Math. Z. 214, 261–271 (1993)

Mathematische
Zeitschrift

Fano manifolds and quadric bundles

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Received 4 June 1992; in final form 28 October 1992

0 Introduction

Let X be a smooth projective variety defined over the field of complex numbers. Throughout the present paper we will assume that X is of dimension $2r$ and its first Chern class $c_1(X)$ (or, equivalently, anticanonical divisor $-K_X$) is linearly equivalent to rH where H is an ample divisor on X. If r is the largest integer dividing $c_1(X)$ then such an X will be called a Fano manifold of index r. In the present paper we prove the following (the notions of *elementary contractions*, *scrolls* etc. will be recalled in the subsequent section).

Theorem I *Let X be a Fano manifold of index r and dimension $2r$. Assume that $r \geqq 3$ and the second Betti number of X, $b_2(X)$, is at least 2. Then one of the following holds*:

(a) *X has a projective bundle structure over a Fano manifold of dimension $r+1$;*
(b) *X has a quadric bundle structure over a smooth variety Y;*
(c) *X has two elementary contractions: one of them makes X a non-equidimensional scroll, the other is birational and of divisorial type (see also Lemma 1.5 below for a description of the divisorial contraction);*
(d) *X admits two elementary contractions and any of them gives a structure of a nonequidimensional scroll on X.*

Remark. If X is a Fano surface of index 1 (the case $r=1$ in the above theorem) then it is called del Pezzo surface and is obtained by blowing-up $b_2(X)-1$ points on $\mathbf{P}^2$ (where $2 \leqq b_2(X) \leqq 8$). Also Fano 4-folds of index 2 are well understood, see [Mu] and [W4]. For Fano manifolds of index larger than half of their dimension we have the following

Theorem. [W2, W3] *Let X be a Fano manifold of index* $r > \dim X/2$. *Then either $b_2(X)=1$ or X has a projective bundle structure.*

As the result of the Theorem I we obtain the following

Corollary. *Let X be Fano manifold of index r and dimension $2r$. If $r \geqq 3$ then $b_2(X) \leqq 2$ unless $X = \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$.*

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Remark. For 6-folds this was proved by Kontani [Ko].

Fano manifolds of index r and dimension $2r$ which have projective bundle structure were studied in [PSW] where a thorough classification of them was obtained. It was proved that, except a few well understood examples which occur for $r \leqq 3$, any of them is a projectivisation of a decomposable bundle on a quadric $\mathbf{Q}^{r+1}$ or on a projective space $\mathbf{P}^{r+1}$, or it is just a product of $\mathbf{P}^{r-1}$ with a del Pezzo manifold, i.e. a Fano manifold of index r and dimension $r+1$. This settles the description of the case (a) of the above theorem.

In the present paper we discuss also the case (b) of the theorem i.e. Fano manifolds of dimension $2r$ and index r which are quadric bundles. A complete description of these is obtained:

Theorem II *Let X and Y be as in the Theorem I, case* (b). *Then one of the following holds*:

(A) $X \simeq \mathbf{Q}^r \times \mathbf{Q}^r$;

(B) $Y \simeq \mathbf{P}^r$ *and either*

(a) *X is a divisor of bidegree* (1, 2) *in the product* $\mathbf{P}^r \times \mathbf{P}^{r+1}$, *or*

(b) *X is a divisor of bidegree* (1, 1) *in the product* $\mathbf{P}^r \times \mathbf{Q}^{r+1}$, *or*

(c) *X is double covering of the product $\mathbf{P}^r \times \mathbf{P}^r$ branched along a divisor of bidegree* (2, 2), *or*

(d) *X is a blow-up of a smooth quadric $\mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$ along its linear section* $\mathbf{Q}^{r-1} = \mathbf{P}^r \cap \mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$.

As for the cases (c) and (d) from the Theorem I, we have the following examples which show that these cases can really occur

Example. Let X be a blow-up of a smooth quadric $\mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$ along a linear subspace $\mathbf{P}^{r-1} \subset \mathbf{Q}^{2r}$. Then X is the graph of a projection of $\mathbf{Q}^{2r}$ from $\mathbf{P}^{r-1}$ onto $\mathbf{P}^{r+1}$. The manifold X is Fano manifold of index $2r$ and the two maps – the projection onto $\mathbf{P}^{r+1}$ and the blow-down morphism onto $\mathbf{Q}^{2r}$ – are as in the point (c) of the theorem.

Example. Let X be a smooth complete intersection of two divisors of bidegree (1, 1) on the product $\mathbf{P}^{r+1} \times \mathbf{P}^{r+1}$. Then X has a structure described in the case (d) of the theorem: any of the two projections of X onto $\mathbf{P}^{r+1}$ is a non-equidimensional scroll.

