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The homology groups of the links of quasi-ordinary singularities

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1 Introduction

A d-dimensional complex analytic hypersurface germ $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ is called a *quasi-ordinary singularity* if there is a branched covering $\pi: (X, x) \to (\mathbb{C}^d, 0)$, which is unbranched outside the union of the coordinate hyperplanes in $\mathbb{C}^d$. We may assume that π is given by the projection of the first d-coordinates. It turns out that (X, x) can be represented by the image of an open neighborhood of 0 in $\mathbb{C}^d$ under the map $(s_1, \ldots s_d) \mapsto (s_1^n, \ldots, s_d^n, \zeta(s_1, \ldots, s_d))$, $n > 0$ where ζ is a convergent power series. ζ is called a *parametrization* of (X, x). Set

$$D_i = \{t \in \mathbb{C} \mid |t| \leqq \varepsilon_i\}, \quad \varepsilon_i > 0,$$
$$\partial D^{d+1} = \partial(D_1 \times \ldots \times D_{d+1}),$$
$$L = X \cap \partial D^{d+1}.$$

By the conic structure lemma [BV], the local topology of $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ is determined by the pair $(\partial D^{d+1}, L)$. L is called the *link* of the singularity (X, x). The collection $\mathcal{U} = \{U_i = \pi^{-1}(V_i)\}$, where $V_i = (D_1 \times \ldots \times D_{i-1} \times D_i^* \times D_{i+1} \times \ldots \times D_d) \cap \partial D^d$, forms an open covering of L. The Čech resolution of singular chain complexes of L with respect to the covering U gives rise to a spectral sequence

$$E^2_{p,q}(L, \mathcal{U}) = H^\delta_p\Big(\bigoplus_{i_0 < \ldots < i_p} H_q(U_{i_0 \ldots i_p}, \mathbb{Z})\Big) \Rightarrow H_{p+q}(L, \mathbb{Z}),$$

where $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$. Now the map π induces a morphism π_* between this spectral sequence and the following spectral sequence:

$$E^2_{p,q}(\partial D^d, \mathcal{V}) = H^\delta_p\Big(\bigoplus_{i_0 < \ldots < i_p} H_q(V_{i_0 \ldots i_p}, \mathbb{Z})\Big) \Rightarrow H_{p+q}(\partial D^d, \mathbb{Z}),$$

where $\mathcal{V} = \{V_i\}$ and $V_{i_0 \ldots i_p} = V_{i_0} \cap \ldots \cap V_{i_p}$. Our particular choice of the covering $\mathcal{V}$ forces that the spectral sequence $E^2_{p,q}(\partial D^d, \mathcal{V})$ degenerates at E^2. In fact,

$E^2_{p,q}(\partial D^d, \mathcal{V})=0$ unless $(p, q)=(0, 0)$ or $(d-1, d)$. Hence $E^2_{p,q}(L, \mathcal{U})$ can be calculated in terms of the quotient groups

$$\wedge^q G_{i_0\ldots i_p}=H_q(i_{0\ldots i_p}, \mathbb{Z})/\pi_*(H_q(U_{i_0\ldots i_p}, \mathbb{Z})).$$

It will be shown in § 7 that these quotient groups are presented by matrices involving the exponents of the terms of the parametrization ζ.

We begin by introducing the Čech resolution of singular chain complexes. In § 3 and § 4, we review some basic properties of the quasi-ordinary singularities and study the covering spaces associated to these singularities. In § 5, we show the spectral sequence $E^2_{p,q}(\partial D^d, \mathcal{V})$ degenerates at E^2. In § 6, we get various results on the homology groups of the link by combining the outcomes of § 4 and § 5. In § 7, we give presentations of the Galois groups $G_{i_0\ldots i_p}$ and show that our computation of the homology group $H_{2d-3}(L, \mathbb{Z})$ via the spectral sequences agrees with Lipman's in his memoir [L]. Finally, we conclude with some examples in § 8.

2 The Čech resolution of singular chain complexes

Let X be a topological space and $\mathcal{U}=\{U_i\}_{i\in I}$ be an open covering of X. Let $S_n(X)$ be an abelian group generated by singular n-chains in X and $S^{\mathcal{U}}_n(X)$ the subgroup of $S_n(X)$ generated by the singular n-simplices which lie in some elements of $\mathcal{U}$. Then

$$S^{\mathcal{U}}_\cdot(X):=\ldots \xleftarrow{\partial} S^{\mathcal{U}}_{q-1}(X) \xleftarrow{\partial} S^{\mathcal{U}}_q(X) \xleftarrow{\partial} S^{\mathcal{U}}_{q+1}(X) \xleftarrow{\partial} \ldots \tag{1}$$

forms a chain complex with the usual topological boundary map ∂. It is clear that the natural inclusion map

$$i\colon\ S^{\mathcal{U}}_\cdot(X)\to S_\cdot(X) \tag{2}$$

is a chain map. Indeed, it is a chain equivalence [V, Appendix I]. Thus the induced map

$$H_q(i)\colon\ H_q(S^{\mathcal{U}}_\cdot(X))\to H_q(S_\cdot(X))$$

is an isomorphism for each q.

Now fix, once and for all, a well-ordering of the index set I. For any finite set of indices $i_0, \ldots, i_p\in I$ we denote the intersection $U_{i_0}\cap\ldots\cap U_{i_p}$ by $U_{i_0\ldots i_p}$. For each $p, q\geqq 0$, let

$$C_{p,q}:=C_{p,q}(X, \mathcal{U})=\bigoplus_{i_0<\ldots<i_p} S_q(U_{i_0\ldots i_p}). \tag{3}$$

Thus an element $c\in C_{p,q}$ is determined by giving an element

$$c_{i_0\ldots i_p}\in S_q(U_{i_0\ldots i_p}),$$

for each $(p+1)$-tuple $i_0 < \ldots < i_p$ of elements of I. Now we define the Čech boundary map $\delta\colon C_{p,q} \to C_{p-1,q}$ by the "alternating sum formula"

$$(\delta c)_{i_0 \ldots i_{p-1}} = \sum_i c_{i i_0 \ldots i_{p-1}}. \tag{4}$$

Here we adopt the convention that interchanging two indices in $c_{i_1 \ldots i_p}$ introduces a minus sign. Moreover, the usual boundary map ∂ on the singular chain complex naturally extends to $\partial\colon C_{p,q} \to C_{p,q-1}$ by $(\partial c)_{i_1 \ldots i_p} = \partial(c_{i_1 \ldots i_p})$. Also, it is easy to see that $\partial^2 = 0$ and $\partial\delta = \delta\partial$. Thus a bigraded group

$$C_{\cdot\cdot} = \bigoplus_{p,q \geq 0} C_{p,q}$$

together with differentials

$$\delta\colon C_{p,q} \to C_{p-1,q}$$
$$\partial' := (-1)^p \partial\colon C_{p,q} \to C_{p,q-1}$$

forms a double complex.

Lemma 2.1 *The following sequence of complexes is exact:*

$$0 \leftarrow S^{\mathcal{U}}_{\cdot}(X) \xleftarrow{\delta} C_{0\cdot}(X, \mathcal{U}) \xleftarrow{\delta} C_{1\cdot}(X, \mathcal{U}) \xleftarrow{\delta} \ldots \tag{5}$$

Hence the spectral sequence associated with the double complex $C_{\cdot\cdot} = \oplus C_{pq}$ converges to $H_{p+q}(X, \mathbb{Z})$. In other words, we have the following spectral sequence:

$$E^2_{p,q}(X, \mathcal{U}) = H^{\delta}_p\Big(\bigoplus_{i_0 < \ldots < i_p} H_q(U_{i_0 \ldots i_p}, \mathbb{Z})\Big) \Rightarrow H_{p+q}(X, \mathbb{Z}), \tag{6}$$
$$d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

Proof. The proof follows from the above discussion and Proposition 15.2 on p. 186 in [BT]. □

Now we will consider the relative situation. Let X and Y be topological spaces, and $\pi\colon X \to Y$ be a continuous map from X to Y. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X, and $\mathcal{V} = \{V_i\}_{i \in I}$ be a covering of Y such that $\pi(U_i) \subset V_i$ for all $i \in I$. Then the map p induces a chain map

$$\pi_*\colon C_{p\cdot}(X, \mathcal{U}) \to C_{p\cdot}(Y, \mathcal{V}),$$

and $\pi_* \delta = \delta \pi_*$. Hence Lemma 2.1 can be generalized in the following way.

Lemma 2.2 *The map π induces a morphism π_* between the following spectral sequences.*

$$\begin{array}{ccc} E^2_{p,q}(X,\mathscr{U}) = H^\delta_p(\bigoplus_{i_0<\ldots<i_p} H_q(U_{i_0\ldots i_p},\mathbb{Z})) & \Rightarrow & H_{p+q}(X,\mathbb{Z}) \\ \downarrow \pi_* & & \downarrow \pi_* \\ E^2_{p,q}(Y,\mathscr{V}) = H^\delta_p(\bigoplus_{i_0<\ldots<i_p} H_q(V_{i_0\ldots i_p},\mathbb{Z})) & \Rightarrow & H_{p+q}(Y,\mathbb{Z}). \end{array}$$

Notation 1 For the simplicity, we denote by $H_n(X)$ the integral singular homology group $H_n(X,\mathbb{Z})$ in the sequel.

