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Titel: A Harnack inequality for solutions of difference differential equations of ellipt...

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Jahr: 1993

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0213|log31

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Math. Z. 213, 393–423 (1993)

Mathematische Zeitschrift
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A Harnack inequality for solutions of difference differential equations of elliptic-parabolic type

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Received 12 July 1991; in final form 16 December 1991

1 Introduction

In treating the regularity problem for solutions of nonlinear elliptic and parabolic equations, we need to consider the corresponding linear equations with only measurable coefficients. Hölder estimates for bounded weak solutions of equations have been obtained in the papers [8–12]. It was shown by Moser that a Harnack inequality holds for solutions of elliptic and parabolic equations with bounded and measurable coefficients [10, 11].

In this paper we shall study difference-differential equations of elliptic-parabolic type with bounded and measurable coefficients. It is our aim to derive a Harnack inequality for solutions of such equations uniformly with respect to approximations. Previously, such problems had been studied by Kikuchi [6], who treated Hölder estimates for equations independently of approximations. Depending on the relation between the size of a local cube and a time-discrete mesh, the equations show a "parabolic" or "elliptic" behavior, respectively. We also think that the estimates in this paper will be fundamental and useful for time-discrete approximations of evolution equations (see [7]) and will play an essential role in constructing Morse flows for certain functionals in the calculus of variations (refer to [1, 5]).

Let Ω be a bounded open set in Euclidean space R^m, $m \geqq 2$, u be a function: $\Omega \to R$ and $Du = (D_1 u, D_2 u, \ldots, D_m u)$, $D_\alpha u = \partial u / \partial x^\alpha$ $(1 \leqq \alpha \leqq m)$ be the gradient of u. Let T be a positive number arbitrarily given and set $Q = (0, T) \times \Omega$. We use the usual Lebesgue space $L_p(\Omega)$, Sobolev spaces; $W_p^k(\Omega) = W_p^k(\Omega, R)$, $W_{p,0}^k(\Omega) = W_{p,0}^k(\Omega, R)$, $V_2(Q) = L^\infty((0, T);\ L^2(\Omega)) \cap L^2((0, T);\ W_2^1(\Omega))$ and $V_{2,0}(Q) = L^\infty((0, T);\ L^2(\Omega)) \cap L^2((0, T);\ W_{2,0}^1(\Omega))$.

For a positive integer N, $N \geqq 2$, we put $h = T/N$ and $t_n = nh$ $(0 \leqq n \leqq N)$. Let u_0 be a function belonging to $W_2^1(\Omega)$. We shall be concerned with a family of linear elliptic partial differential equations:

$$(1.1) \qquad \frac{u_n - u_{n-1}}{h} = D_\alpha(a_n^{\alpha\beta}(x) D_\beta u_n) \qquad (1 \leqq n \leqq N).$$

In the summation convention over repeated ones, the Greek indices run from 1 to m. The coefficients $a_n^{\alpha\beta}(\cdot)$ $(1 \leqq \alpha, \beta \leqq m,\ 1 \leqq n \leqq N)$ are measurable functions defined in Ω satisfying the uniform ellipticity and boundedness condition with positive constants λ and μ:

(1.2) $\lambda|\xi|^2 \leqq a_n^{\alpha\beta}(x)\xi^\alpha\xi^\beta \leqq \mu|\xi|^2$ for $\xi=(\xi^\alpha)\in R^m$, $1\leqq n\leqq N$ and any $x\in\Omega$.

Consider a family of weak solutions of (1.1) with initial data u_0, that is, a family $\{u_n\}$ $(1\leqq n\leqq N)$ of functions $u_n\in W_2^1(\Omega)$ which satisfy

(1.3) $$\int_\Omega \frac{u_n-u_{n-1}}{h}\varphi\,dx+\int_\Omega a_n^{\alpha\beta}D_\beta u_n D_\alpha\varphi\,dx=0 \quad \text{for any } \varphi=(\varphi^i)\in W_{2,0}^1(\Omega).$$

For such a family $\{u_n\}(1\leqq n\leqq N)$, define a function $u_h(t,\cdot)$: $t\in[0,T]\to u_h(t,\cdot)\in W_2^1(\Omega)$ as follows:

(1.4) $$u_h(0,\cdot)=u_0(\cdot),$$
$$u_h(t,\cdot)=u_n(\cdot) \quad \text{for } t_{n-1}<t\leqq t_n \quad (1\leqq n\leqq N).$$

Also define

(1.5) $$a^{\alpha\beta}(t,\cdot)=a_n^{\alpha\beta}(\cdot), \quad \text{for } t_{n-1}<t\leqq t_n \quad (1\leqq n\leqq N).$$

We deduce from (1.3) and the definitions (1.4), (1.5) that u_h satisfies the identity

(1.6) $$\int_\Omega \frac{u_h(t,\cdot)-u_h(t-h,\cdot)}{h}\varphi(\cdot)\,dx+\int_\Omega a^{\alpha\beta}(t,\cdot)D_\beta u_h(t,\cdot)D_\alpha\varphi(\cdot)\,dx=0$$

for any $\varphi=(\varphi^i)\in W_{2,0}^1(\Omega)$ and all $t\in(0,T]$. (1.4) understood, we use the notations u_h and $\{u_n\}$ interchangeably.