The paper is divided into three sections: first we recall some pertinent results, then in subsequent sections we prove Theorems I and II. The proof of Theorem I involves studying deformations of rational curves on Fano manifolds and so-called Mori's *breaking-up technique.* A similar approach was used in [W2] and [W3] to deal with Fano manifolds of larger index. The proof of Theorem II depends on several structural results on *elementary contractions* which are recalled in the subsequent section.

1 Preliminaries

In this section we recall some definitions and results which will be used in the sequel. Our terminology is compatible with the one used in *minimal model theory* (cf. [KMM]) and *adjunction theory* (cf. [BSW]).

1.0 The set-up. Let X be a projective manifold of dimension n and L an ample line bundle on X. Assume that K_X is not *nef* (*nef* means *numerically effective*) and that for some integer r the divisor K_X+rL is nef but not ample. Then, according to *Contraction Theorem* (see [KMM]) a linear system $|m(K_X+rL)|$ is base-point-free for $m \gg 0$ and thus it defines a projective map $\phi: X \to Y$; we assume that ϕ is onto a normal projective variety Y and has connected fibers. The divisor K_X+rL is isomorphic to a pull-back, via ϕ, of an ample divisor from Y; we call K_X+rL a *good supporting divisor* of ϕ. Such a ϕ is called *elementary contraction* if all curves contracted by ϕ to points are numerically proportional.

We say that (X, L, ϕ) is a *scroll* (resp. a *quadric fibration*) ir $r-1=\dim X - \dim Y$ (resp. $r=\dim X-\dim Y$).

The exceptional locus of an elementary contraction ϕ, that is the set on which ϕ is not bijective (if $\dim Y<\dim X$ then the set is equal to X), will be denoted by $E(\phi)$. If $E(\phi)=X$ then ϕ is called *fiber type*, if $E(\phi)$ is of codimension 1 in X then ϕ is called *divisorial*, otherwise it is called *small*.

The first result we need is about the dimension of $E(\phi)$.

Proposition 1.1 ([I, Theorem 0.4; W, Theorem 1.1] *Assume that a map ϕ from* (1.0) *is an elementary contraction. Then for any irreducible component F of any non-trivial fiber of ϕ the following inequality holds*

$$\dim F+\dim E(\phi) \geqq n+r-1.$$

Now we recall structural results about equidimensional maps, i.e. maps whose fibers are of equal dimension.

Proposition 1.2 [F1, Lemma 2.12] *If (X, L, ϕ) is an equidimensional scroll* (i.e. *all fibers of ϕ are of dimension* $r-1$) *and ϕ is an extremal ray contraction then Y is smooth and $\phi: X \to Y$ is a projective bundle over Y, that is, there exists a vector bundle $\mathscr{E}$ over Y such that $X=\mathbf{P}(\mathscr{E})$ (one may assume $\mathscr{E} \simeq \phi_* L$).*

Corollary 1.3 *Let us assume that (X, L, ϕ) is as in* (1.0) *and moreover assume that ϕ is an elementary contraction. If all fibers of ϕ are of dimension $\leqq r-1$ then $X \simeq \mathbf{P}(\phi_*(L))$.*

Proposition 1.4 [ABW, Theorem C] *If (X, L, ϕ) is an equidimensional quadric bundle (i.e. all fibers of ϕ are of dimension r) and ϕ is an extremal ray contraction then Y is smooth and X embedds over Y into a projective bundle $\mathbf{P}(\mathscr{E})$ as a divisor of the relative degree 2 (as above, one may take $\mathscr{E} \simeq \phi_* L$; $\mathscr{E}$ is of rank $r+2$).*

Remark. The original statement of the Theorem C in [ABW] is about the smoothness of Y; the rest then follows easily because μ is then flat and L has a constant number of section on every fiber so that, by a theorem of Grauert, $\phi_* L$ is locally free; the embedding $X \subset \mathbf{P}(\phi_* L)$ is then given by the evaluation $\phi^* \phi_* L \to L$.

We will also need the following easy fact on divisorial elementary contractions (for its proof see e.g. [PSW, Lemma 7.1]).

Lemma 1.5 *Let (X, L, ϕ) be as at the beginning of the section. Assume that ϕ is an elementary contraction such that $E=E(\phi)$ is a divisor. Let $Z=\phi(E)$. If* $\dim Z=\dim E-r$ *then*

(a) *Y is smooth at a general point of Z,*
(b) *a general fiber F of $\phi_{|E}$ is isomorphic to $\mathbf{P}^r$ and $\mathcal{O}_F(E) \simeq O_{\mathbf{P}^r}(-1)$,*
(c) *K_Y is a Cartier divisor and $K_X = \phi^* K_Y + rE$.*

Next we will also need a result on exceptional fibers of a contraction morphism.