3 Quasi-ordinary singularities

Definition 3.1 An irreducible hypersurface analytic germ $(X,x)\subset(\mathbb{C}^{d+1},0)$ is called a *quasi-ordinary* singularity if there exists a branched covering map of analytic germs $\pi:(X,x)\to(\mathbb{C}^d,0)$ with a normal crossing discriminant Δ at 0. The map π is called a quasi-ordinary projection.

After a suitable coordinate change, we may assume that the quasi-ordinary projection π is the restriction of the projection map $\mathbb{C}^{d+1}\to\mathbb{C}^d$ sending $(t_1,\ldots,t_d,z)\mapsto(t_1,\ldots,t_d)$ and that its discriminant locus Δ is contained in the coordinate hyperplanes of $\mathbb{C}^d$. For such a coordinate system, the Weierstrass preparation theorem implies that the germ $(X,x)\subset(\mathbb{C}^{d+1},0)$ is defined by an equation

$$F(t_1,\ldots t_d,z)=z^m+f_1(t_1,\ldots,t_d)\,z^{m-1}+\ldots+f_m(t_1,\ldots t_d)=0$$

where $(t_1,\ldots t_d,z)$ is a local coordinate system of $\mathbb{C}^{d+1}$ and the f_i are non-unit convergent power series. From the Riemann Extension Theorem and the fact that the fundamental group of the complement of the discriminant locus is a free abelian group, it follows that the roots ζ_i of F as a polynomial in z belong to a fractional convergent power series ring $\mathbb{C}\{t_1^{1/n},\ldots,t_d^{1/n}\}$ for some $n>0$ (cf. [O, Theorem 1.6.5]). Thus the germ (X,x) can be represented as the image of an open neighborhood of 0 in $\mathbb{C}^d$ under the map $(s_1,\ldots,s_d)\mapsto(s_1^n,\ldots,s_d^n,\zeta(s_1,\ldots,s_d))$, $n>0$ where ζ is a convergent power series. In this sense, ζ is called a *parametrization* of the germ (X,x). Since the discriminant Δ of F has a normal crossing, Δ is of the form

$$\prod_{i\neq j}(\zeta_i-\zeta_j)=t_1^{e_1}\ldots t_d^{e_d}\cdot\varepsilon(t_1,\ldots,t_d),\qquad \varepsilon(0,\ldots,0)\neq 0.$$

Since $t_1^{1/n},\ldots,t_d^{1/n}$ are irreducible elements in the unique factorization domain $\mathbb{C}\{t_1^{1/n},\ldots,t_d^{1/n}\}$, we have

$$\zeta_i-\zeta_j=M_{ij}\,\varepsilon_{ij}(t_1^{1/n},\ldots,t_d^{1/n}),\qquad \varepsilon_{ij}(0,\ldots,0)\neq 0 \tag{7}$$

where

$$M_{ij}=t_1^{a_{1i}/n}\, t_2^{a_{2i}/n}\ldots t_d^{a_{di}/n} \tag{8}$$

with integers $a_l \geqq 0$ depending on (i, j).

In general, a fractional convergent power series $\zeta \in \mathbb{C}\{t_1^{1/n}, \ldots, t_d^{1/n}\}$ is called a *quasi-ordinary branch* if any two conjugates $\zeta_i \neq \zeta_j$ of ζ over $\mathbb{C}\{t_1, \ldots, t_d\}$ satisfy the Eq. (7). The fractional monomials M_{ij} are called the *characteristic monomials* of ζ and the exponent vectors $(a_1/n, a_2/n, \ldots, a_d/n)$ of the characteristic monomials $M_{ij}=t_1^{a_{1i}/n}\, t_2^{a_{2i}/n}\ldots t_d^{a_{di}/n}$ are called the *characteristic exponent vectors.* It turns out that the characteristic monomials determine the embedded topology $(X, x)\subset(\mathbb{C}^{d+1}, 0)$. The converse was proved by Gau [G] using Lipman's result on the topological invariance of the branching sequences [L1]. We refer to [L1] for more detailed exposition. The following description of the characteristic exponent vectors will be used in § 7.

Lemma 3.1 ([L1, Lemma 5.7] and [G, Proposition 1.3]) *Let*

$$\zeta=\sum_{(\alpha_1,\ldots,\alpha_d)} c_{\alpha_1,\ldots,\alpha_d}\, t_1^{\alpha_1}\ldots t_d^{\alpha_d}\in\mathbb{C}\{t_1^{1/n}, \ldots, t_d^{1/n}\}$$

be a convergent fractional power series. Assume that ζ is a quasi-ordinary branch. Then the set $\{(\lambda_{1i}, \ldots, \lambda_{di}) | c_{\lambda_{1i},\ldots,\lambda_{di}} \neq 0,\ 1 \leqq i \leqq g\}$, consisting of characteristic exponent vectors of ζ satisfies (after reindexing if necessary):

(1) $(0, \ldots, 0)<(\lambda_{11}, \ldots, \lambda_{d1})<\ldots<(\lambda_{1g}, \ldots, \lambda_{dg})$ *where* $(\lambda_{1i}, \ldots, \lambda_{di})<(\lambda_{1j}, \ldots, \lambda_{dj})$ *means that* $\lambda_{1i}\leqq\lambda_{1j}, \ldots, \lambda_{di}\leqq\lambda_{dj}$ *and* $(\lambda_{1i}, \ldots, \lambda_{di})\neq(\lambda_{1j}, \ldots, \lambda_{dj})$.
(2) *For* $1 \leqq k \leqq g$,

$$(\lambda_{1k}, \ldots, \lambda_{dk})\notin\mathbb{Z}^d+\sum_{i<k}\mathbb{Z}(\lambda_{1i}, \ldots, \lambda_{di}).$$

(3) *If* $c_{\alpha_1,\ldots,\alpha_d}\neq 0$, *then*

$$(\alpha_1, \ldots, \alpha_d)\in\mathbb{Z}^d+\sum_{(\lambda_{1i},\ldots,\lambda_{di})\leqq(\alpha_1,\ldots,\alpha_d)}\mathbb{Z}(\lambda_{1i}, \ldots, \lambda_{di}).$$

Conversely, if there is a set $\{(\lambda_{1i}, \ldots, \lambda_{di}) | c_{\lambda_{1i},\ldots,\lambda_{di}} \neq 0,\ 1 \leqq i \leqq g\}$ of exponent vectors of ζ satisfying the above conditions 1, 2 *and* 3, *then ζ is a quasi-ordinary branch. Moreover the set $\{(\lambda_{1i}, \ldots, \lambda_{di})\}$ is the set of characteristic exponent vectors of ζ. In particular, there is a chain of subfields of $\mathscr{L}(\zeta)$:*

$$\mathscr{L}\subsetneqq\mathscr{L}(M_1)\subsetneqq\mathscr{L}(M_1, M_2)\subsetneqq\ldots\subsetneqq\mathscr{L}(M_1, M_2, \ldots, M_g)=\mathscr{L}(\zeta)$$

where $M_i=t_1^{\lambda_{1i}}\ldots t_d^{\lambda_{di}}$ are the characteristic monomials and $\mathscr{L}$ is the fractional field of $\mathbb{C}(t_1, \ldots, t_d\}$. The monomial M_1 will be called the minimal characteristic monomial.