Some standard notations: For a point $z_0=(t_0,x_0)\in Q$, we put

(1.7) $$B_r(x_0)=\{x\in R^m: |x^\alpha-x_0^\alpha|<r(1\leqq\alpha\leqq m)\},$$
$$C_{r,\tau}(z_0)=\{t\in R: |t-t_0|<\tau\}\times B_r(x_0),$$
$$C_{r,\tau}^+(z_0)=\{t\in R: t_0-\tau<t<t_0\}\times B_r(x_0),$$
$$C_{r,\tau}^-(z_0)=\{t\in R: t_0<t<t_0+\tau\}\times B_r(x_0).$$

These domains are referred to as cubes. For simplicity, we shall use abbreviations:

$$C_r(z_0)=C_{r,r^2}(z_0), \quad C_r^+(z_0)=C_{r,r^2}^+(z_0), \quad C_r^-(z_0)=C_{r,r^2}^-(z_0).$$

In the above notations, the centre x_0 and z_0 will be omitted when no confusion may arise. For $z_i=(t_i,x_i)$ $(i=1,2)$, we introduce the parabolic metric

(1.8) $$\delta(z_1,z_2)=\max\{|t_1-t_2|^{1/2}, |x_1^\alpha-x_2^\alpha|(1\leqq\alpha\leqq m)\}.$$

For a measurable set A in R^k, we denote the k-dimensional measure of A by $|A|$ and for an integrable function f on A, we put

$$\bar{f}_A=\frac{1}{|A|}\int_A f(z)\,dz.$$

For a positive number l we denote by $[l]$ the greatest non-negative integer not greater than l. We also set

(1.9) $$\tilde{n}_l = \text{the greatest non-negative integer less than } l^2/h.$$

The same letter γ will be used to denote different constants depending on the same parameters.

Now let N_0 be a positive integer satisfying

$$N_0 > \log(1+m/2)/\log(1+2/m)$$

and let h_0 be a sufficiently small positive number. From now on, we take N in (1.1) sufficiently large satisfying

$$N \geqq \max\{N_0, T/h_0\}.$$

We also define a cube $\tilde{Q}_{h_0}$ as follows:

$$\tilde{\Omega}_{h_0} = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \sqrt{(N_0+1)h_0}\}, \qquad \tilde{Q}_{h_0} = ((N_0+1)h_0, T) \times \tilde{\Omega}_{h_0}.$$

Now we shall describe our main results:

Theorem 1.1 (Weak Harnack inequality, parabolic regime) *Let u_h be a weak solution of* (1.1). *If u_h is nonnegative in a cube $C_r^+(t_{n_0}, x_0) \subset Q$ with $r^2 > h$, then, for any $0 < p < 1 + 2/m$, there exists a positive constant γ depending only on λ, μ, m and p such that*

(1.10) $$\left(\frac{1}{|C^-_{\sqrt{\tilde{n}_r h}/2}|} \iint_{C^-_{\sqrt{\tilde{n}_r h}/2}(t_{n_0} - \tilde{n}_r, x_0)} (u_h)^p\, dx\, dt\right)^{1/p} \leqq \gamma \inf_{C^+_{\sqrt{\tilde{n}_r h}/2}(t_{n_0}, x_0)} u_h .$$

Theorem 1.2 (Weak Harnack inequality, elliptic regime) *Let u_h be a weak solution of* (1.1) *satisfying*

$$\iint_Q (u_h)^2\, dx\, dt \leqq \gamma_1$$

with a uniform constant γ_1. If $u_n \geqq 0$ $(N_0+1 \leqq n \leqq N)$ in $B_{2r}(x_0) \subset \tilde{\Omega}_{h_0}$ with $r^2 \leqq h$, then, for any $0 < p < m/(m-2)$, there exist positive constants γ and α, $0 < \alpha < 1$, depending only on λ, μ, m, γ_1 and $\operatorname{dist}(x_0, \partial\Omega)$ such that, for the same n, $N_0+1 \leqq n \leqq N$ as above, there holds

(1.11) $$\left(\frac{1}{|B_{r/2}|} \int_{B_{r/2}(x_0)} (u_n)^p\, dx\right)^{1/p} \leqq \gamma \big[\inf_{B_r(x_0)} u_n + r^\alpha\big].$$

Theorem 1.3 (Local boundedness of solutions) *Let u_h be a weak solution of* (1.1) *satisfying*

$$\iint_Q (u_h)^2\, dx\, dt \leqq \gamma_1$$

with a uniform positive number γ_1. *Then, for all* $(\bar{t}, \bar{x}) \in \tilde{Q}_{h_0}$ *with* $d = (1/4)\min\{|\bar{t} - N_0 h_0|^{1/2}, \operatorname{dist}(\bar{x}, \partial\Omega)\}$ *and any* $p > 1$, *there exist positive constants* γ *and* α, $0 < \alpha < 1$, *depending only on* λ, μ, γ_1, p *and* d *such that, setting* $u_h^{\pm} = \max\{\pm u_h, 0\}$,

$$\sup_{C^+_{r/2}(t_{n_0}, x_0)} u_h^{\pm} \leqq \gamma \left[\left(\frac{1}{|C_r^+|} \iint_{C_r^+(t_{n_0}, x_0)} (u_h^{\pm})^p \, dx \, dt \right)^{1/p} + r^{\alpha} \right] \tag{1.12}$$

holds for any $(t_{n_0}, x_0) \in C^+_{d/2}(\bar{t}, \bar{x})$ *and all* $0 < r < d/2$.

We emphasize that the above theorems hold uniformly with respect to h and u_h.

This paper is arranged as follows: In Sect. 2 we shall derive the so-called Caccioppoli inequality for $u_h^{p/2} (p \neq -1)$. Here we use a special cut-off function with respect to time-variable t, which was introduced in the paper [6]. Section 3 is devoted to obtaining an estimate for $\sup u_h$. It seems impossible to obtain the boundedness of solutions of (1.1) only by Moser's iteration. Instead we exploit De Girgi's iterative technique. In Sect. 4 we give a variation of John-Nirenberg estimate for a time-step function and estimate $\log u_h$, the most important and difficult step in our proof. In Sect. 5 we shall prove Theorem 1.1, 1.2 and 1.3 and also obtain Hölder estimates for weak solutions of (1.1).