Proposition 1.6 [YZ, F1, F2] *Assume that X, L, r and $\phi: X \to Y$ are as at the beginning of this section. Let F be an irreducible component of a fiber of ϕ. If* $\dim F = r > \dim$*(general fiber of ϕ) then the normalization of F is isomorphic to $\mathbf{P}^r$.*

Remark. The result is proved in [YZ] (although it is not formulated there this way) and is based on a vanishing whose idea is from [F1]; to get a reference: use vanishing from Lemma 4 in [YZ] and apply Theorem 2.2 from [F2].

We will depend on the following result on maps of projective spaces and quadrics

Proposition 1.7 [L, CS, PS] *Assume that Y is a smooth variety of dimension n.*

(a) *If there exists a dominant regular map $\psi: \mathbf{P}^n \to Y$ then Y is isomorphic to $\mathbf{P}^n$;*

(b) *If there exists a dominant regular map $\psi: \mathbf{Q}^n \to Y$ then $Y = \mathbf{P}^n$ or $Y = \mathbf{Q}^n$ and in the latter case the map ψ is biregular.*

We will also use a result on extending sections of line bundles from a divisor to the ambient space. The following result is a version of a theorem of Sommese: its proof is the same as the proof of Proposition III from [S].

Proposition 1.8 (cf. [S, Proposition III]) *Let X be an irreducible and smooth divisor on a smooth projective variety V of dimension n. Assume that L is a line bundle on V such that $L_X := L_{|X}$* is spanned by global sections on X. Assume also that

(i) *the evaluation map $X \to \mathbf{P}^{\dim|L_X|}$* is onto variety of dimension $\leqq n-3$,
(ii) *the line bundle $\mathcal{O}_X(X)$ is ample, and*
(iii) *$\mathcal{O}_V(mX) \otimes L^{-1}$ is ample on V for $m \gg 0$,*

then every section of L_X extends to a section of L on V (in particular $Bs|L| \cap X = \emptyset$).

Finally, we will use a result about *uniform* vector bundles on $\mathbf{P}^r$. A vector bundles $\mathcal{E}$ of rank k on $\mathbf{P}^r$ is called *uniform* if its restriction to any line on $\mathbf{P}^r$ is isomorphic to $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_k)$ with fixed integers $a_1 \leqq \ldots \leqq a_k$; the sequence $(a_1 \ldots a_k)$ is called *splitting type* of $\mathcal{E}$. It is well-known (see e.g. [OSS, 3.2.1]) that if $\mathcal{E}$ is uniform of splitting type $(a, a, \ldots, a)$ then $\mathcal{E} \simeq \mathcal{O}(a) \oplus \ldots \mathcal{O}(a)$. We also need the following

Proposition 1.9 ([E] and [OSS, 3.4]) *If $\mathcal{E}$ is a uniform vector bundle on $\mathbf{P}^r$ of rank k with the splitting type $(a, \ldots, a, a+1)$ (resp. $(a-1, a \ldots a)$) then $\mathcal{E}$ is either decomposable into a sum of line bundles or (if $k \geqq r$) isomorphic to $T\mathbf{P} \otimes \mathcal{O}(a-1) \oplus \mathcal{O}(a)^{\oplus(k-r)}$ (to $\Omega\mathbf{P} \otimes \mathcal{O}(a+1) \oplus \mathcal{O}(a)^{\oplus(k-r)}$, respectively), where $T\mathbf{P}$* (resp. $\Omega\mathbf{P}$) *denotes the tangent bundle (resp. the cotangent bundle) of $\mathbf{P}^r$.*

Proof. For $k \leqq r$ the result is known (see e.g. [OSS, 3.4]). For $r=2$ and arbitrary k the result is due to Elencwajg [E]. The extension of these results to arbitrary

r and k is proved by induction on $k \geqq 3$. We may assume that the bundle $\mathscr{E}$ is of splitting type $(0, 0 \ldots, 0, 1)$ therefore the restriction of $\mathscr{E}$ to any hyperplane $H \simeq \mathbf{P}^{r-1}$ either splits into a sum of line bundles (in this case $\mathscr{E}$ splits because of 2.3.2 from [OSS]) or is isomorphic to $T\mathbf{P}^{r-1} \otimes \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(k-r-1)}$. Call this restriction $\mathscr{E}_H$. The bundle $\mathscr{E}_H$ is spanned and $H^i(H, \mathscr{E}_H(-1)) = 0$ for any $i \geqq 0$.