4 Quasi-ordinary singular spaces as covering spaces

Let $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ be a quasi-ordinary singularity with a parametrization $\zeta = H(t_1^{1/n}, \ldots, t_d^{1/n})$ and let π be the quasi-ordinary projection from (X, x) to $(\mathbb{C}^d, 0)$. Let

$$D_i := \{t \in \mathbb{C} \mid |t| \leqq \varepsilon_i\}. \quad \varepsilon_i > 0,$$
$$N_i := D_1 \times \ldots \times D_{i-1} \times \partial D_i \times D_{i+1} \times \ldots \times D_d,$$
$$\partial D^d := \bigcup_{i=1}^{d} N_i,$$
$$V_i := (D_1 \times \ldots \times D_{i-1} \times D_i^* \times D_{i+1} \times \ldots \times D_d) \cap \partial D^d$$

where $D_i^* := D_i - \{0\}$. Hence ∂D^d is homeomorphic to the standard sphere of dimension $2d-1$ and $N_{i_0} \cap N_{i_1} \cap \ldots \cap N_{i_p}$ is a strong deformation retract of $V_{i_0} \cap V_{i_1} \cap \ldots \cap V_{i_p}$ for $1 \leqq p \leqq d$. We will adopt the following notations.

$$N_{i_0 \ldots i_p} := N_{i_0} \cap \ldots \cap N_{i_p}, \tag{9}$$

$$V_{i_0 \ldots i_p} := V_{i_0} \cap \ldots \cap V_{i_p} \tag{10}$$

for $1 \leqq i_0 < \ldots < i_p \leqq d$. The topological space $\partial D^{d+1} \cap X$ is called the link of the singularity (X, x). By the conic structure lemma [BV], the local topology of the singularity is determined by the pair $(\partial D^{d+1}, \partial D^{d+1} \cap X)$. Since π is a finite map, we may take $\pi^{-1}(\partial D^d)$ as the link of X at x by choosing $\varepsilon_1, \ldots, \varepsilon_d$ relatively small compared to ε_{d+1}. Set

$$L = \pi^{-1}(\partial D^d), \tag{11}$$

$$M_{i_0 \ldots i_p} = \pi^{-1}(N_{i_0 \ldots i_p}), \tag{12}$$

$$U_{i_0 \ldots i_p} = \pi^{-1}(V_{i_0 \ldots i_p}). \tag{13}$$

Note that the collection $\{U_i\}_{i=1,\ldots,d}$ is an open covering of L. It is easy to see that the space $M_{i_0 \ldots i_p}$ is a strong deformation retract of the open set $U_{i_0 \ldots i_p}$ (cf. [O, Lemma 3.4.1]). We let

$$\zeta_{i_0 \ldots i_p} := H(0, \ldots, 0, t_{i_0}^{1/n}, 0, \ldots, 0, t_{i_1}^{1/n}, 0, \ldots, 0, t_{i_p}^{1/n}, 0, \ldots, 0). \tag{14}$$

Then the map $(s_{i_0}, \ldots, s_{i_p}) \mapsto (s_{i_0}^n, \ldots, s_{i_p}^n, \zeta_{i_0 \ldots i_p}(s_{i_0}, \ldots, s_{i_p}))$ parametrizes a quasi-ordinary singularity in $\mathbb{C}^{p+2}$. This singularity is called an *induced quasi-ordinary singularity* by $\zeta_{i_0 \ldots i_p}$. Let

$$\mathscr{L}' \subset \mathscr{L}'(\zeta_{i_0 \ldots i_p}) \subset \mathscr{L}'_n$$

be the respective fraction fields of

$$\mathbb{C}\{t_{i_0}, t_{i_1}, \ldots, t_{i_p}\} \subset \mathbb{C}\{t_{i_0}, t_{i_1}, \ldots, t_{i_p}\}[\zeta_{i_0 \ldots i_p}] \subset \mathbb{C}\{t_{i_0}^{1/n}, \ldots, t_{i_p}^{1/n}\}.$$

Then $\mathscr{L}'(\zeta_{i_0 \ldots i_p})$ is a Galois extension of $\mathscr{L}'$ since $\mathscr{L}'_n$ (and hence $\mathscr{L}'(\zeta_{i_0 \ldots i_p})$) is an abelian extension of $\mathscr{L}'$. Now let $\mathscr{L}$ be the fractional field of $\mathbb{C}\{t_1, t_2, \ldots, t_d\}$. Then since $\mathscr{L}' = \mathscr{L}'(\zeta_{i_0 \ldots i_p}) \cap \mathscr{L}$, it follows from Theorem 1.12 on p. 305 in [La] that the restriction map gives a canonical isomorphism between

the Galois group $\mathrm{Gal}[\mathscr{L}'(\zeta_{i_0\ldots i_p})\colon \mathscr{L}']$ and the Galois group $\mathrm{Gal}[\mathscr{L}(\zeta_{i_0\ldots i_p})\colon \mathscr{L}]$. Set

$$G_{i_0\ldots i_p} := \mathrm{Gal}[\mathscr{L}(\zeta_{i_0\ldots i_p})\colon \mathscr{L}]. \tag{15}$$

Lemma 4.1 *There is an exact sequence:*

$$0 \to H_1(U_{i_0\ldots i_p}) \xrightarrow{H_1(\pi)} H_1(V_{i_0\ldots i_p}) \to G_{i_0\ldots i_p} \to 0. \tag{16}$$

Proof. Let $(X_{i_0\ldots i_p}, x) \subset (\mathbb{C}^{p+2}, 0)$ be the induced quasi-ordinary singularity by $\zeta_{i_0\ldots i_p}$ and $\pi\colon X_{i_0\ldots i_p} \to \mathbb{C}^{p+1}$ be the quasi-ordinary projection. Then there is the following commutative diagram:

$$\begin{array}{ccccc} U_{i_0\ldots i_p} & \longleftarrow & M_{i_0\ldots i_p} & \longrightarrow & X^*_{i_0\ldots i_p} \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ V_{i_0\ldots i_p} & \longleftarrow & N_{i_0\ldots i_p} & \longrightarrow & (D^*)^{p+1} \end{array}$$

where $(D^*)^{p+1} = \{(t_{i_0}, \ldots, t_{i_p}) \in \mathbb{C}^{p+1} \mid 0 < t_{i_j} < \varepsilon_{i_j}\}$ and $X^* = \pi^{-1}((D^*)^{p+1})$. Here the top (resp. bottom) right horizontal arrow is induced by the projection sending $(t_1, \ldots, t_d, z) \in \mathbb{C}^{d+1}$ to $(t_{i_0}, \ldots, t_{i_p}, z) \in \mathbb{C}^{p+2}$ (resp. sending $(t_1, \ldots, t_d) \in \mathbb{C}^d$ to $(t_{i_0}, \ldots, t_{i_p}) \in \mathbb{C}^{p+1}$). Note that all horizontal arrows are homotopy equivalences. Since $\mathscr{L}'(\zeta_{i_0\ldots i_p})$ is a Galois extension of $\mathscr{L}'$, $\pi\colon X^*_{i_0\ldots i_p} \to (D^*)^{p+1}$ is an unbranched Galois covering. Thus the map $\pi_*\colon \pi_1(X_{i_0\ldots i_p}) \to \pi_1((D^*)^{p+1})$ between the fundamental groups is injective and its cokernel is $G_{i_0\ldots i_p}$ (See [O, Theorem 1.7.1] for the detailed proof.) Since both $\pi_1(U_{i_0\ldots i_p})$ and $\pi_1(V_{i_0\ldots i_p})$ are abelian, the proof follows from this. □

Let $E = \bigoplus_{i=1}^{p} \mathbb{Z} e_i$ be a free abelian group. Recall that the q-th exterior product $\wedge^q E$ of E is the free abelian group $\bigoplus_{1 \le i_1 < \ldots < i_q \le p} \mathbb{Z} e_{i_1} \wedge \ldots \wedge e_{i_q}$ of rank $\binom{p}{q}$. To each group homomorphism $f\colon E \to F$ between free abelian groups, we obtain a group homomorphism

$$\wedge^q(f)\colon \wedge^q E \to \wedge^q F$$

which is defined by

$$\wedge^q(f)(\textstyle\sum a_i e_{i_1} \wedge \ldots \wedge e_{i_q}) = \sum a_i f(e_{i_1}) \wedge \ldots \wedge f(e_{i_q}), \qquad a_i \in \mathbb{Z}.$$

Definition 4.1 Let G be a finite abelian group with a presentation

$$0 \to E \xrightarrow{f} F \to G \to 0,$$

where E and F are free abelian groups. Then the abelian group $\wedge^q G$ with a presentation

$$0 \to \wedge^q E \xrightarrow{\wedge^q(f)} \wedge^q F \to \wedge^q G \to 0$$

is called the *q-th exterior product* of G, where $\wedge^q E$ and $\wedge^q F$ denote the usual exterior powers of free abelian groups.

Remark 1 Assume that E and F have the same rank. Let $A=(a_{ij})$ be the $p \times p$ matrix representing the map f with respect to the ordered basis $\{e_1, \ldots, e_p\}$ (resp. $\{f_1, \ldots, f_p\}$) of E (resp. F). The matrix A will be called a *presentation matrix* of the group G. Note that the presentation matrix of $\wedge^q G$ is given by a matrix whose entries are $q \times q$-minors of A. In particular, the order of the group $\wedge^q G$ is $|G|^{\binom{p-1}{q-1}}$. By a direct computation, one can see that the presentation matrix of $\wedge^{p-1} G$ is given by the adjoint matrix Ad/A^t of the transpose of A with respect to the ordered bases $\{(-1)^{i+1} e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_p\}_{i=1,\ldots,p}$ of E and $\{(-1)^{i+1} f_1 \wedge \ldots \wedge f_{i-1} \wedge f_{i+1} \wedge \ldots \wedge f_p\}_{i=1,\ldots,p}$ of F.

We will always assume that the Galois group $G_{i_0 \ldots i_p} = \mathrm{Gal}[\mathscr{L}(\zeta_{i_0 \ldots i_p}): \mathscr{L}]$ is presented by the sequence (16).