2 Estimates for u^p

In this section we derive a Caccioppoli inequality analogous to Moser [11].

Lemma 2.1 *Let* u_h *be a weak solution of* (1.1) *and let us take* $C^-_{\rho,\tau}(t_{n_0}, x_0)$, $C^+_{\rho,\tau}(t_{n_0}, x_0) \subset Q$ *arbitrarily. Then there exists a positive constant* γ *depending only on* λ, μ *and* m *such that, if* u_h *is nonnegative in* $C^+_{\rho,\tau}(t_{n_0}, x_0)$ *and* $u_{n_0 - [\tau/h] - 1} \geqq 0$ *in* $B_\rho(x_0)$, *then*

$$\begin{aligned} &\sup_{t_{n_0} - \tau(1-\sigma_2) \leqq t \leqq t_{n_0}} \int_{B_{\rho(1-\sigma_1)}(x_0)} (u_h + \varepsilon)^p(t, \cdot) \, dx \\ &\quad + \iint_{C^+_{\rho(1-\sigma_1), \tau(1-\sigma_2)}(t_{n_0}, x_0)} |D(u_h + \varepsilon)^{p/2}|^2 \, dx \, dt \\ &\leqq \gamma((\sigma_1 \rho)^{-2} + (\sigma_2 \tau)^{-1}) \iint_{C^+_{\rho,\tau}(t_{n_0}, x_0)} (u_h + \varepsilon)^p \, dx \, dt \end{aligned} \tag{2.1}$$

holds for any $p < 0$, *all* σ_1, $\sigma_2 \in (0, 1)$ *and any* $\varepsilon > 0$. *If* u_h *is nonnegative in* $C^-_{\rho,\tau}(t_{n_0}, x_0)$ *and* $u_{n_0} \geqq 0$ *in* $B_\rho(x_0)$,

$$\begin{aligned} &\sup_{t_{n_0} \leqq t \leqq t_{n_0} + \tau(1-\sigma_2)} \int_{B_{\rho(1-\sigma_1)}(x_0)} (u_h + \varepsilon)^p(t, \cdot) \, dx \\ &\quad + \iint_{C^-_{\rho(1-\sigma_1), \tau(1-\sigma_2)}(t_{n_0}, x_0)} |D(u_h + \varepsilon)^{p/2}|^2 \, dx \, dt \\ &\leqq \gamma((\sigma_1 \rho)^{-2} + (\sigma_2 \tau)^{-1}) \iint_{C^-_{\rho,\tau}(t_{n_0}, x_0)} (u_h + \varepsilon)^p \, dx \, dt \end{aligned} \tag{2.2}$$

holds for any $0 < p < 1$, *all* σ_1, $\sigma_2 \in (0, 1)$ *and any* $\varepsilon > 0$.

Remark. For $p = 0$, the above estimates are trivial.

Proof. In the arguments we omit writing a center point of $B_\rho(x_0)$ or vertex of $C^+_{\rho,\tau}(t_{n_0}, x_0)$.

We demonstrate only the proof of (2.1). Let $\eta \in C_0^\infty(B_\rho)$ be a cut-off function such that $0 \leqq \eta \leqq 1$, $\eta = 1$ on $B_{\rho(1-\sigma_1)}$ and $|D\eta| \leqq 2(\sigma_1 \rho)^{-1}$. Also we take some appropriate cut-off function $\sigma(t)$ defined on $[t_{n_0} - \tau, t_{n_0}]$, the definition of which is given later. We remark that, since $u_h(t,\cdot)$ is nonnegative in $C^+_{\rho,\tau}$, $(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\sigma(t)$, $p<0$, $\varepsilon>0$, is admissible as a test function in the identity (1.6) in $C^+_{\rho,\tau}$. Integrating the resulting inequality with respect to t in $(t_{n_0}-\tau, t_{n_0}]$, we have

$$\iint_{C^+_{\rho,\tau}} \frac{u_h(t,\cdot) - u_h(t-h,\cdot)}{h} (u_h(t,\cdot)+\varepsilon)^{p-1} \eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$+ \iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot) D_\beta u_h(t,\cdot) D_\alpha[(u_h(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)]\,\sigma(t)\,dx\,dt = 0.$$

Put $v(t,\cdot) = u_h(t,\cdot)+\varepsilon$, then the above becomes

$$\text{(2.3)} \qquad \iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)-v(t-h,\cdot)}{h} (v(t,\cdot))^{p-1}\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$+ \iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot) D_\beta v(t,\cdot) D_\alpha[(v(t,\cdot))^{p-1}\eta^2(\cdot)]\,\sigma(t)\,dx\,dt = 0.$$

Now we separately estimate each term in (2.3). To do so, we need to distinguish two cases.