We claim that also $H^i(\mathbf{P}^r, \mathscr{E}(-1)) = 0$ for any $i \geqq 0$. Indeed, take a linear pencil of hyperplanes Λ and by Γ denote the incidence variety of Λ (the blow up of $\mathbf{P}^r$ at the base point locus of Λ) with projections $p: \Gamma \to \mathbf{P}^r$ (the blow-down map) and $q: F \to \Lambda$. Then $R^i q_*(p^* \mathscr{E}) = 0$ for $i \geqq 0$, $p_*(p^* \mathscr{E}) = \mathscr{E}$ and $R^i p_*(p^* \mathscr{E}) = 0$ for $i > 0$ and our claim follows by Leray spectral sequence.

Now using the cohomology of a sequence

$$0 \to \mathscr{E}(-1) \to \mathscr{E} \to \mathscr{E}_H \to 0$$

we find out that sections of $\mathscr{E}_H$ extend to sections of $\mathscr{E}$ so that the latter bundle is spanned by global sections. Being spanned it fits to a sequence

$$0 \to \mathcal{O}^{\oplus(k-r)} \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is a uniform vector bundle of rank r and splitting type $(0, 0 \ldots 0, 1)$. Thus $\mathscr{E}'$ is as we desire, so is $\mathscr{E}$.

2 Proof of Theorem I

Throughout the present section we assume that X is a Fano manifold of dimension $n = 2r \geqq 6$ and index $r = n/2$; let H be an ample divisor on X such that $-K_X = rH$. We will also assume that Picard number of X, $\rho(X)$, is at least two. We will examine elementary contractions of X: each elementary contraction ϕ of X fits to the description from (1.0) (for a suitable choice of L), so that its fibers should be rather large (of dimension $\geqq r-1$). On the other hand, we will frequently use the fact that no two fibers of two different elementary contractions can intersect along a curve, so that their fibers – if they meet – should not be too large. This is an easy observation:

Lemma 2.0 *Assume that F_1 and F_2 are fibers of two different elementary contractions of X. If $F_1 \cap F_2 \neq \emptyset$ then* $\dim F_1 + \dim F_2 \leqq n$ *and therefore, in view of* (1.1), $r-1 \leqq \dim F_i \leqq r+1$, *for* $i = 1, 2$.

First we have to show that at least one of the elementary contractions is of fiber type – so that its fibers meet fibers of other contractions.

Lemma 2.1 *There exists a rational curve C_0, $C_0 \cdot H = 1$ whose deformations sweep out the manifold X.*

Proof is very similar to this of Lemma 1 from [W3], we only sketch it here. We are to show that through every point of X there passes a curve whose intersection $\cdot$ with H is equal to 1. Assume the contrary. Then there exists a point $x \in X$ such that every rational curve pasing through this point has intersection at least 2 with H. From Mori theory, [M, Theorem 6], we know that there exists a rational curve C_1 passing through x such that $C_1 \cdot H = 2$. From deformation theory, [ibid, Sect. 1], (and our assumption that C_1 has minimal intersection with H among curves passing through x) we know that deformation of C_1 containing x sweep out a subvariety of X of codimension at most 1. If the variety coincides with X then we conclude as the proof of Lemma 1 in [W3] or (3) in [W2]. Otherwise we note that the effective divisor swept out by deforming C_1 must have positive intersection with some extremal rational curve on X (it is clear that the curve is not a multiplicity of C_1: a divisor contracted by an elementary contraction has non-positive intersection with curves contracted by the contraction) and we conclude similarly.

2.2 Now as in [W2], or [W3] we construct a complete variety T parametrising deformations of C_0 and incidence variety V with proper maps $p: V \to X$ and $q: V \to T$, p onto X, fibers of q being rational curves. From deformation theory it follows that $\dim T \geqq n+r-3=3(r-1)$ so that fibers of p are of dimension $\geqq r-2$.

For a closed subvariety $Y \subset X$ we will denote $T_Y = q(p^{-1}(Y))$, $X_Y = p(q^{-1}(T_Y))$. The variety X_Y is closed in X.

For a closed subvariety $F \subset X$ by $NE(F) \subset NE(X)$ we will denote a cone spanned by numerical classes of curves contained in F. Similarly as Lemma (1.4.5) in [BSW] one proves the following

Lemma 2.3 *If $F \subset X$ is a fiber of a projective map of X then*

$$NE(X_F) = NE(F) + \mathbf{R}^+ [C_0].$$

Lemma 2.4 *If $F \subset X$ is a fiber of a projective map which does not contract C_0 then*

$$\dim X_F \geqq \dim F + (r-1).$$

Proof. Fibers of p are od dimension $\geqq r-2$, thus we will be done if we prove that the induced map

$$p_{|q^{-1}(T_F)}: q^{-1}(T_F) \to X_F$$

is finite-to-one outside $p^{-1}(F)$. This, however, follows from Mori's *breaking-up technique*, cf. [M, proof of Theorem 4]. Namely, otherwise we can find in T_F a curve parametrizing curves passing through a point outside F, then via normalisation we can produce a ruled surface having two non-intersecting multisections contracted by different morphisms: see e.g. (3) in [W2] for details.

