=\frac{x_1^4x_3}{x_2x_4^3}=\frac{x_1^2x_2x_4}{x_3^3}=\frac{x_1^3}{x_3x_4}. \tag{6.2}$$

Using Theorem 6.1, we now give a polyhedral description of the two toric degenerations $(X\cap\mathbf{P}^{n-1})*e_0$ and $Y+Z*e_0$ of the toric variety X. The initial forms of the $\mathscr{A}$-resultant are the denominator and numerator in (5.4). They correspond to the two faces of the secondary polytope $\Sigma(\mathscr{A})$, which maximize the two antipodal linear functionals w and $-w$.

See Fig. 1 for an illustration of these two toric degenerations for the toric surface in $\mathbf{P}^4$ presented in Example 3.2.

To describe the two polyhedral subdivisions of $\mathscr{A}$ we employ the terminology introduced by Lee [13]. The cycle $(X\cap\mathbf{P}^{n-1})*e_0$ corresponds to the regular subdivision of $\mathscr{A}$ gotten by *pulling* the point a_0. In this subdivision, which is denoted $p^-_{a_0}(\{\mathscr{A}\})$ in [13, §2], the point a_0 is joined with each facet of $\mathrm{conv}(\mathscr{A})$

not containing a_0. The cycle $Y+Z*e_0$ corresponds to the regular subdivision of $\mathscr{A}$ gotten by *placing* the point a_0. This subdivision is denoted $p^+_{a_0}(\{\mathscr{A}\})$ in [13, §2]. It includes the subpolytope $\mathrm{conv}(\tilde{\mathscr{A}})$ as one of its cells, while the other cells are gotten by joining the point a_0 with each facet of $\mathrm{conv}(\tilde{\mathscr{A}})$ which is visible from a_0. See [16, §5] for a discussion of the operations of pulling and placing in terms of Gröbner bases.

Proof of Theorem 3.1 The degree d equals the normalized volume of $\mathrm{conv}(\mathscr{A})$, while $\tilde{d}$ is the normalized volume of the subpolytope $\mathrm{conv}(\tilde{\mathscr{A}})$. Over the algebraically closed field K, the zero-dimensional variety

$$Y' = \{\beta = (\beta_1 : \ldots : \beta_n) \in Y \mid \tilde{\ell}_1(\beta) = \ldots = \tilde{\ell}_k(\beta) = 0\} \tag{6.3}$$

consists of $\tilde{d}$ distinct points. None of the coordinates of these points is zero. Therefore we can uniquely coordinatize each point of Y' by the requirement that $\beta^\lambda = \prod_{i=1}^{n} \beta_i^{\lambda_i} = 1$, with $\lambda_1, \ldots, \lambda_n$ as in (6.1). Using this normalization of coordinates, we find that the assignment

$$(\mathbf{C}^*)^k \to \mathbf{P}^{n-1}, \gamma \mapsto (\gamma^{a_1}, \ldots, \gamma^{a_n}) = \beta \tag{6.4}$$

defines a bijection from the zero-dimensional variety $V(f_1, \ldots, f_k)$ in (3.2) to the zero-dimensional variety Y'. This follows from the fact that a_0 equals the origin and hence

$$\beta^\lambda = \prod_{i=1}^{n} (\gamma^{a_i})^{\lambda_i} = \gamma \sum_{i=1}^{n} \lambda_i a_i = \gamma^{a_0} = 1. \tag{6.5}$$

Under this bijection between $V(f_1, \ldots, f_k)$ and Y', each scalar $\tilde{\ell}_0(\beta)$ appearing on the left hand side of (5.3) equals the scalar $f_0(\gamma)$ appearing on the left hand side of (3.3). Therefore the left hand side of (3.3) equals (5.3).

By our polyhedral interpretation arising from Theorem 6.1, the right hand side of (5.4) is an alternating product of $\mathscr{A}$-resultants corresponding to the maximal cells in two "antipodal" subdivisions of $\mathscr{A}$. If we apply the limit (5.5) to that alternating product we get precisely the right hand side of (3.3). Therefore the right hand side of (3.3) equals (5.3). This completes our derivation of Theorem 3.1 from Theorem 5.1. □

7 Factorization of mixed resultants

We now return to the original setting of $k+1$ forms f_i supported on $k+1$ distinct subsets $\mathscr{A}_i \subset \mathbf{Z}^k$, $i = 0, \ldots, k$. We will prove Theorem 1.1 in parallel with the following result.