Case 1 $\sigma_2 \tau > 3h$. Then we take $\sigma(t)$ as follows (see [6] or [7]):

$$\sigma(t) = \sigma_n \quad \text{for } t_{n-1} < t \leqq t_n \ (1 \leqq n \leqq N)$$

$$\text{(2.4)} \quad \sigma_n = \begin{cases} 1 & \text{for } n_0 - [(1-\sigma_2)\tau/h] \leqq n \leqq n_0, \\ \dfrac{n - n_0 + [\tau/h] - 1}{[\tau/h] - 1 - [(1-\sigma_2)\tau/h]} & \text{for } n_0 - [\tau/h] + 1 \leqq n \leqq n_0 - [(1-\sigma_2)\tau/h], \\ 0 & \text{for } n \leqq n_0 - [\tau/h]. \end{cases}$$

We first estimate the quotient term of (2.3). Using Young's inequality and noting that $p<1$ and $p \neq 0$, we have

$$(v(t,\cdot) - v(t-h,\cdot))(v(t,\cdot))^{p-1} \leqq (v^p(t,\cdot) - v^p(t-h,\cdot))/p,$$

so that

$$\iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)-v(t-h,\cdot)}{h} (v(t,\cdot))^{p-1}\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$\leqq \iint_{C^+_{\rho,\tau}} \frac{v^p(t,\cdot) - v^p(t-h,\cdot)}{ph}\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt.$$

By the definition of u and σ, the latter equals to

$$\frac{1}{p}\sum_{n=n_0-[(1-\sigma_2)\tau/h]+1}^{n_0}\int_{B_\rho}(v_n^p-v_{n-1}^p)\,\eta^2\,dx \tag{2.5}$$

$$+\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}(v_n^p-v_{n-1}^p)\,\sigma_n\,\eta^2\,dx$$

$$=\frac{1}{p}\int_{B_\rho}v_{n_0}^p\,\eta^2\,dx-\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\,\eta^2\,dx$$

$$+\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}(v_n^p-v_{n-1}^p)\,\sigma_n\,\eta^2\,dx.$$

Since $\sigma_{n_0-[\tau/h]+1}=0$, we have,

$$\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}(v_n^p-v_{n-1}^p)\,\sigma_n\,\eta^2\,dx$$

$$=\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}\int_{B_\rho}(v_n^p\,\sigma_n-v_{n-1}^p\,\sigma_{n-1})\,\eta^2\,dx$$

$$-\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}(\sigma_n-\sigma_{n-1})\int_{B_\rho}v_{n-1}^p\,\eta^2\,dx$$

$$=\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\,\eta^2\,dx-\frac{1}{p}\sum_{n=n_0-[\tau/h]+2}^{n=n_0-[(1-\sigma_2)\tau/h]}(\sigma_n-\sigma_{n-1})\int_{B_\rho}v_{n-1}^p\,\eta^2\,dx.$$

Noting that $\sigma_n-\sigma_{n-1}\leqq 3h/(\sigma_2\tau)$ and $p<0$, the latter is

$$\leqq\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\,\eta^2\,dx-\frac{3}{p}(\sigma_2\tau)^{-1}h\sum_{n=n_0-[\tau/h]+1}^{n=n_0-[(1-\sigma_2)\tau/h]-1}\int_{B_\rho}v_n^p\,\eta^2\,dx \tag{2.6}$$

$$\leqq\frac{1}{p}\int_{B_\rho}v_{n_0-[(1-\sigma_2)\tau/h]}^p\,\eta^2\,dx-\frac{3}{p}(\sigma_2\tau)^{-1}\int_{t_{n_0}-\tau}^{t_{n_0}}\int_{B_\rho}v^p\,\eta^2\,dx\,dt.$$

Substituting (2.6) into (2.5) gives that

$$\iint_{C_{\rho,\tau}^+}\frac{v(t,\cdot)-v(t-h,\cdot)}{h}(v(t,\cdot))^{p-1}\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt \tag{2.7}$$

$$\leqq\frac{1}{p}\int_{B_\rho}v_{n_0}^p\,\eta^2\,dx-\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C_{\rho,\tau}^+}v^p\,\eta^2\,dx\,dt.$$

Next we shall deal with the term including spatial derivatives.

Noting that $p<1$ and using Young's inequality with $\varepsilon=-\lambda(p-1)/\mu|p|>0$, we have

$$(2.8)\quad \iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)\, D_\beta v(t,\cdot)\, D_\alpha[(v(t,\cdot))^{p-1}\eta^2(\cdot)]\,\sigma(t)\,dx\,dt$$

$$=\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta} D_\beta v^{p/2} D_\alpha v^{p/2}\eta^2\sigma\,dx\,dt$$

$$+\frac{4}{p}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta} D_\beta v^{p/2} v^{p/2}\eta D_\alpha\eta\sigma\,dx\,dt$$

$$\leqq\frac{4\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} |Dv^{p/2}|^2\eta^2\sigma\,dx\,dt+\frac{4}{p}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}D_\beta v^{p/2}v^{p/2}\eta D_\alpha\eta\sigma\,dx\,dt$$

$$\leqq\left(\frac{4(p-1)}{p^2}\lambda+\frac{2\varepsilon}{|p|}\mu\right)\iint_{C^+_{\rho,\tau}} |Dv^{p/2}|^2\eta^2\sigma\,dx\,dt$$

$$+\frac{2\mu}{|p|\varepsilon}\iint_{C^+_{\rho,\tau}}(v^{p/2})^2|D\eta|^2\sigma\,dx\,dt$$

$$\leqq\frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}|Dv^{p/2}|^2\eta^2\sigma\,dx\,dt-\frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma\,dx\,dt.$$

Combining (2.8) with (2.7) gives that

$$\frac{1}{p}\int_{B_\rho} v^p_{n_0}\eta^2\,dx-\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2\,dx\,dt$$

$$+\frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}|Dv^{p/2}|^2\eta^2\sigma\,dx\,dt-\frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^2|D\eta|^2\sigma\,dx\,dt\geqq 0.$$

From this inequality, it follows that

$$(2.9)\quad \frac{1}{p}\int_{B_\rho} v^p_{n_0}\eta^2\,dx\geqq\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2\,dx\,dt+\frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma\,dx\,dt,$$

$$(2.10)\quad \frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}|Dv^{p/2}|^2\eta^2\sigma\,dx\,dt$$

$$\geqq\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2\,dx\,dt+\frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma\,dx\,dt.$$