Lemma 2.5 *Let ϕ_1 be an elementary contraction of X which does not contract C_0. If ϕ_1 has a fiber of dimension $\geqq r+1$ then $\rho(X)=2$ and X has a $\mathbf{P}^{r-1}$-bundle structure over a smooth Fano variety. Therefore, in view of* (1.1), *X admits no small contraction unless it has a projective bundle structure.*

Proof. Let F be a fiber in question, $\dim F \geqq r+1$. According to the above lemma $X_F = X$ so that $NE(X) = NE(F) + \mathbf{R}^+ [C_0]$. Thus $\rho(X)=2$ and C_0 is an extremal rational curve. The contraction of the ray spanned on the class of C_0 has all fibers of dimension $\leqq r-1$, (2.0), so it gives a projective bundle structure to X by (1.2).

Lemma 2.6 *X has at least one extremal ray contraction of fibre type.*

Proof. Assume the contrary. Take an extremal ray contraction with exceptional locus being a divisor, say E_1. Being effective E_1 has positive intersection with some other extremal ray and thus it must meet its locus, say E_2. Thus we can find two fibers F_1 and F_2 of two extremal ray contractions which meet, $\dim F_1 = \dim F_2 = r$ because of (1.1) and (2.0). As above, we find out that X_{F_1} is a divisor meeting F_2 so that we have a curve from F_2 whose numerical class is in $NE(X_{F_1}) = NE(F_1) + \mathbf{R}^+[C_0]$, a contradiction.

From now on we may assume that C_0 is an extremal rational curve. Let ϕ_0 denote the contraction of the ray spanned by C_0.

Lemma 2.7 *If $\rho(X) \geqq 3$ then $n=6$ and all extremal ray contractions of X have all fibers of dimension $\leqq 2$, so that X has a projective bundle structure.*

Proof. Assume $\rho(X) \geqq 3$. We will be done if we prove that no contraction of X is of divisorial type. Indeed, in such a case we take a contraction of a 2-dimensional face of $NE(X)$. The contraction is onto a variety of dimension $\geqq r-1$ (because no fiber of a contraction of the remaining extremal ray is contracted) and has a general fiber of dimension $\geqq 2r-2$ (because Picard number of a general fiber is $\geqq 2$ and we can apply the main result of [W2]) thus $n=6$ and any fiber of a contraction of a remaining extremal ray is of dimension 2.

Now we can apply an argument from the previous lemma. Assume that we have two rays not spanned by C_0 with loci E_1 and E_2, where E_1 is a divisor. Since E_1 is effective, it has positive intersection with some extremal curve; thus we can assume that either $E_1 \cdot C_0 > 0$ or, perhaps changing E_2, $E_1 \cdot C_2 > 0$, where the exeptional locus of the contraction of an extremal ray spanned on C_2 is E_2. If E_1 meets E_2 then arguing as in the previous lemma we arrive to the contradiction. Therefore we may assume that both E_1 and E_2 are divisors and E_1 has positive intersection with C_0, But then either E_2 contains a fiber of the contraction of C_0 or has a positive-dimensional intersection with a general fiber of ϕ_0, so E_1 and E_2 have to meet anyway.

Now we can improve Lemma 2.5 to get

Lemma 2.8 *If an extremal ray contraction of X has a fiber of dimension $\geqq r+1$ then X has a projective bundle structure.*

Proof. In view of (2.5)–(2.7) we can assume $\rho(X)=2$ and the contraction ϕ_0 of C_0 has a fiber of dimension $\geqq r+1$ while another contraction ϕ_1 of X is of divisorial type. But then the exceptional divisor of ϕ_1 has positive intersection with C_0 so that any two fibers of ϕ_0 and ϕ_1 meet, which yields contradiction because of (2.0) and (1.1).

Now we arrive to the conclusion of this section:

Proof of Theorem I. Let us assume that X does not have a projective bundle structure. Then, according to (2.7), $\rho(X)=2$ and X has two elementary contractions ϕ_0 and ϕ_1. According to Lemma 2.6 we may assume that ϕ_0 is of fiber type. Then, in view of (2.8), ϕ_0 has all fibers of dimension $\leqq r$, so either it is an equidimensional quadric bundle or non-equidimensional scroll. The same argument applies to ϕ_1 if it is of fiber type. If not, then, according to (1.1), it is divisorial and has all fibers of dimension r.