Proposition 7.1 *The mixed resultant satisfies the following multiplicativity relation:*

$$\begin{aligned} R_{\mathscr{A}_0,\ldots,\mathscr{A}_i+\mathscr{A}_{i'},\ldots,\mathscr{A}_k}(f_0, \ldots, f_i' \cdot f_i'', \ldots, f_k) \\ = R_{\mathscr{A}_0,\ldots,\mathscr{A}_i,\ldots,\mathscr{A}_k}(f_0, \ldots, f_i', \ldots, f_k)^{[L(\mathscr{A}):L(\mathscr{A}')]} \\ \times R_{\mathscr{A}_0,\ldots,\mathscr{A}_{i'},\ldots,\mathscr{A}_k}(f_0, \ldots, f_i'', \ldots, f_k)^{[L(\mathscr{A}):L(\mathscr{A}'')]} \end{aligned} \tag{7.1}$$

where $L(\mathscr{A})$, $L(\mathscr{A}')$ and $L(\mathscr{A}'')$ are lattices affinely generated by $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{A}''$.

Example 7.2 We illustrate Proposition 7.1 in a small one-dimensional example. Let $k=1$, $\mathscr{A}_0=\{0,6\}$, $\mathscr{A}_1'=\{0,2\}$, $\mathscr{A}_1''=\{0,3\}$, $\mathscr{A}_1=\{0,2,3,5\}$. Then $L(\mathscr{A}_1)=\mathbf{Z}^1$, $[L(\mathscr{A}):L(\mathscr{A}')]=3$ and $[L(\mathscr{A}):L(\mathscr{A}'')]=2$. We consider the generic polynomials

$$f_0 := c_{01}\,x^6 + c_{02},$$
$$f_1 := c_{11}\,x^5 + c_{12}\,x^3 + c_{13}\,x^2 + c_{14},$$
$$f_1' := a_0\,x^2 + a_1, \qquad f_1'' := b_0\,x^3 + b_1.$$

The mixed resultant of f_0 and f_1 coincides with the Sylvester resultant:

$$\begin{aligned}(7.2)\qquad R_{\mathscr{A}_0,\mathscr{A}_1}(f_0,f_1) = {} & c_{01}^5\,c_{14}^6 + 3c_{01}^4\,c_{02}\,c_{12}^2\,c_{14}^4 - 2c_{01}^4\,c_{02}\,c_{13}^3\,c_{14}^3 \\ & + 6c_{01}^3\,c_{02}^2\,c_{11}^2\,c_{13}\,c_{14}^3 - 18c_{01}^3\,c_{02}^2\,c_{11}\,c_{12}\,c_{13}^2\,c_{14}^2 \\ & + 3c_{01}^3\,c_{02}^2\,c_{12}^4\,c_{14}^2 + 6c_{01}^3\,c_{02}^2\,c_{12}^2\,c_{13}^3\,c_{14} \\ & + c_{01}^3\,c_{02}^2\,c_{13}^6 + 6c_{01}^2\,c_{02}^3\,c_{11}^3\,c_{12}\,c_{14}^2 \\ & - 18c_{01}^2\,c_{02}^3\,c_{11}^2\,c_{12}^2\,c_{13}\,c_{14} + 3c_{01}^2\,c_{02}^3\,c_{11}^2\,c_{13}^4 \\ & + 6c_{01}^2\,c_{02}^3\,c_{11}\,c_{12}^3\,c_{13}^3 + c_{01}^2\,c_{02}^3\,c_{12}^6 \\ & + 3c_{01}\,c_{02}^4\,c_{11}^4\,c_{13}^2 - 2c_{01}\,c_{02}^4\,c_{11}^3\,c_{12}^3\,c_{14} + c_{02}^5\,c_{11}^6.\end{aligned}$$

We now replace f_1 by the product $f_1'\cdot f_1''$, or equivalently, we replace $c_{11}\mapsto a_0\,b_0$, $c_{12}\mapsto a_1\,b_0$, $c_{13}\mapsto a_0\,b_1$, $c_{14}\mapsto a_1\,b_1$ in (7.2). As the result we obtain

$$\begin{aligned}(7.3)\qquad R_{\mathscr{A}_0,\mathscr{A}_1}(f_0,f_1'\cdot f_1'') &= (c_{01}\,a_1^3 - c_{02}\,a_0^3)^2\cdot(c_{01}\,b_1^2 + c_{02}\,b_0^2)^3 \\ &= R_{\mathscr{A}_0,\mathscr{A}_1}(f_0,f_1')^2\cdot R_{\mathscr{A}_0,\mathscr{A}_1''}(f_0,f_1'')^3,\end{aligned}$$

as predicted by Proposition 7.1.