Lemma 4.2 *For $1 \leqq q \leqq p+1$, the following sequence is exact:*

$$0 \to H_q(U_{i_0 \ldots i_p}) \xrightarrow{H_q(\pi)} H_q(V_{i_0 \ldots i_p}) \to \wedge^q G_{i_0 \ldots i_p} \to 0. \tag{17}$$

Proof. Note that $V_{i_0 \ldots i_p}$ and $U_{i_0 \ldots i_p}$ are homotopic to a $(p+1)$ dimensional torus. Thus, by the naturality of Künneth isomorphism [Do, (2.7) on p. 190], there is a commutative diagram:

$$\begin{array}{ccc} H_q(U_{i_0 \ldots i_p}) & \xrightarrow{\sim} & \wedge^q H_1(U_{i_0 \ldots i_p}) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \wedge^q H_1(\pi)} \\ H_q(V_{i_0 \ldots i_p}) & \xrightarrow{\sim} & \wedge^q H_1(V_{i_0 \ldots i_p}) \end{array}$$

where the horizontal arrows are Künneth isomorphisms. The lemma follows from this and Lemma 4.1. □

Notation 2 Since both $U_{i_0 \ldots i_p}$ and $V_{i_0 \ldots i_p}$ are connected spaces, the map $H_0(\pi)$: $H_0(U_{i_0 \ldots i_p}) \to H_0(V_{i_0 \ldots i_p})$ is an isomorphism. In light of this, we set

$$\wedge^0 G_{i_0 \ldots i_p} = 0. \tag{18}$$

The complex $E^1_{\cdot q}(\partial D^d, \mathscr{V})$, i.e.

$$\ldots \leftarrow \bigoplus_{i_0, \ldots, i_p} H_q(V_{i_0 \ldots i_p}) \xleftarrow{\delta} \bigoplus_{i_0, \ldots, i_{p+1}} H_q(V_{i_0 \ldots i_{p+1}}) \leftarrow \ldots$$

induces the following complex of quotient groups

$$(19) \qquad \cdots \leftarrow \bigoplus_{i_0,\ldots,i_p} \wedge^q G_{i_0\ldots i_p} \xleftarrow{\delta} \bigoplus_{i_0,\ldots,i_{p+1}} \wedge^q G_{i_0\ldots i_{p+1}} \leftarrow \cdots$$

since the map $H_q(\pi)$ commutes with δ.

Lemma 4.3 *For $0<q<d$, the following map δ is surjective.*

$$\bigoplus_{i_0,\ldots,i_{q-1}} \wedge^q G_{i_0\ldots i_{q-1}} \xleftarrow{\delta} \bigoplus_{i_0,\ldots,i_q} \wedge^q G_{i_0\ldots i_q}.$$

Proof. It will be shown in (the proof of) Lemma 5.1 that the map δ: $\bigoplus_{i_0,\ldots,i_q} H_q(V_{i_0\ldots i_q}) \to \bigoplus_{i_0,\ldots,i_{q-1}} H_q(V_{i_0\ldots i_{q-1}})$ is surjective. Thus the induced map on the quotient groups is also surjective. □

Let

$$(20) \qquad S_q(U_{i_0\ldots i_p} \xrightarrow{\pi} V_{i_0\ldots i_p}) := S_q(V_{i_0\ldots i_p})/\pi_* S_q(U_{i_0\ldots i_p}),$$

$$(21) \qquad C_{p,q}(L \xrightarrow{\pi} \partial D^d) := \bigoplus_{i_0<\ldots<i_p} S_q(U_{i_0\ldots i_p} \xrightarrow{\pi} V_{i_0\ldots i_p}).$$

Then the bigraded group

$$C..(L \to \partial D^d) = \bigoplus_{p,q\geqq 0} C_{pq}(L \xrightarrow{\pi} \partial D^d)$$

with the induced differentials from the double complex $C..(\partial D^d, \mathscr{V})$, forms a double complex. Let $\mathrm{Tot}(C..(L \to \partial D^d))$ be the associated single complex. Then there is a spectral sequence:

$$(22) \qquad G^1_{p,q} = H^\partial_q(C_p.(L \to \partial D^d)) \Rightarrow H_{p+q}(\mathrm{Tot}(C..(L \to \partial D^d))).$$

This spectral sequence will be called a *Galois spectral sequence.*

Theorem 4.4 *For $0 \leqq q \leqq p+1$, the complex of G^1-terms of the Galois spectral sequence* (22) *is isomorphic to the complex* (19). *In particular, we have* $G^1_{p,q} \cong \bigoplus_{i_0,\ldots,i_p} \wedge^q G_{i_0\ldots i_p}$.

Proof. This follows from Lemma 4.2. □

5 The degeneration of $E^2_{p,q}(\partial D^d, \mathcal{V})$

Recall that $\mathcal{V}=\{V_i\}_{i=1,\dots,d}$ is a covering of a $(2d-1)$-dimensional sphere ∂D^d. Thus we have a spectral sequence associated to this covering by Lemma 2.1:

$$E^2_{p,q}(\partial D^d, \mathcal{V}) = H^\delta_p\Big(\bigoplus_{i_0<\dots<i_p} H_q(V_{i_0\dots i_p}, \mathbb{Z})\Big) \Rightarrow H_{p+q}(\partial D^d, \mathbb{Z}). \tag{23}$$

Lemma 5.1 *$E^2_{p,q}(\partial D^d), \mathcal{V})=0$ unless $(p, q)=(0, 0)$ or $(d-1, d)$. In particular, the spectral sequence* (23) *degenerates at E^2.*

Proof. We will prove this claim by induction on d. For $d=2$, the claim follows from the Mayer-Vietoris sequence:

$$0=H_1(S^3)\leftarrow H_1(V_1)\oplus H_1(V_2)\leftarrow H_1(V_{12})\leftarrow H_2(S^3)=0.$$

Assume that the claim is true for $d-1$ and then we will prove the claim for d. First of all, it is clear that δ induces an isomorphism

$$0\leftarrow \bigoplus_i H_{d-1}(V_{1\dots\hat{i}\dots d})\leftarrow H_{d-1}(V_{1\dots d})\leftarrow 0.$$

This proves for the case $q=d-1$. The claim is also true for $p=d-1$, since the map $\delta\colon \bigoplus_i H_q(V_{1\dots\hat{i}\dots d})\leftarrow H_q(V_{1\dots d})$ is injective. Now for $2\leqq p\leqq d-2$ and $0\leqq q\leqq d-2$, the sequence $E^1_{\cdot q}\Big(\bigcup_{i=2}^d V_i, \mathcal{V}'\Big)$, $\mathcal{V}'=\{V_i\}_{i=2,\dots,d}$ naturally injects into $E^1_{\cdot q}(\partial D^d, \mathcal{V})$. Thus we have the following diagram of the sequences with exact columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\dots\leftarrow & \bigoplus_{2\leqq i_0<\dots<i_p} & H_q(V_{i_0\dots i_p}) & \leftarrow & \bigoplus_{2\leqq i_0<\dots<i_{p+1}} & H_q(V_{i_0\dots i_{p+1}})\leftarrow\dots & \\
 & & \downarrow & & \downarrow & & \\
\dots\leftarrow & \bigoplus_{1\leqq i_0<\dots<i_p} & H_q(V_{i_0\dots i_p}) & \leftarrow & \bigoplus_{1\leqq i_0<\dots<i_{p+1}} & H_q(V_{i_0\dots i_{p+1}}\leftarrow\dots & \\
 & & \downarrow & & \downarrow & & \\
\dots\leftarrow & \bigoplus_{2\leqq i_0<\dots<i_{p-1}} & H_q(V_{1\,i_0\dots i_{p-1}})\leftarrow & & \bigoplus_{2\leqq i_0<\dots<i_p} & H_q(V_{1\,i_0\dots i_p})\leftarrow\dots & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0. & &
\end{array}$$

Here the top row is $E^1_{\cdot q}\Big(\bigcup_{i=2}^d V_i, \mathcal{V}'\Big)$ and the middle row is $E^1_{\cdot q}(\partial D^d, \mathcal{V})$. We now prove the exactness of the middle row by showing that of the top and

the bottom rows. Since we can contract the first coordinate to the origin in the open sets V_i for $2 \leqq i \leqq d$, the top row is exact by the induction hypothesis. Since $V_{1\, i_0 \ldots i_{p-1}}$ is homotopic to $V_1 \times V_{i_0 \ldots i_{p-1}}$, the bottom row for $1 \leqq q \leqq d-2$ is naturally isomorphic to

$$H_1(V_1) \otimes E^1_{\cdot q-1}\left(\bigcup_{i=2}^{d} V_i, \mathscr{V}'\right) \oplus E^1_{\cdot q}\left(\bigcup_{i=2}^{d} V_i, \mathscr{V}'\right).$$