Dividing the both sides of (2.9) and (2.10) by $1/p<0$ and $2\lambda(p-1)/p^2<0$ respectively, we obtain

$$(2.11)\quad \int_{B_\rho} v^p_{n_0}\eta^2\,dx\leqq 3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\eta^2\,dx\,dt+\frac{2\mu^2 p}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma\,dx\,dt,$$

$$(2.12)\quad \iint_{C^+_{\rho,\tau}}|Dv^{p/2}|^2\eta^2\sigma\,dx\,dt$$

$$\leqq\frac{3p(\sigma_2\tau)^{-1}}{2\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p\eta^2\,dx\,dt+\left(\frac{\mu p}{\lambda(p-1)}\right)^2\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\sigma\,dx\,dt.$$

Noting that $p-1<p<0$, (2.11) and (2.12) become, respectively,

$$\int_{B_\rho} v_{n_0}^p \eta^2\, dx \leqq \max\left(3, \frac{8\mu^2}{\lambda}\right)((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}) \iint_{C^+_{\rho,\tau}} v^p\, dx\, dt, \tag{2.13}$$

$$\iint_{C^+_{\rho,\tau}} |D v^{p/2}|^2 \eta^2 \sigma\, dx\, dt \leqq \max\left(\frac{3}{2\lambda}, \frac{4\mu^2}{\lambda}\right)((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}) \iint_{C^+_{\rho,\tau}} v^p\, dx\, dt. \tag{2.14}$$

Estimating similarly as in (2.13), for $n_0-[(1-\sigma_2)\tau/h]\leqq n\leqq n_0$ we obtain

$$\int_{B_\rho} v_n^p \eta^2\, dx \leqq \max\left(3, \frac{8\mu^2}{\lambda}\right)((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}) \iint_{C^+_{\rho,\tau}} v^p\, dx\, dt. \tag{2.15}$$

Thus we have

$$\begin{aligned}\sup_{t_{n_0}-(1-\sigma_2)\tau\leqq t\leqq t_{n_0}} &\int_{B_\rho} v^p(t,\cdot)\,\eta^2(\cdot)\, dx \\ &\leqq \max\left(3, \frac{8\mu^2}{\lambda}\right)((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}) \iint_{C^+_{\rho,\tau}} v^p\, dx\, dt.\end{aligned} \tag{2.16}$$

Next we shall consider the *Case 2* $\sigma_2\tau\leqq 3h$. Then we put $\sigma(t)$ as $\sigma\equiv 1$ on $[t_{n_0}-\tau, t_{n_0}]$, so that we have (2.3) with $\sigma\equiv 1$. Let's remark that since u_h is nonnegative in $C^+_{\rho,\tau}$ and $u_{n_0-[\tau/h]-1}\geqq 0$ in B_ρ, $v=u_h+\varepsilon$ is also nonnegative in $C^+_{\rho,\tau}$ and $v_{n_0-[\tau/h]-1}=u_{n_0-[\tau/h]-1}+\varepsilon\geqq 0$ in B_ρ. Thus

$$\begin{aligned}&\iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)-v(t-h,\cdot)}{h}(v(t,\cdot))^{p-1}\eta^2(\cdot)\,\sigma(t)\, dx\, dt \\ &= \iint_{C^+_{\rho,\tau}} \frac{v^p(t,\cdot)-v^{p-1}(t,\cdot)\,v(t-h,\cdot)}{h}\eta^2(\cdot)\, dx\, dt \leqq \frac{1}{h}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\,\eta^2(\cdot)\, dx,\end{aligned}$$

so that, with $\varepsilon=-\lambda(p-1)/\mu|p|(>0)$ and $\sigma_2\tau<3h$, from (2.3), (2.8) we obtain

$$\begin{aligned}&\frac{3}{\sigma_2\tau}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\,\eta^2(\cdot)\, dx\, dt+\frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} |D v^{p/2}|^2\eta^2\, dx\, dt \\ &+\frac{2\mu^2}{\lambda(1-p)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\, dx\, dt\geqq 0.\end{aligned}$$

That is, we have

$$\begin{aligned}&\frac{2\lambda(1-p)}{p^2}\iint_{C^+_{\rho,\tau}} |D v^{p/2}|^2\eta^2\, dx\, dt \\ &\leqq \frac{3}{\sigma_2\tau}\iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\,\eta^2(\cdot)\, dx\, dt+\frac{2\mu^2}{\lambda(1-p)}\iint_{C^+_{\rho,\tau}} v^p|D\eta|^2\, dx\, dt.\end{aligned} \tag{2.17}$$

Dividing the both side of (2.17) by $2\lambda(1-p)/p^2(>0)$, we have

(2.18)

$$\iint_{C^+_{\rho,\tau}} |D v^{p/2}|^2 \eta^2 \, dx\, dt \leqq \max\left(\frac{3}{2\lambda}, \frac{4\mu^2}{\lambda^2}\right)((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}) \iint_{C^+_{\rho,\tau}} v^p(t,\cdot)\, dx\, dt.$$

Moreover, trivially, for $\sigma_2\tau \leqq 3h$ and $n_0-[(1-\sigma_2)\tau/h] \leqq n \leqq n_0$, there holds

$$\int_{B_{\rho(1-\sigma_1)}} v_n^p \, dx \leqq 3(\sigma_2\tau)^{-1} \iint_{C^+_{\rho,\tau}} v^p \, dx\, dt. \tag{2.19}$$

The proof of (2.2) is similar and can be omitted.