Proof of Theorem 1.1 *and Proposition* 7.1 We will prove both statements simultaneously by induction on k, the number of variables. The main idea is to reduce the mixed identiy (1.5) to the already established unmixed identity (3.3).

We form $k+1$ duplicates of each form using new indeterminate coefficients, and multiply these together as follows:

$$\begin{aligned}(7.4)\qquad u_0 &= f_{00}f_{01}f_{02}\ldots f_{0k}, & v_0 &= f_{01}f_{02}\ldots f_{0k}, \\ u_1 &= f_{10}f_{11}f_{12}\ldots f_{1k}, & v_1 &= f_{11}f_{12}\ldots f_{1k}, \\ &\vdots = \quad \vdots & &\vdots = \quad \vdots \\ u_k &= f_{k0}f_{k1}f_{k2}\ldots f_{kk}, & v_k &= f_{k1}f_{k2}\ldots f_{kk}.\end{aligned}$$

Each polynomial f_{ij}, $0\leqq i\leqq k$, appearing in the j-th column is supported on $\mathscr{A}_j$. Therefore each row product u_i is supported on $\mathscr{A}=\mathscr{A}_0+\mathscr{A}_1+\ldots+\mathscr{A}_k$. Similarly, each row product v_i is supported on $\bar{\mathscr{A}}=\mathscr{A}_1+\mathscr{A}_2+\ldots+\mathscr{A}_k$.

Step 1 Multiplicativity. We show how formula (1.5) in the statement of the main theorem implies (7.1). Since the resultant is symmetric up to sign in its inputs, we may assume without loss of generality that $i=0$ and let

$$\begin{aligned} \mathscr{A} &= (\mathscr{A}_0' + \mathscr{A}_0'') + \mathscr{A}_1 + \ldots + \mathscr{A}_k, \\ \mathscr{A}' &= \mathscr{A}_0' + \mathscr{A}_1 + \ldots + \mathscr{A}_k, \\ \mathscr{A}'' &= \mathscr{A}_0'' + \mathscr{A}_1 + \ldots + \mathscr{A}_k. \end{aligned} \tag{7.5}$$

Then

$$R(f_0' \cdot f_0'', f_1, \ldots, f_k) = \prod_{\gamma \in V} f_0'(\gamma) f_0''(\gamma) \prod_w R(f_1^w, \ldots, f_k^w)^{-\langle w[\bar{\mathscr{A}}], \mathscr{A}_0^w \rangle}. \tag{7.6}$$

The number of roots $|V(f_1, \ldots, f_k)|$ depends on which lattice $L(\mathscr{A})$, $L(\mathscr{A}')$, or $L(\mathscr{A}'')$ is used to compute the mixed volumes, since we are normalizing with respect to different unit simplices. Let V, V', and V'' denote the three possible varieties arising from the three different lattices $L(\mathscr{A})$, $L(\mathscr{A}')$, and $L(\mathscr{A}'')$. (See the construction in (1.6)). We have the relations

$$|V| = [L(\mathscr{A}) : L(\mathscr{A}')] \cdot |V'| = [L(\mathscr{A}) : L(\mathscr{A}'')] \cdot |V''|. \tag{7.7}$$

Therefore

$$\prod_{\gamma \in V} f_0'(\gamma) f_0''(\gamma) = \prod_{\gamma \in V'} f_0'(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}')]} \cdot \prod_{\gamma \in V''} f_0''(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}'')]}. \tag{7.8}$$

We have $\mathscr{A}_0^w = (\mathscr{A}_0')^w + (\mathscr{A}_0'')^w$ and consequently

$$\langle w[\bar{\mathscr{A}}], \mathscr{A}_0^w \rangle = \langle w[\bar{\mathscr{A}}], (\mathscr{A}_0')^w \rangle + \langle w[\bar{\mathscr{A}}], (\mathscr{A}_0'')^w \rangle. \tag{7.9}$$