3 Proof of Theorem II

In this section we assume that X is a Fano manifold of index r and dimension $2r$, and has a quadric bundle structure, that is, there exists an elementary contraction $\phi: X \to Y$ which is a quadric bundle. The manifold X has two elementary contractions so let ψ denote "the other" contraction of X. From the classification of such Fano manifolds which have projective bundle structure (Main Theorem and Proposition 7.4 in [PSW]) it follows that we may assume that ψ is not a projective bundle so that it has fibers of dimension r. We will prove the following refined version of the Theorem II:

Theorem 3.0 *Let X, Y, ϕ and ψ be as above. Then one of the following holds:*

(a) *$X \simeq \mathbf{Q}^r \times \mathbf{Q}^r$ and both ϕ and ψ are projections onto factors;*

(B) *$Y \simeq \mathbf{P}^r$ and either*

(a) *X is a divisor of bidegree $(1,2)$ in the product $\mathbf{P}^r \times \mathbf{P}^{r+1}$, ϕ and ψ are projections onto $\mathbf{P}^r$ and $\mathbf{P}^{r+1}$, respectively, or*

(b) *X is a divisor of bidegree $(1,1)$ in the product $\mathbf{P}^r \times \mathbf{Q}^{r+1}$, ϕ and ψ are projections onto $\mathbf{P}^r$ and $\mathbf{Q}^{r+1}$, respectively, or*

(c) *X is double covering of the product $\mathbf{P}^r \times \mathbf{P}^r$ branched along a divisor of bidegree $(2,2)$, ϕ and ψ are projections onto each of $\mathbf{P}^r$'s (equivalently: X is an intersection of a cone over a Segre imbedding $\mathbf{P}^r \times \mathbf{P}^r \subset \mathbf{P}^{(r+1)(r+1)-1}$ with a smooth quadric in $\mathbf{P}^{(r+1)(r+1)}$), or*

(d) *X is a blow-up of a smooth quadric $\mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$ along its linear section $\mathbf{Q}^{r-1} = \mathbf{P}^r \cap \mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$, the map ϕ is the projection from $\mathbf{P}^r \subset \mathbf{P}^{2r+1}$ and ψ is the blow-down morphism.*

The proof of the above theorem will occupy the rest of this section. First we will show that Y is either $\mathbf{P}^r$ of $\mathbf{Q}^r$ with the latter possibility only if $X \simeq \mathbf{Q}^r \times \mathbf{Q}^r$.

Lemma 3.1 *If X is as above then Y is isomomorphic to either $\mathbf{Q}^r$ (if ψ is a quadric bundle) or to $\mathbf{P}^r$.*

Proof. First note that we may assume ψ has a fiber of dimension r and thus ϕ is equidimensional, but then by a result of [ABW] Y is smooth. If ψ is equidimensional, then it is a quadric bundle and its general quadric is isomorphic to $\mathbf{Q}^r$. But then $\mathbf{Q}^r$ is mapped onto smooth Y and thus, by (1.7), Y is either $\mathbf{P}^r$ or $\mathbf{Q}^r$. The latter case is possible only if the map from the fiber onto Y is biregular.

If ψ is not equidimensional then by (1.6) an exceptional fiber of ψ has a component whose normalisation is $\mathbf{P}^r$. Then, again, we have $\mathbf{P}^r$ mapped onto smooth Y so by (1.7) $Y = \mathbf{P}^r$.

Lemma 3.2 *In the above situation, if $Y = \mathbf{Q}^r$ then $X = \mathbf{Q}^r \times \mathbf{Q}^r$.*

Proof. By the symmetry, we have a finite map $\phi \times \psi: X \to \mathbf{Q}^r \times \mathbf{Q}^r$. But, by (1.7), fibers of one of the maps (ϕ or ψ) are mapped biholomorphically onto "the other" $\mathbf{Q}^r$, so that the product map is actually biholomorphic.

In the remaining part of this section we deal with the case $Y = \mathbf{P}^r$; so that $\phi: X \to \mathbf{P}^r$. Let $\mathscr{E} := \phi_* \mathcal{O}_X(H)$. The sheaf $\mathscr{E}$ is then locally free of rank $r+2$ and X embedds into $\mathbf{P}(\mathscr{E})$ as a divisor of relative degree 2 so that $\mathcal{O}_{\mathbf{P}(\mathscr{E})}(1)_{|X} \simeq \mathcal{O}(H)$. Let η denote the pullback to $\mathbf{P}(\mathscr{E})$ of a hyperplane section of $\mathbf{P}^r$ and let ξ the divisor class of $\mathcal{O}_{\mathbf{P}(\mathscr{E})}(1)$. From the adjunction formula for $X \subset \mathbf{P}(\mathscr{E})$ we find out that $X \in |2\xi + (r+1-c)\eta|$, where $c_1(\mathscr{E}) = c_1(\mathcal{O}(c))$. The proof of the part (B) of the theorem is now divided into some lemmata.