Hence it is exact by the induction hypothesis again. For $q=0$, note that the inclusion $V_{1\, i_0 \ldots i_{p-1}} \hookrightarrow V_{i_0 \ldots i_{p-1}}$ induces the isomorphisms

$$H_0(V_{1\, i_0 \ldots i_{p-1}}) \cong H_0(V_{i_0 \ldots i_{p-1}}),$$

which is compatible with δ. Thus by the induction hypothesis, $E^2_{p,0}=0$ for $p \geqq 2$. So far, we have proved $E^2_{p,q}=0$ unless $p=0$, 1 or $(p, q)=(d-1, d)$. Hence $E^2_{p,q} = E^\infty_{p,q}$. This completes the proof since the abutting term $H_n(\partial D^d, \mathbb{Z})$ is equal to 0 unless $n=0$ or $2d-1$. □

6 The homology groups of the link

Recall that a topological space L is the link of the quasi-ordinary singularity (X, x) and $\mathscr{U}=\{U_i\}$ is an open covering of L. By Lemma 2.1, there is an induced spectral sequence:

$$E^2_{p,q}(L, \mathscr{U}) = H^\delta_p\Big(\bigoplus_{i_0 < \ldots < i_p} H_q(U_{i_0 \ldots i_p}, \mathbb{Z})\Big) \Rightarrow H_{p+q}(L, \mathbb{Z}). \tag{24}$$

Remark 2 Since $U_{i_0 \ldots i_p}$ is homotopic to a $(p+1)$-dimensional torus, $H_q(U_{i_0 \ldots i_p}, \mathbb{Z}) \neq 0$ only if $0 \leqq q \leqq p+1$. Therefore, we have $E^2_{p,q}(L, \mathscr{U}) \neq 0$ only if $0 \leqq p \leqq d-1$ and $0 \leqq q \leqq p+1$.

Theorem 6.1 *The homology groups $H_i(L, \mathbb{Z})$ of the link of a quasi-ordinary singularity (X, x) are torsion unless $i=2d-1$ or 0.*

Proof. Consider the following diagram of spectral sequences (cf. Lemma 2.2):

$$\begin{array}{ccc}
E^2_{p,q}(L, \mathscr{U}) \otimes \mathbb{Q} = H^\delta_p\Big(\bigoplus_{i_0 < \ldots < i_p} H_q(U_{i_0 \ldots i_p}, \mathbb{Z})\Big) \otimes \mathbb{Q} & \Rightarrow & H_{p+q}(L, \mathbb{Z}) \otimes \mathbb{Q} \\
\Big\downarrow \pi_* \otimes 1 & & \Big\downarrow \pi_* \otimes 1 \\
E^2_{p,q}(\partial D^d, \mathscr{V}) \otimes \mathbb{Q} = H^\delta_p\Big(\bigoplus_{i_0 < \ldots < i_p} H_q(V_{i_0 \ldots i_p}, \mathbb{Z})\Big) \otimes \mathbb{Q} & \Rightarrow & H_{p+q}(\partial D^d, \mathbb{Z}) \otimes \mathbb{Q}.
\end{array}$$

By Lemma 4.2,

$$\pi_* \otimes 1 \colon H_q(U_{i_0 \ldots i_p}, \mathbb{Z}) \otimes \mathbb{Q} \to H_q(V_{i_0 \ldots i_p}, \mathbb{Z}) \otimes \mathbb{Q}$$

is an isomorphism. Hence we have an induced isomorphism $\pi_* \otimes 1$: $H_i(L, \mathbb{Z}) \otimes \mathbb{Q} \to H_i(\partial D^d, \mathbb{Z}) \otimes \mathbb{Q}$. But $H_i(\partial D^d, \mathbb{Z})=0$ unless $i=2d-1$ or 0. □

Theorem 6.2 *Unless* $(p, q)=(0, 0)$ *or* $(d-1, d)$, *there is an isomorphism between the spectral sequence* (24) *and the Galois spectral sequence* (22) *with shift in bidegree* $(1, 0)$. *In other words, the vertical arrows are isomorphisms in the following commutative diagram:*

$$\begin{array}{ccc} E^r_{p,q}(L, \mathscr{U}) & \xrightarrow{d^r_E} & E^r_{p-r,q+r-1}(L, \mathscr{U}) \\ \downarrow & & \downarrow \\ G^r_{p+1,q} & \xrightarrow{d^r_G} & G^r_{p-r+1,q+r-1} \end{array}$$

where $r \geqq 2$ *and* $(p, q) \neq (0, 0), (d-1, d)$.

Proof. By Theorem 4.4 and Lemma 4.2, there is an exact sequence of complexes:

$$0 \to E^1_{\cdot q}(L, \mathscr{U}) \to E^1_{\cdot q}(\partial D^d, \mathscr{V}) \to G^1_{\cdot q} \to 0.$$

But by Lemma 5.1, $E^2_{p,q}(\partial D^d, \mathscr{V})=0$ unless $(p, q)=(0, 0)$ or $(d-1, d)$. The theorem follows from this. □

Corollary 6.3 *The alternating product* $\prod_{i=1}^{2d-2} |H_i(L, \mathbb{Z})|^{(-1)^{i+1}}$ *of the orders of singular homology groups of the link is equal to* $|G|/\prod_{i=1}^{d} |G_i|$.

Proof. Note that the alternating product of the order of terms of a bounded complex of finite groups is the same as that of its homology. Hence we have

$$\prod_{i=1}^{2d-2} |H_i(L, \mathbb{Z})|^{(-1)^{i+1}} = \prod_{p+q=1}^{2d-2} |E^\infty_{p,q}|^{(-1)^{p+q+1}} = \prod_{p+q=1}^{2d-2} |E^2_{p,q}|^{(-1)^{p+q+1}},$$

because $d^2_E: E^2_{p,q} \to E^2_{p-r,q+r-1}$ are the zero maps if $p+q=1$ or $2d-1$. Here $E^r_{p,q}$ denotes $E^r_{p,q}(L, \mathscr{U})$. On the other hand, we have

$$\begin{aligned} &\prod_{p+q=1}^{2d-2} |E^2_{p,q}|^{(-1)^{p+q+1}} \\ &= \prod_{q=1}^{d-1} |E^2_{0,q}|^{(-1)^{q+1}} \cdot \prod_{p=1}^{d-1} \prod_{q=0}^{d-1} |E^2_{p,q}|^{(-1)^{p+q+1}} \quad \text{(by Remark 2)} \\ &= \prod_{q=1}^{d-1} |G^2_{1,q}|^{(-1)^{q+1}} \cdot \prod_{p=1}^{d-1} \prod_{q=0}^{d-1} |G^2_{p+1,q}|^{(-1)^{p+q+1}} \quad \text{(by Theorem 6.2)} \\ &= \prod_{p=-1}^{d-2} \prod_{q=1}^{d-1} |G^1_{p+1,q}|^{(-1)^{p+q+1}} \quad \text{(since } G^1_{d,q} = G^1_{p,0} = 0\text{)} \end{aligned}$$

$$= \prod_{q=1}^{d-1} |G^1_{d-1,q}|^{(-1)^{d+q-1}} \cdot \prod_{q=-1}^{d-3} \prod_{q=1}^{d-2} |G^1_{p+1,q}|^{(-1)^{p+q+1}}$$

$$= \prod_{q=1}^{d-1} |\wedge^q G|^{(-1)^{d+q-1}} \cdot \left(\prod_{p=0}^{d-2} \left(\prod_{1 \leqq i_0 < \ldots < i_p \leqq d} \left(\prod_{q=1}^{p+1} |\wedge^q G_{i_0 \ldots i_p}|^{(-1)^{p+q}} \right) \right) \right)$$

(by Theorem 4.4)

$$= \prod_{q=1}^{d-1} |G|^{(-1)^{d+q-1}\binom{d-1}{q}} \cdot \left(\prod_{p=0}^{d-2} \left(\prod_{1 \leqq i_0 < \ldots < i_p \leqq d} \left(\prod_{q=1}^{p+1} |G_{i_0 \ldots i_p}|^{(-1)^{p+q}\binom{p}{q-1}} \right) \right) \right)$$

(by Remark 1)

$$= |G| / \prod_{i=1}^{d} |G_i|.$$

For the last equality, we used the following combinatorial equality:

$$\sum_{q=0}^{p} (-1)^q \binom{p}{q} = 0 \quad \text{if } p \geqq 1. \quad \square$$

Lemma 6.4 *For $p+q \leqq 2$ or $p+q \geqq 2d-5$, we have*

$$E^2_{p,q}(L, \mathscr{U}) = E^\infty_{p,q}(L, \mathscr{U}).$$

Proof. This follows from Lemma 4.3, Theorem 6.2 and Remark 2. $\square$

Corollary 6.5 *Let L be the link of a quasi-ordinary singularity (X, x) with a parametrization $\zeta = H(t_1^{1/n}, \ldots, t_d^{1/n})$. Then we have*