We rewrite the relation (7.6) as follows:

$$\begin{aligned} R(f_0' \cdot f_0'', f_1, \ldots, f_k) = & \prod_{\gamma \in V'} f_0'(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}')]} \cdot \prod_w R(f_1^w, \ldots, f_k^w)^{\langle w[\bar{\mathscr{A}}], (\mathscr{A}_0')^w \rangle} \\ & \times \prod_{\gamma \in V''} f_0''(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}'')]} \cdot \prod_w R(f_1^w, \ldots, f_k^w)^{\langle w[\bar{\mathscr{A}}], (\mathscr{A}_0'')^w \rangle}. \end{aligned} \tag{7.10}$$

In the formula (7.10) the facet resultants $R(f_1^w, \ldots, f_k^w)$ are regarded with respect to the lattice affinely generated by $\bar{\mathscr{A}}$, $\mathscr{A}_0$. In order for (1.5) to be applicable, the facet resultants must be regarded with respect to the possibly coarser lattices generated by $\bar{\mathscr{A}}$, $\mathscr{A}_0'$, and $\bar{\mathscr{A}}$, $\mathscr{A}_0''$ respectively. Therefore we must introduce extra exponents $[L(\mathscr{A}) : L(\mathscr{A}')]$ and $[L(\mathscr{A}) : L(\mathscr{A}'')]$. We obtain the following identity:

$$\begin{aligned} R(f_0' \cdot f_0'', f_1, \ldots, f_k) = & R(f_0', f_1, \ldots, f_k)^{[L(\mathscr{A}) : L(\mathscr{A}')]} \\ & \times R(f_0'', f_1, \ldots, f_k)^{[L(\mathscr{A}) : L(\mathscr{A}'')]}. \end{aligned} \tag{7.11}$$

Step 2 The induction. As before, let $\bar{\mathscr{A}}=\mathscr{A}_1+\ldots+\mathscr{A}_k$ and $\mathscr{A}=\mathscr{A}_0+\bar{\mathscr{A}}$. Each u_i is supported on $\mathscr{A}$, and each v_j is supported on $\bar{\mathscr{A}}$. Applying the "unmixed" Theorem 3.1 to the $\mathscr{A}$-resultant and the $\bar{\mathscr{A}}$-resultant, we get

$$(7.12)\quad R_{\mathscr{A}}(u_0,u_1,\ldots,u_k)=\prod_{\gamma\in V(u_1,\ldots,u_k)} u_0(\gamma)\cdot\prod_{w\in S^{k-1}} R_{\mathscr{A}^w}(u_1^w,\ldots,u_k^w)^{\langle w[\mathscr{A}],\mathscr{A}^w\rangle},$$

$$(7.13)\quad R_{\bar{\mathscr{A}}}(v_0,v_1,\ldots,v_k)=\prod_{\gamma\in V(v_1,\ldots,v_k)} v_0(\gamma)\cdot\prod_{w\in S^{k-1}} R_{\bar{\mathscr{A}}^w}(v_1^w,\ldots,v_k^w)^{\langle w[\bar{\mathscr{A}}],\bar{\mathscr{A}}^w\rangle}.$$

We now apply Proposition 7.1 inductively for lattices of dimension $\leqq k-1$. The right hand sides of (7.12) and (7.13) may be completely factored as products of lower dimensional mixed resultants with various exponents. We collect those factors which are functions only in the coefficients of u_0 and of the diagonal polynomials f_{ii}. This includes the factors

$$(7.14)\qquad \prod_{\gamma\in V(f_{11},\ldots,f_{kk})} u_0(\gamma),\quad\text{and}\quad \prod_{\gamma\in V(f_{11},\ldots,f_{kk})} v_0(\gamma),$$

since only the "diagonal" roots $\gamma\in V(f_{11},\ldots,f_{kk})$ depend *only* on the stated coefficients.

We now factor the left hand side of (7.12). One of the appearing factors (conceivably with an exponent $\nu>1$) must be the mixed resultant $R(u_0,f_{11},\ldots,f_{kk})$. This follows directly from our geometric definition in Section 1, because the solvability of the system $u_0=f_{11}=\ldots=f_{kk}=0$ implies the solvability of $u_0=u_1=\ldots=u_k=0$.