Lemma 2.2 *Let u_h be a weak solution of (1.1). If $u_h \geqq 0$ in $C^+_{\rho_0,\tau_0}(t_{n_0}, x_0) \subset Q$ and $u_{n_0-[\tau_0/h]-1} \geqq 0$ in $B_{\rho_0}(x_0)$, then there exists a positive constant γ depending only on λ, μ and m such that*

$$\left(\frac{1}{|C^+_{\rho_0,\tau_0}|} \iint_{C^+_{\rho_0,\tau_0}(t_{n_0},x_0)} u_h^p(t,x)\, dt\, dx\right)^{1/p} \leqq \gamma(2+\rho_0^{-2}\tau_0)^{-p^{-1}(2+m/2)} \inf_{(t,x)\in C^+_{\rho_0/2,\tau_0/2}(t_{n_0},x_0)} u_h(t,x) \tag{2.20}$$

holds for any $p<0$. If $u_h \geqq 0$ in $C^-_{\rho_0,\tau_0}(t_{n_0}, x_0) \subset Q$ and $u_{n_0} \geqq 0$ in $B_{\rho_0}(x_0)$, then, for any $0<p<1+2/m$, there exists a positive constant γ depending only on λ, μ, m and p such that

$$\left(\frac{1}{|C^-_{\rho_0/2,\tau_0/2}|} \iint_{C^-_{\rho_0/2,\tau_0/2}(t_{n_0},x_0)} u_h^p(t,x)\, dt\, dx\right)^{1/p} \leqq \gamma\left(\frac{1}{|C^-_{\rho_0,\tau_0}|} \iint_{C^-_{\rho_0,\tau_0}(t_{n_0},x_0)} u_h^q(t,x)\, dt\, dx\right)^{1/q} \tag{2.21}$$

holds for any $0<q<p$.

Proof. We proceed by iteration as in [9, pp. 105–110]. We here remark only the following. Making a changing of variables:

$$\begin{cases} x-x_0=\rho_0 y, \\ t-t_{n_0}=\rho_0^2 s \end{cases} \tag{2.22}$$

and putting

$$\tilde{u}_h(s,y)=u_h(t_{n_0}+\rho_0^2 s, x_0+\rho_0 y),$$

we find that, for any s, $-\tau_0 \leqq \rho_0^2 s \leqq 0$, and all $\varphi=(\varphi^i) \in W^1_{2,0}(B_1)$ $\tilde{u}_h$ satisfies

$$\int_{B_1} \frac{\tilde{u}_h(s,\cdot)-\tilde{u}_h(s-h/\rho_0^2,\cdot)}{h/\rho_0^2} \varphi\, dy + \int_{B_1} \tilde{a}^{\alpha\beta}(s,\cdot) D_\beta \tilde{u}_h(s,\cdot) D_\alpha \varphi\, dy = 0. \tag{2.23}$$

Thus, noticing that $\tilde{u}_{n_0-[\tau_0/h]-1}\geqq 0$ in B_1, from (2.1) it follows that

$$(2.24)\qquad \sup_{\tilde{\tau}(1-\sigma_2)\leqq t\leqq 0}\int_{B_{\rho(1-\sigma_1)}}(\tilde{u}_h+\varepsilon)^p(t,\cdot)\,dy + \iint_{C^+_{\tilde{\rho}(1-\sigma_1),\tilde{\rho}(1-\sigma_2)}}|D(\tilde{u}_h+\varepsilon)^{p/2}|^2\,dy\,ds \leqq \gamma((\sigma_1\tilde{\rho})^{-2}+(\sigma_2\tilde{\tau})^{-1})\iint_{C^+_{\tilde{\rho},\tilde{\tau}}}(\tilde{u}_h+\varepsilon)^2\,dy\,ds$$

holds for $0<\tilde{\rho}<1$, $0<\tilde{\tau}<\theta=\rho_0^{-2}\tau_0$, σ_1, $\sigma_2\in(0,1)$, all $p<0$ and for any $\varepsilon>0$.

Lemma 2.3 *Let u_h be a weak solution of* (1.1). *Then there exists a positive constant γ depending only on λ, μ and m such that, setting $u_h^{\pm}=\max\{\pm u_h, 0\}$,*

$$(2.25)\qquad \operatorname{Sup}_{t_{n_0}-\tau(1-\sigma_2)\leqq t\leqq t_{n_0}}\int_{B_{\rho(1-\sigma_1)}(x_0)}(u_h^{\pm})^p(t,\cdot)\,dx + \iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)}|D(u_h^{\pm})^{p/2}|^2\,dx\,dt$$
$$\leqq \gamma\frac{p}{p-1}\left(1+\frac{p}{p-1}\right)\{((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1})\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}(u_h^{\pm})^p\,dx\,dt + (\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)}|u_h|^p(t-h,\cdot)\,dx\,dt\}$$

holds for all $1<p\leqq m+2$, any $C^+_{\rho,\tau}(t_{n_0}, x_0)\subset\tilde{Q}_{h_0}$ and all σ_1, $\sigma_2\in(0,1)$.