(1) $H_{2d-1}(L, \mathbb{Z}) = \mathbb{Z}$, $H_{2d-2}(L, \mathbb{Z}) = 0$.
(2) *The following sequences are exact:*

$$(25) \qquad 0 \leftarrow \bigoplus_{i=1}^{d} \wedge^{d-1} G_{12\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{d-1} G \leftarrow H_{2d-3}(L, \mathbb{Z}) \leftarrow 0$$

$$\bigoplus_{i=1}^{d} \wedge^{d-2} G_{12\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{d-2} G \leftarrow H_{2d-4}(L, \mathbb{Z}) \leftarrow 0$$

where $12 \ldots \hat{i} \ldots d$ means omitting i from the index $12 \ldots d$.
(3) *The homology group $H_{2d-5}(L, \mathbb{Z})$ is equal to a direct sum of the homology group of the following short sequence at the middle term*

$$\bigoplus_{1 \leqq i < j \leqq d} \wedge^{d-2} G_{1\ldots\hat{i}\ldots\hat{j}\ldots d} \xleftarrow{\delta} \bigoplus_{i=1}^{d} \wedge^{d-2} G_{12\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{d-2} G,$$

and the kernel of the following map

$$\bigoplus_{i=1}^{d} \wedge^{d-3} G_{12\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{d-3} G.$$

(4) $H_2(L, \mathbb{Z})$ *(resp.* $H_1(L, \mathbb{Z})$*) is isomorphic to the homology group of the following sequence at the third (resp. second) term from the left*

$$\bigoplus_i G_i \xleftarrow{\delta} \bigoplus_{ij} G_{ij} \xleftarrow{\delta} \bigoplus_{ijk} G_{ijk} \xleftarrow{\delta} \bigoplus_{ijkl} G_{ijkl}. \tag{26}$$

(5) $H_0(L, \mathbb{Z}) = \mathbb{Z}$.

Proof. All these follow from Theorem 6.2, Lemma 6.4 and Lemma 4.3. □

Lemma 6.6 *Assume that* $G_{1\ldots\hat{i}\ldots\hat{j}\ldots d} = 0$ *for all* $1 \leqq i < j \leqq d$. *Then the spectral sequence* $E^2_{p,q}(L, \mathcal{U}) \Rightarrow H_{p+q}(L, \mathbb{Z})$ *degenerates at* E^2. *Hence, we have*

(1) $H_{2d-1}(L, \mathbb{Z}) = H_0(L, \mathbb{Z}) = \mathbb{Z}$, $H_{2d-2}(L, \mathbb{Z}) = 0$.
(2) *The following sequences are exact:*

$$0 \leftarrow \bigoplus_{i=1}^{d} \wedge^{d-1} G_{12\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{d-1} G \leftarrow H_{2d-3}(L, \mathbb{Z}) \leftarrow 0,$$

$$\bigoplus_{i=1}^{d} \wedge^{d-2} G_{12\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{d-2} G \leftarrow H_{2d-4}(L, \mathbb{Z}) \leftarrow 0.$$

(3) *For* $d-1 \leqq n \leqq 2d-5$, $H_n(L, \mathbb{Z})$ *is isomorphic to the direct sum of the kernel of the map*

$$\bigoplus_{i=1}^{d} \wedge^{n-d+2} G_{1\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{n-d+2} G$$

and the cokernel of the map

$$\bigoplus_{i=1}^{d} \wedge^{n-d+3} G_{1\ldots\hat{i}\ldots d} \xleftarrow{\delta} \wedge^{n-d+3} G.$$

(4) $H_{d-2}(L, \mathbb{Z})$ *is isomorphic to the cokernel of the map*

$$\bigoplus_{i=1}^{d} G_{1\ldots\hat{i}\ldots d} \xleftarrow{\delta} G.$$

(5) *For* $1 \leqq n \leqq d-3$, *we have* $H_n(L, \mathbb{Z}) = 0$.

Proof. The assumption implies that $G_{i_0\ldots i_p} = 0$ for $0 \leqq p \leqq d-3$. Hence $E^2_{p,q} = 0$ unless $p = d-3$ or $d-2$. The lemma follows from this. □

The assumption in the above lemma is equivalent to requiring that the minimal characteristic monomial is missing at most one variable.

Corollary 6.7 *Assume that* $G_{1\ldots\hat{i}\ldots d}=0$ *for all* $i=1, \ldots, d$. *Then we have*

$$H_{2d-1}(L, \mathbb{Z})=H_0(L, \mathbb{Z})=\mathbb{Z},$$
$$H_n(L, \mathbb{Z})=\wedge^{n-d+2}G \quad \text{for } d-1\leqq n\leqq 2d-3,$$
$$H_n(L, \mathbb{Z})=0 \quad \text{otherwise.}$$

7 Presentation of the Galois groups and H_{2d-3}

Let $(X, x)\subset(\mathbb{C}^{d+1}, 0)$ be a quasi-ordinary singularity with a parametrization $\zeta=H(t_1^{1/n}, \ldots, t_d^{1/n})$ and L be its link. Let Γ be the additive subgroup of $((1/n)\mathbb{Z})^d$ consisting of all d-tuples $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_d)$ such that $t^\alpha=t_1^{\alpha_1}t_2^{\alpha_2}\ldots t_d^{\alpha_d}\in\mathscr{L}(\zeta)$, where $\mathscr{L}$ is the fractional field of $\mathbb{C}\{t_1, \ldots, t_d\}$. By Lemma 3.1, we have

$$\Gamma=\mathbb{Z}^d+\sum\mathbb{Z}(\lambda_{1i}, \ldots, \lambda_{di}), \tag{27}$$

where the sum is over all characteristic exponent vectors $(\lambda_{1i}, \ldots, \lambda_{di})$ of ζ. Hence Γ is generated by characteristic exponent vectors and the standard unit vectors $(1, 0, 0, \ldots), (0, 1, 0, \ldots), \ldots, (\ldots, 0, 0, 1)$. We introduce a matrix whose rows generate Γ.

Definition 7.1 The matrix $\Lambda(\zeta)$, whose rows consist of the characteristic exponent vectors and the standard unit vectors, is called the *characteristic marix.*

By performing only elementary row operations, we may assume that $\Lambda(\zeta)$ is a $d\times d$ triangular matrix after discarding rows consisting of only 0 entries if necessary. Such a $\Lambda(\zeta)$ is called a *triangulated characteristic matrix.* Note that the monomials corresponding to the row vectors of the triangulated characteristic matrix $\Lambda(\zeta)$ also generate $\mathscr{L}(\zeta)$ over $\mathscr{L}$. In other words, we have

$$\mathscr{L}(\zeta)=\mathscr{L}(M_1, M_2, \ldots, M_d)$$

where $M_i=t_1^{\alpha_{i1}}t_2^{\alpha_{i2}}\ldots t_d^{\alpha_{id}}$ and $(\alpha_{ij})=\Lambda(\zeta)$. We set

$$m_i=|G|/|G_{1\ldots\hat{i}\ldots d}|=[\mathscr{L}(\zeta)\colon \mathscr{L}]/[\mathscr{L}(\zeta^{(i)})\colon \mathscr{L}], \tag{28}$$

where $\zeta^{(i)}=\zeta(t_1^{1/n}, \ldots, t_{i-1}^{1/n}, 0, t_{i+1}^{1/n}, \ldots, t_d^{1/n})$. In this section, we will show that $H_{2d-3}(L, \mathbb{Z})$ calculated via the exact sequence (25) has the same presentation as the following Lipman's.

Theorem 7.1 [L1, Theorem 4.1 and (5.9.1)] *Let* $\Lambda(\zeta)$ *be a triangulated characteristic matrix and let* M *be a* $d\times d$ *diagonal matrix with diagonal entries* $m_1, \ldots, m_d$ *as defined by* (28). *Then the group* $H_{2d-3}(L, \mathbb{Z})$ *is presented by the matrix* $\Lambda(\zeta)M$. *In other words,* $H_{2d-3}(L, \mathbb{Z})$ *is isomorphic to the cokernel of the following map*

$$0\to\mathbb{Z}^d\xrightarrow{M\Lambda(\zeta)^{-1}}\mathbb{Z}^d. \tag{29}$$

First we give a presentation of the Galois group $G=\mathrm{Gal}(\mathscr{L}(\zeta)/\mathscr{L})$.