Lemma 3.3 *In the above situation $c \leqq r+3$.*

Proof. This follows from smoothness of X. Namely, X is defined by a section of $S^2(\mathscr{E})$ whose determinant vanishes along the divisor of singular fibers (singular quadrics) of ϕ and is of degree $2c+(r+2)(r+1-c)$.

Lemma 3.4 *If ξ is ample then X is described in either* (a), *or* (b), *or* (d) *of the part* (B) *of* (3.0).

Proof. Since $\mathscr{E}$ is ample of rank $r+2$ and $c_1(\mathscr{E}) \leqq r+3$ it follows that $\mathscr{E}$ is uniform of splitting type either $(1, \dots 1)$ or $(2, 1, \dots 1)$. In the former case $\mathscr{E}$ is trivial twisted by $\mathscr{O}(1)$ and consequently X is as in the case (a) of the theorem. In the latter case from (1.9) it follows that $\mathscr{E}$ is either $\mathscr{O}(2) \oplus \mathscr{O}(1) \oplus \dots \oplus \mathscr{O}(1)$ or $T\mathbf{P}^r \oplus \mathscr{O}(1) \oplus \mathscr{O}(1)$ and thus X is, respectively, as in the case (d) or (b) of the theorem.

Lemma 3.5 *If "the other" contraction ψ is of divisorial type then X is as in the case* (d) *of* (3.0).

Proof. A general description of the contraction morphism ψ in this case is provided in Lemma 1.5. Moreover, let us note that a good supporting divisor D of ψ is a multiple of either $\xi+\eta$ or $2\xi+\eta$. The proof of this is similar to an argument from [PSW]. Namely, let E be the exceptional divisor of ψ, then E has intersection -1 with an extremal rational curve contracted by ψ and thus $\xi+E$ is a good supporting divisor of ψ, and $\psi^* K_Y = r(\xi+E)$. Then, using deformation argument as in the proof of Lemma 7.2 from [PSW] we note that K_V has intersection $\leqq 2r+1$ with the image of a line contained in the fiber of ϕ. Therefore $E=\xi-a\eta$ and $D=2\xi-a\eta$, and consequently $a=1$ or 2 depending on the intersection of η with the extremal rational curve contracted by ψ.

Now we claim that the case $D=2\xi-\eta$ can not occur. For this purpose let us consider the restriction of $\mathscr{E}$ to a line $l \subset \mathbf{P}^r$ and then by X_l denote $\phi^{-1}(l)$. On $\mathbf{P}(\mathscr{E})$ we have the following intersection formulas: $\eta^2 \cdot \mathbf{P}(\mathscr{E}_l) \equiv 0$, $\eta \cdot \xi^{r+1} \cdot \mathbf{P}(\mathscr{E}_l) = 1$ and $\xi^{r+2} \cdot \mathbf{P}(\mathscr{E}_l) = c_1(\mathscr{E})$. Thus we can compute the intersection $E \cdot D^r$ inside X_l:

$$E \cdot D^r \cdot X_l = (\xi-\eta) \cdot (2\xi-\eta)^r \cdot (2\xi+(r+1-c)\eta) \cdot \mathbf{P}(\mathscr{E}_l) = 2^r(c-1).$$

But this intersection is equal to 0 as the image of E under the contraction ψ is of dimension $r-1$. Therefore $c=1$ in this case. To complete this case note that $\mathscr{E}_l$ has a quotient bundle $\mathscr{E}'_l$ of rank $r+1$ such that $c_1(\mathscr{E}'_l) \leqq 0$. The intersection $E \cdot D^{r-1} \cdot X \cdot \mathbf{P}(\mathscr{E}'_l)$ should be then non-negative (as $\mathbf{P}(\mathscr{E}'_l) \subset \mathbf{P}(\mathscr{E}_l)$). But computing it as above we get

$$\begin{aligned} E \cdot D^{r-1} \cdot X \cdot \mathbf{P}(\mathscr{E}') &= (\xi-\eta) \cdot (2\xi-\eta)^{r-1} \cdot (2\xi+(r+1-c)\eta) \cdot \mathbf{P}(\mathscr{E}'_l) \\ &= 2^{r-1}(2c_1(\mathscr{E}'_l)-1) < 0, \end{aligned}$$

a contradiction.