Proof. Let $\eta\in C_0^\infty(B_\rho)$ satisfying $\eta=1$ on $B_{\rho(1-\sigma_1)}$, $|D\eta|\leqq 2/(\sigma_1\rho)$ and σ be some function defined on $[t_{n_0}-\tau, t_{n_0}]$, the definition of which is given later. At first, we consider the case $1<p\leqq 2$. Then we remark that $(u_h^{\pm}(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\,\sigma(t)$ belongs to $W^1_{2,0}(B_\rho)$ for any $\varepsilon>0$ and $t\in[t_{n_0}-\tau, t_{n_0}]$. Testing the identity (1.6) by the function $(u_h^{\pm}(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\,\sigma(t)$ and integrating the resulting equality with respect to t in $(t_{n_0}-\tau, t_{n_0}]$, we have

$$(2.26)\qquad \iint_{C^+_{\rho,\tau}}\frac{\pm u_h(t,\cdot)-\pm u_h(t-h,\cdot)}{h}(u_h^{\pm}(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)\,\sigma(t)\,dx\,dt + \iint_{C^+_{\rho,\tau}}a^{\alpha\beta}(t,\cdot)\,D_\beta(\pm u_h(t,\cdot))\,D_\alpha[(u_h^{\pm}(t,\cdot)+\varepsilon)^{p-1}\eta^2(\cdot)]\,\sigma(t)\,dx\,dt=0.$$

Now we put

$$v=u_h^{\pm}.$$

Noting that, in the set $\{\pm u_h>0\}$

$$\pm u_h(t,\cdot)-(\pm u_h(t-h,\cdot))\geqq v(t,\cdot)-v(t-h,\cdot),$$

we obtain from (2.26)

$$\iint_{C^+_{\rho,\tau}} \frac{v(t,\cdot)+\varepsilon-(v(t-h,\cdot)+\varepsilon)}{h}(v(t,\cdot)+\varepsilon)^{p-1}\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$+4(p-1)/p^2 \iint_{C^+_{\rho,\tau}} a^{\alpha\beta} D_\beta(v(t,\cdot)+\varepsilon)^{p/2} D_\alpha(v(t,\cdot)+\varepsilon)^{p/2}\,\eta^2\,\sigma\,dx\,dt$$
$$+4/p \iint_{C^+_{\rho,\tau}} a^{\alpha\beta} D_\beta(v(t,\cdot)+\varepsilon)^{p/2}(v(t,\cdot)+\varepsilon)^{p/2}\,\eta\,D_\alpha\eta\,\sigma\,dx\,dt$$
$$+\varepsilon^{p-1} \iint_{C^+_{\rho,\tau}\cap\{\pm u_h\leqq 0\}} \frac{\pm u_h(t,\cdot)}{h}\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt\leqq 0.$$

If $\sigma_2\tau>3h$, then we are able to proceed as in the case $p<0$ in the proof of Lemma 2.1. Taking $\sigma(t)$ as in (2.4) in the proof of Lemma 2.1, we conclude that, for $n_0-[(1-\sigma_2)\tau/h]\leqq n\leqq n_0$

$$\text{(2.27)}\qquad \int_{B_\rho}(v_n+\varepsilon)^p\,\eta^2\,dx+\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{\pm u_h\leqq 0\}}\frac{\pm u_h(t,\cdot)}{h}\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$\leqq \max\left(3,\frac{8\mu^2}{\lambda}\right)((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1})\iint_{C^+_{\rho,\tau}}(v+\varepsilon)^p\,dx\,dt$$

and that

$$\text{(2.28)}\qquad \iint_{C^+_{\rho,\tau}}|D(v+\varepsilon)^{p/2}|^2\,\eta^2\,\sigma\,dx\,dt+\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{\pm u_h\leqq 0\}}\frac{\pm u_h(t,\cdot)}{h}\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$\leqq \max\left(3,\frac{8\mu^2}{\lambda}\right)((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1})\iint_{C^+_{\rho,\tau}}(v+\varepsilon)^p\,dx\,dt.$$

If $\sigma_2\tau\leqq 3h$, we let $\sigma\equiv 1$ on $[t_{n_0}-\tau, t_{n_0}]$ and argue again as in the proof of (2.1). Using Young's inequality and noting that $h^{-1}\leqq 3(\sigma_2\tau)^{-1}$, we have, for the quotient term,

$$\iint_{C^+_{\rho,\tau}}\frac{\pm u_h(t,\cdot)+\varepsilon-(\pm u(t-h,\cdot)+\varepsilon)}{h}(u^\pm(t,\cdot)+\varepsilon)^{p-1}\,\eta^2(\cdot)\,dx\,dt$$
$$\geqq \frac{1}{p}\iint_{C^+_{\rho,\tau}}\frac{(v(t,\cdot)+\varepsilon)^p-|\pm u(t-h,\cdot)+\varepsilon|^p}{h}\,\eta^2(\cdot)\,dx\,dt$$
$$\geqq -\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}|v(t-h,\cdot)+\varepsilon|^p\,\eta^2(\cdot)\,dx\,dt.$$

Calculations similar to (2.28) give that

$$\text{(2.29)}\qquad \iint_{C^+_{\rho,\tau}}|D(v+\varepsilon)^{p/2}|^2\,\eta^2\,dx\,dt$$
$$+\frac{p^2}{2\lambda(p-1)}\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{\pm u_h\leqq 0\}}\frac{\pm u_h(t,\cdot)}{h}\,\eta^2(\cdot)\,dx\,dt$$
$$\leqq \frac{\mu^2p^2}{\lambda^2(p-1)^2}\iint_{C^+_{\rho,\tau}}(v+\varepsilon)^p\,|D\eta|^2\,dx\,dt$$
$$+\frac{3p(\sigma_2\tau)^{-1}}{2\lambda(p-1)}\iint_{C^+_{\rho,\tau}}|v(t-h,\cdot)+\varepsilon|^p\,\eta^2\,dx\,dt.$$

Also we remark that the calculation yielding (2.19) is justified in this case since $v+\varepsilon=u_h^{\pm}+\varepsilon\geqq 0$.

Finally letting ε tend to 0 in (2.27), (2.28) and (2.29), by Fatou's lemma, we obtain (2.25) for $1<p\leqq 2$.