Lemma 7.2 *Let $\Lambda(\zeta)$ be a triangulated characteristic matrix. Then the Galois group $G=\mathrm{Gal}(\mathscr{L}(\zeta)/\mathscr{L})$ is presented by the matrix $(\Lambda(\zeta)^{-1})^t$. In other words, there is an exact sequence*

$$0\to\mathbb{Z}^d\xrightarrow{\Lambda(\zeta)^{-1}}\mathbb{Z}^d\to G\to 0. \tag{30}$$

Proof. Let

$$\mathscr{L}\subset\mathscr{L}(\zeta)\subset\mathscr{L}_n$$

be the respective fractional fields of

$$\mathbb{C}\{t_1,\ldots,t_d\}\subset\mathbb{C}\{t_1,\ldots t_d\}[\zeta]\subset\mathbb{C}\{t_1^{1/n},\ldots,t_d^{1/n}\}.$$

Observe that the Galois group $\mathrm{Gal}(\mathscr{L}_n/\mathscr{L})$ is generated by the elements g_i, $i=1,\ldots,d$, which acts on $t_j^{1/n}$ by

$$g_i(t_j^{1/n})=\exp\left(\frac{2\pi\sqrt{-1}}{n}\delta_{ij}\right)t_j^{1/n}$$

where $\delta_{ij}=1$, if $i=j$, and 0 otherwise. Thus the Galois group $G=\mathrm{Gal}(\mathscr{L}(\zeta)/\mathscr{L})$ is generated by the elements g_i, $i=1,\ldots,d$, which acts on the generators $t_1^{\alpha_1}\ldots t_d^{\alpha_d}$ of $\mathscr{L}(\zeta)$ over $\mathscr{L}$ by

$$g_i(t_1^{\alpha_1}\ldots t_d^{\alpha_d})=\exp(2\pi\sqrt{-1}\,\alpha_i)\,t_1^{\alpha_1}\ldots t_d^{\alpha_d}.$$

Thus G acts on the row vectors of the triangulated characteristic matrix $\Lambda(\zeta)$ and the column vectors of the inverse matrix $A:=(\Lambda(\zeta))^{-1}$ give the relations between the generators g_i of G. It remains to see that these generate all possible relations among g_i. After the column operations on $\Lambda(\zeta)$ (which correspond to the change of variables t_i), we may assume that $\Lambda(\zeta)$ is diagonalized. The diagonal entries of the diagonalized matrix $\Lambda(\zeta)$ are necessarily of the form $1/\lambda$ with $\lambda\in\mathbb{Z}$ since the standard unit vectors $(1,0,0,\ldots)$, $(0,1,0,\ldots),\ldots,(\ldots,0,0,1)$ are contained in the group Γ, which is generated by the row vectors of $\Lambda(\zeta)$. Now it is clear that A generates all possible relations among g_i. □

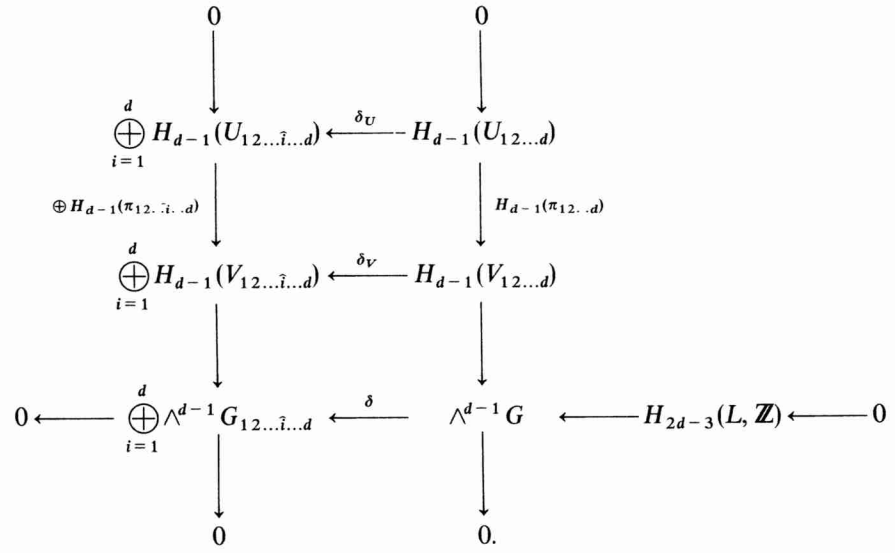
Now we prove that our computation of $H_{2d-3}(L,\mathbb{Z})$ agrees with Lipman's.

Theorem 7.3 *The group $H_{2d-3}(L,\mathbb{Z})$ calculated via the exact sequence* (25):

$$0\leftarrow\bigoplus_{i=1}^{d}\wedge^{d-1}G_{12\ldots\hat{i}\ldots d}\xleftarrow{\delta}\wedge^{d-1}G\leftarrow H_{2d-3}(L,\mathbb{Z})\leftarrow 0$$

is also presented by the matrix $\Lambda(\zeta)\,M$.

Proof. Observe that the sequence (25) is a part of the commutative diagram:



In this diagram, we use the same notation as in § 4. Note that the columns are exact and δ_V is an isomorphism as proved in Lemma 5.1. Hence $H_{2d-3}(L, \mathbb{Z})$ is isomorphic to the cokernel of the map δ_U. It remains to show that δ_U can be represented by the matrix $M\Lambda^t$ with respect to suitable bases of $H_{d-1}(U_{12\ldots d})$ and $H_{d-1}(V_{12\ldots d})$ (cf. Theorem 7.1).

First we fix bases of various free abelian groups in the diagram. We set

$$f_i = \{\varepsilon_1\} \times \ldots \times \{\varepsilon_{i-1}\} \times \partial D_i \times \{\varepsilon_{i+1}\} \times \ldots \times \{\varepsilon_d\}$$

for $i = 1, \ldots, d$. Then the cycles f_i form a basis of the group $H_1(V_{12\ldots d})$. Now as a basis of $H_{d-1}(V_{12\ldots d})$ we choose the ordered set

$$\{(-1)^{i+1} f_{12\ldots\hat{i}\ldots d}\}_{i=1,\ldots,d}$$

where

$$f_{12\ldots\hat{i}\ldots d} = f_1 \wedge f_2 \wedge \ldots \wedge f_{i-1} \wedge f_{i+1} \wedge \ldots \wedge f_d.$$

For a given basis $\{e_1, \ldots, e_d\}$ of $H_1(U_{12\ldots d})$ (which will be chosen later), we choose for the basis of $H_{d-1}(U_{12\ldots d})$ the ordered set

$$\{(-1)^{i+1} e_{12\ldots\hat{i}\ldots d}\}_{i=1,\ldots,d}$$

where

$$e_{12\ldots\hat{i}\ldots d} = e_1 \wedge e_2 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_d.$$

Finally, as a basis of $H_{d-1}(U_{12\ldots\hat{i}\ldots d})$ (resp. $H_{d-1}(V_{12\ldots\hat{i}\ldots d})$) we choose the fundamental class $[U_{12\ldots\hat{i}\ldots d}]$ (resp. $[V_{12\ldots\hat{i}\ldots d}]$).

Now we will give matrix representations of the maps $H_{d-1}(\pi_{12\ldots d})$, δ_V, and $\oplus H_{d-1}(\pi_{12\ldots\hat{i}\ldots d})$ with respect to these bases to get the desired matrix presentation of the map δ_U. By Remark 1, the map $H_{d-1}(\pi_{12\ldots d})$ is represented by the

adjoint matrix $\mathrm{Ad}\, A^t$ of the transpose of A, where A is the matrix representing the following map:

$$0 \longrightarrow H_1(U_{12\ldots d}) \xrightarrow{H_1(\pi_{12\ldots d})} H_1(V_{12\ldots d}).$$

By the correspondence between the covering group and the Galois group, the group $H_1(V_{12\ldots d}) = \pi_1(V_{12\ldots d})$ acts on the field $\mathscr{L}(\zeta)$ by

$$f_i(t_1^{\alpha_1} \ldots t_d^{\alpha_d}) = \exp(2\pi\sqrt{-1}\,\alpha_i)\, t_1^{\alpha_1} \ldots t_d^{\alpha_d},$$

where $t_1^{\alpha_1} \ldots t_d^{\alpha_d} \in \mathscr{L}(\zeta)$ (cf. Theorem 1.7.1 in [O]). Hence, with respect to a suitable basis $\{e_1, \ldots, e_d\}$ of $H_1(U_{12\ldots d})$, A can be identified with the inverse of the triangulated characteristic matrix $\Lambda(\zeta)^{-1}$ by Lemma 7.2 (without changing the basis $\{f_1, \ldots, f_d\}$ of $H_1(V_{i\ldots d})$). Thus the matrix representing $H_1(\pi_{12\ldots d})$ is $\det(\Lambda(\zeta)^{-1})\, \Lambda(\zeta)^t = |G|\, \Lambda(\zeta)^t$. Since $\delta_V((-1)^{i+1} f_{12\ldots\hat{i}\ldots d}) = [V_{12\ldots\hat{i}\ldots d}]$, the identity matrix represents the map δ_V with respect to the bases chosen as above. Finally, the map $H_1(\pi_{12\ldots\hat{i}\ldots d})$ is given by the multiplication by $|G_{12\ldots\hat{i}\ldots d}| = |G|/m_i$ by Remark 1. Thus we can conclude that δ_U is represented by the matrix $M\Lambda(\zeta)^t$ with respect to the bases chosen as above. □

8 Examples

Definition 8.1 Let

$$f_i = \{\varepsilon_1\} \times \ldots \times \{\varepsilon_{i-1}\} \times \partial D_i \times \{\varepsilon_{i+1}\} \times \ldots \times \{\varepsilon_d\}.$$

Then the ordered set $\{f_{i_0}, \ldots, f_{i_p}\}$ forms a basis of $H_1(V_{i_0\ldots i_p})$, called the *canonical basis*. The matrix presenting the group $G_{i_0\ldots i_p}$ with respect to the canonical basis is called a *canonical matrix*, which will be denoted by $A(G_{i_0\ldots i_p})$.