To complete the proof of the lemma we deal with the case when $E=\xi-2\eta$ and $D=\xi-\eta$. Then we compute as above that $c=r+3$. Consequently $X \in |2\xi-2\eta| = |2D|$ and thus ξ is ample on $\mathbf{P}(\mathscr{E})$ because of the following lemma and consequently we are done because of (3.3).

Lemma 3.6 *If $\mathscr{O}_X(X)$ is not ample then ξ is ample on $\mathbf{P}(\mathscr{E})$. Also, if $c > r+1$ then ξ is ample on $\mathbf{P}(\mathscr{E})$.*

Proof. The cone of effective 1-cycles on $\mathbf{P}(\mathscr{E})$ has two edges. Since $\mathcal{O}_X(X)$ is not ample X must contain curves generating the edge not contracted by $\mathbf{P}(\mathscr{E}) \to \mathbf{P}^r$. Since $\xi_{|X}$ is ample it follows that ξ has positive intersection with this edge and consequently is ample. The second part of the lemma one gets similarly.

Lemma 3.7 *Assume that $X \subset \mathbf{P}(\mathscr{E})$ is a quadric bundle such that its "other contraction" ψ is of fibre type and $\mathscr{E}$ is not ample. Then X is as in the case* (c) *of the Theorem* (3.0).

Proof. In view of (3.6) $c \leqq r+1$ and therefore $X-2\xi=(r+1-c)\eta$ is nef on $\mathbf{P}(\mathscr{E})$. Let L_X be a pull-back – via ψ – of a very ample line bundle on the target of ψ. Since $\rho(\mathbf{P}(\mathscr{E}))=\rho(X)$ we may assume that the line bundle L_X extends to a line bundle L on $\mathbf{P}(\mathscr{E})$. Since ξ is ample on X, $X \geqq 2\xi$ on $\mathbf{P}(\mathscr{E})$ and X is effective, X is also nef on $\mathbf{P}(\mathscr{E})$. Therefore, by a similar argument as in the above lemma, it follows that for $m \gg 0$ the line bundle $\mathcal{O}_{\mathbf{P}(\mathscr{E})}(mX) \otimes L^{-1}$ is ample on $\mathbf{P}(\mathscr{E})$. Moreover, (3.6) yields that $\mathcal{O}_X(X)$ is ample on X. Thus we are in the situation of Sommese's theorem (1.8) and consequently every section of L_X extends to $\mathbf{P}(\mathscr{E})$ and base locus of L does not meet X.

Let $l \subset \mathbf{P}^r$ be a line. We have the following decomposition of $\mathscr{E}$ on l

$$\mathscr{E}_l = \mathcal{O}(v) \oplus \mathscr{E}'_l$$

where v is the smallest number in the splitting type of $\mathscr{E}$ on l. Let $C_0 \subset \mathbf{P}(\mathscr{E}_l)$ be a line such that $\xi \cdot C_0 = v$, then the bundle L is base-point-free on $\mathbf{P}(\mathscr{E}_l)$ outside C_0 (because the base-point-set of L on $\mathbf{P}(\mathscr{E}_l)$ does not meet X and thus is of dimension 1 at most). Therefore $\mathscr{E}'_l$ is ample so that $c_1(\mathscr{E}'_l)=c-v \geqq r+1$. From (3.6) it follows that $v \leqq 0$. But then C_0 is in the base locus of L. Thus X does not meet the curve C_0 and thus $X \cdot C_0 = 0$. On the other hand we find out that

$$0 = X \cdot C_0 = 2\xi \cdot C_0 + (r+1-c)\eta \cdot C_0 = 2v + r + 1 - c \leqq v$$

and therefore $v=0$ and $c=r+1$. Consequently, the splitting type of $\mathscr{E}$ on (every line) l is (0, 1, ..., 1). Now we use (1.9) to complete the proof. Namely, we verify easily that only $\mathscr{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)^{\oplus r+1}$ satisfies our assumptions in this case, so that we are in the case (c) of (3.0) (B).

Conclusion. Note that lemmata 3.4, 3.5 and 3.7 yield the part (B) of Theorem 3.0.

Acknowledgements. I would like to acknowledge a partial support from a Polish grant KBN GR-54. I am grateful to Professors Ballico and Spindler for providing me with a reference to (1.9).

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Note added in proof.
A classification of Fano manifolds of index r, dimension $2r$ and $b_2 \geqq 2$ has been completed recently by the author of the present paper and Edoardo Ballico. The result is as follows: such manifolds have either projective or quadric bundle structure (and therefore are know – see [PSW] and Theorem II of the present paper, respectively), or they are isomorphic to one of the two types of varieties discussed in the examples following the Theorem II.