A degree count using Lemma 2.2 shows that $\nu=1$ and that $R(u_0,f_{11},\ldots,f_{kk})$ is the only factor on the left hand side which depends only on the coefficients of u_0 and of the diagonal polynomials f_{ii}. The same arguments apply to (7.13). Collecting the factors of interest on both sides of (7.12) and (7.13), we obtain a new pair of equations:

$$(7.15)\qquad R(u_0,f_{11},\ldots,f_{kk})=\prod_{\gamma} u_0(\gamma)\cdot\prod_{w} R(f_{11}^w,\ldots,f_{kk}^w)^{[L(\mathscr{A}^w):L(\bar{\mathscr{A}}^w)]\cdot\langle w[\mathscr{A}],\mathscr{A}^\gamma+\bar{\mathscr{A}}^w\rangle}$$

$$(7.16)\qquad R(v_0,f_{11},\ldots,f_{kk})=\prod_{\gamma} v_0(\gamma)\cdot\prod_{w} R(f_{11}^w,\ldots,f_{kk}^w)^{\langle w[\bar{\mathscr{A}}],\bar{\mathscr{A}}^w\rangle},$$

where γ runs over $V(f_{11},\ldots,f_{kk})$ and w runs over the $(k-1)$-sphere S^{k-1}. The index $[L(\mathscr{A}^w):L(\bar{\mathscr{A}}^w)]$ appears in (7.15) because when extracting the diagonal factors $R(f_{11}^w,\ldots,f_{kk}^w)$ from $R(u_1^w,\ldots,u_k^w)$, we must take into account the refinement of the lattice caused by the additional dependence of the u_i on $\mathscr{A}_0$. Here we have inductively used the multiplicative formula (7.1) in lower dimensions. The same index for the second equation (7.16) is simply $1=[L(\bar{\mathscr{A}}^w):L(\bar{\mathscr{A}}^w)]$.

We divide Eq. (7.15) by Eq. (7.16). Using the facts that $[L(\mathscr{A}^w):L(\bar{\mathscr{A}}^w)]\cdot w[\mathscr{A}]=w[\bar{\mathscr{A}}]$ and $u_0(\gamma)/v_0(\gamma)=f_{00}(\gamma)$, we obtain the identity

$$(7.17)\qquad \frac{R(u_0,f_{11},\ldots,f_{kk})}{R(v_0,f_{11},\ldots,f_{kk})}=\prod_{\gamma\in V(f_{11},\ldots,f_{kk})} f_{00}(\gamma)\cdot\prod_{w\in S^{k-1}} R_{\mathscr{A}^w,\ldots,\mathscr{A}^w}(f_{11}^w,\ldots,f_{kk}^w)^{\langle w[\bar{\mathscr{A}}],\mathscr{A}^\gamma\rangle}.$$

When deriving (7.17) from (7.15) and (7.16), the exponents cancel as follows:

$$(7.18)\qquad [L(\mathscr{A}^w): L(\bar{\mathscr{A}}^w)]\cdot\langle w[\mathscr{A}], \mathscr{A}_0^w+\bar{\mathscr{A}}^w\rangle-\langle w[\bar{\mathscr{A}}], \bar{\mathscr{A}}^w\rangle$$
$$=\langle w[\bar{\mathscr{A}}], \mathscr{A}_0^w+\bar{\mathscr{A}}^w\rangle-\langle w[\bar{\mathscr{A}}], \bar{\mathscr{A}}^w\rangle=\langle w[\bar{\mathscr{A}}], \mathscr{A}_0^w\rangle.$$

All that remains is to observe that the quotient on the left hand side of (7.17) is a polynomial, since $v_0=0$ implies $u_0=0$. The denominator exactly divides, and the result is $R_{\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_k}(f_{00},f_{11},\ldots,f_{kk})$ by another degree count. We conclude that the product formula (1.5) holds for the "diagonal" polynomials f_{00}, $f_{11},\ldots,f_{kk}$. These polynomials are entirely generic, so we are done.

We note that our induction begins in dimension $k=0$ with constants, in which case both Theorem 1.1 and Proposition 7.1 are trivially satisfied. □

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