Example 1 Let ζ be the quasi-ordinary branch

$$\zeta = w^{3/2} x^{1/3} y^{3/4} z + w^{13/6} x^{2/3} y^{11/12} z^{6/5}$$

and let $\mathscr{L}$ be the fractional field of $\mathbb{C}\{w, x, y, z\}$. The characteristic matrix $\Lambda(\zeta)$ is given by

$$\begin{pmatrix} 3/2 & 1/3 & 3/4 & 1 \\ 13/6 & 2/3 & 11/12 & 6/5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

After elementary row and column operations, $\Lambda(\zeta)$ can be written as a 4×4 diagonal matrix with diagonal entries (1/12, 1/5, 1/3, 1). Hence the Galois group $G = \mathrm{Gal}(\mathscr{L}(\zeta)/\mathscr{L})$ is isomorphic to

$$G \cong \mathbb{Z}/3 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/12 = \mathbb{Z}/3 \oplus \mathbb{Z}/2^2 \cdot 3 \cdot 5 \tag{32}$$

and its canonical presentation matrix is the 6×6 diagonal matrix with diagonal entries (12, 5, 3, 1) after renaming the variables. Since $G_{1\ldots\hat{i}\ldots 6}=0$ for $i=1, \ldots, 6$, by Corollary 6.7, the homology groups of the link L of the quasi-ordinary singularity parametrized by ζ are given by

$$
\begin{aligned}
&H_7(L,\mathbb{Z})=\mathbb{Z},\\
&H_6(L,\mathbb{Z})=0,\\
&H_5(L,\mathbb{Z})=\wedge^3 G=\mathbb{Z}/3\cdot 5\oplus\mathbb{Z}/3\cdot 12\oplus\mathbb{Z}/5\cdot 12\oplus\mathbb{Z}/3\cdot 5\cdot 12\\
&\qquad=(\mathbb{Z}/3)^{\oplus 3}\oplus(\mathbb{Z}/2^2\cdot 3\cdot 5)^{\oplus 3},\\
&H_4(L,\mathbb{Z})=\wedge^2 G=\mathbb{Z}/3\oplus\mathbb{Z}/5\oplus\mathbb{Z}/12\oplus\mathbb{Z}/3\cdot 5\oplus\mathbb{Z}/3\cdot 12\oplus\mathbb{Z}/5\cdot 12\\
&\qquad=(\mathbb{Z}/3)^{\oplus 3}\oplus(\mathbb{Z}/2^2\cdot 3\cdot 5)^{\oplus 3},\\
&H_3(L,\mathbb{Z})=G=\mathbb{Z}/3\oplus\mathbb{Z}/2^2\cdot 3\cdot 5,\\
&H_2(L,\mathbb{Z})=H_1(L,\mathbb{Z})=0,\\
&H_0(L,\mathbb{Z})=\mathbb{Z}.
\end{aligned}
$$

Thus we have $|H_5|=2^6\cdot 3^6\cdot 5^3$, $|H_4|=2^6\cdot 3^6\cdot 5^3$, and $|H_3|=2^2\cdot 3^2\cdot 5$. Hence the alternating product of the order of the group H_i is equal to

$$\prod_{i=1}^{6}|H_i(L,\mathbb{Z})|^{(-1)^{i+1}}=|G|=2^2\cdot 3^2\cdot 5$$

as expected by Corollary 6.3.

Example 2 [L1, Example 7.5] Let L be the link of the quasi-ordinary singularity parametrized by

$$\zeta(x^{1/8}, y^{1/6}, z^{1/8})=x^{3/2}y^{2/3}+x^{7/4}y^{7/6}+x^{9/4}y^{9/6}z^{1/4}+x^{19/8}y^2z^{11/8},$$

and let $\mathscr{L}$ be the fractional field of the convergent power series ring $\mathbb{C}\{x, y, z\}$. The characteristic matrix $\Lambda(\zeta)$ is given by

$$\Lambda(\zeta)=\begin{pmatrix}3/2 & 2/3 & 0\\ 7/4 & 7/6 & 0\\ 9/4 & 9/6 & 1/4\\ 19/8 & 2 & 11/8\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{33}$$

By elementary row operations, $\Lambda(\zeta)$ can be triangulated as follows:

$$\Lambda(\zeta)=\begin{pmatrix}1/8 & 0 & 1/8\\ 0 & 1/6 & 0\\ 0 & 0 & 1/4\end{pmatrix}. \tag{34}$$

Thus a canonical matrix $A(G)$ of the Galois group $G=\mathrm{Gal}(\mathscr{L}(\zeta)/\mathscr{L})$ is given by

$$A(G)=\begin{pmatrix}8 & 0 & -4\\ 0 & 6 & 0\\ 0 & 0 & 4\end{pmatrix}. \tag{35}$$

Thus

$$G\cong\mathbb{Z}/2\oplus\mathbb{Z}/4\oplus\mathbb{Z}/24. \tag{36}$$

As they are calculated in [L1, Example 7.5], we have

$$(m_x, m_y, m_z)=(192, 192, 16), \tag{37}$$

$$H_3(L,\mathbb{Z})=\mathbb{Z}/2\oplus\mathbb{Z}/16\oplus\mathbb{Z}/96. \tag{38}$$

In order to calculate $H_2(L,\mathbb{Z})$ and $H_1(L,\mathbb{Z})$, we need to know the canonical presentation matrix for G_{ij}, $1\leqq i<j\leqq 3$ and G_i, $i=1, 2, 3$. Since $\zeta(0, y^{1/6}, z^{1/8})=\zeta(x^{1/8}, 0, z^{1/8})=0$, the canonical matrices $A(G_{23})$ and $A(G_{13})$ are the 2×2 identity matrices. Since we have

$$\zeta_{12}=\zeta(x^{1/8}, y^{1/6}, 0)=x^{3/2}y^{2/3}+x^{7/4}y^{7/6}, \tag{39}$$

the characteristic matrix $\Lambda(\zeta_{12})$ is equal to

$$\Lambda(\zeta_{12})=\begin{pmatrix}3/2 & 2/3\\ 7/4 & 7/6\\ 1 & 0\\ 0 & 1\end{pmatrix}. \tag{40}$$

After performing elementary row operations, this matrix is equivalent to

$$\begin{pmatrix}1/4 & -1/6\\ 0 & 1/3\end{pmatrix}. \tag{41}$$

Thus a canonical matrix for G_{12} is given by

$$A(G_{12})=\begin{pmatrix}4 & 2\\ 0 & 3\end{pmatrix}. \tag{42}$$

Since $\zeta_1=\zeta_2=\zeta_3=0$, we have $G_1=G_2=G_3=0$ and $A(G_1)=A(G_2)=(A(G_3)=1$.

Now, by the sequence (26), $H_2(L, \mathbb{Z})$ (resp. $H_1(L, \mathbb{Z})$) is the kernel (resp. cokernel) of the bottom row in the following diagram.

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
H_1(U_1)\oplus H_1(U_2)\oplus H_1(U_3) & \leftarrow & H_1(U_{23})\oplus H_1(U_{13})\oplus H_1(U_{12}) & \leftarrow & H_1(U_{123}) & & & & \\
\downarrow & & \downarrow{\scriptstyle \oplus H_1(\pi_{ij})} & & \downarrow{\scriptstyle H_1(\pi_{123})} & & & & \\
0 \leftarrow H_1(V_1)\oplus H_1(V_2)\oplus H_1(V_3) & \leftarrow & H_1(V_{23})\oplus H_1(V_{13})\oplus H_1(V_{12}) & \overset{\delta}{\leftarrow} & H_1(V_{123}) & \leftarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & \leftarrow & G_{23}\oplus G_{13}\oplus G_{12} & \overset{\delta}{\leftarrow} & G_{123} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

In this diagram, all columns and the middle row are exact. With respect to the canonical bases of $H_1(V_{ij})$ and $H_1(V_{123})$, we have $H_1(\pi_{123})=A(G)$ and $H_1(\pi_{ij})=A(G_{ij})$. Moreover the map δ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{43}$$

It is easy to see that $\bar{\delta}$ is surjective. Thus we have

$$H_1(L, \mathbb{Z})=0. \tag{44}$$

Finally a tedious computation through the above diagram shows that

$$H_2(L, \mathbb{Z})=\mathbb{Z}/4\oplus\mathbb{Z}/4. \tag{45}$$

Hence we get

$$\frac{|H_3|\cdot|H_1|}{|H_4|\cdot|H_2|}=|G|=192. \tag{46}$$

Thus this agrees with Corollary 6.3.

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