Next we deal with the case $p>2$. Remark that $[(u_h^{\pm}(t,\cdot))^{(M)}]^{p-1}\,\eta^2(\cdot)\,\sigma(t)$ is admissible as a test function in the identity (1.1) for any $t\in[t_{n_0}-\tau,\,t_{n_0}]$ and $M>0$, where $v^{(M)}$ is defined as

$$v^{(M)}=\begin{cases} M & v\geqq M,\\ v & v<M,\end{cases}$$

and $\eta(\cdot)$ and σ are as above. Arguing as above, for $\sigma_2\tau>3h$ we obtain the following estimate for $t_{n_0}-(1-\sigma_2)\,\tau\leqq t\leqq t_{n_0}$

$$\begin{aligned}(2.30)\qquad &\int_{B_\rho(x_0)} v(t,\cdot)(v^{(M)})^{p-1}(t,\cdot)\,\eta^2\,dx\\ &+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}\,D_\beta(v^{(M)})^{p/2}\,D_\alpha(v^{(M)})^{p/2}\,\eta^2\,\sigma\,dx\,dt\\ &+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)\,D_\beta\, v(v^{(M)})^{p-1}\,\eta\,D_\alpha\eta\,\sigma\,dx\,dt\\ &\leqq 3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v(v^{(M)})^{p-1}\,\eta^2\,dx\,dt+\frac{p-1}{p}\int_{B_\rho(x_0)}(v^{(M)})^p(t,\cdot)\,\eta^2\,dx\\ &+\frac{3(p-1)}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}(v^{(M)})^p(t,\cdot)\,\eta^2(\cdot)\,dx\,dt.\end{aligned}$$

Similarily, if $\sigma_2\tau\leqq 3h$, we deduce that, for $t_{n_0}-\tau(1-\sigma_2)\leqq t\leqq t_{n_0}$,

$$\begin{aligned}(2.31)\qquad &\int_{B_\rho}(v^{(M)})^p(t,\cdot)\,\eta^2\,dx-3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}(v^{(M)})^p\,dx\,dt\\ &+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)\,D_\beta(v^{(M)})^{p/2}\,D_\alpha(v^{(M)})^{p/2}\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt\\ &+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)\,D_\beta\, v(v^{(M)})^{p-1}\,\eta\,D_\alpha\eta(\cdot)\,\sigma(t)\,dx\,dt\\ &-\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}|u_h|^p(t-h,\cdot)\,\eta^2(\cdot)\,dx\,dt\leqq 0,\end{aligned}$$

where note (2.19).

As a result we obtain that (2.30) and (2.31) are valid in a case of $\sigma_2\tau>3h$ and $\sigma_2\tau\leqq 3h$, respectively.

Now, since by Young's inequality

$$\begin{aligned}(2.32)\qquad &\Big|\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}\,D_\beta\, v(v^{(M)})^{p-1}\,\eta\,D_\alpha\eta\,dx\,dt\Big|\\ &\leqq\frac{1}{p}\mu\iint_{C^+_{\rho,\tau}}|D(v^{(M)})^{p/2}|^2\,\eta^2\,dx\,dt+\frac{1}{p}\mu\iint_{C^+_{\rho,\tau}}(v^{(M)})^p\,|D\eta|^2\,dx\,dt,\end{aligned}$$

we are able to pass M to the limit in (2.30) and (2.31) if $p=2$. From it, we obtain that, for any $t_{n_0}-(1-\sigma_2)\tau \leqq t \leqq t_{n_0}$,

$$\frac{1}{2}\int_{B_\rho} v^2(t,\cdot)\,\eta^2\,dx+\frac{\lambda}{2}\iint_{C^+_{\rho,\tau}} |Dv|^2\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt \tag{2.33}$$
$$\leqq 3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^2\,\eta^2\,dx\,dt+\frac{3}{2}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} |u_h|^2(t-h,\cdot)\,\eta(\cdot)\,dx\,dt$$
$$\cdot+\frac{2\mu^2}{\lambda}\iint_{C^+_{\rho,\tau}} v^2\,|D\eta|^2\,dx\,dt.$$

Then Sobolev's type inequality (see [9, p. 76]) implies that $v\in L^{2(1+2/m)}_{\mathrm{loc}}$. Noting (2.32) again, we can pass to the limit in (2.30) and (2.31) for $2<p\leqq 2(1+2/m)$ respectively. Repeating the above procedure inductively, we deduce that, for any $t_{n_0}-(1-\sigma_2)\tau\leqq t\leqq t_{n_0}$ and all $2<p\leqq m+2$,

$$\frac{1}{p}\int_{B_\rho} v^p(t,\cdot)\,\eta^2\,dx-3(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} v^p\,\eta^2\,dx\,dt \tag{2.34}$$
$$+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)\,D_\beta v^{p/2}\,D_\alpha v^{p/2}\,\eta^2\,\sigma\,dx\,dt$$
$$+2\iint_{C^+_{\rho,\tau}} a^{\alpha\beta}(t,\cdot)\,D_\beta v\,v^{p-1}\,\eta\,D_\alpha\eta\,\sigma\,dx\,dt$$
$$-\frac{3}{p}(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}} |u_h|^p(t-h,\cdot)\,\eta(\cdot)\,dx\,dt\leqq 0.$$

We conclude from (2.34) that

$$\frac{1}{p}\int_{B_\rho} v^p(t,\cdot)\,\eta^2\,dx+\frac{2\lambda(p-1)}{p^2}\iint_{C^+_{\rho,\tau}} |Dv^{p/2}|^2\,\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$\leqq\frac{3}{\sigma_2\tau}\iint_{C^+_{\rho,\tau}} v^p\,\eta^2\,dx\,dt+\frac{2\mu^2}{\lambda(p-1)}\iint_{C^+_{\rho,\tau}} v^p\,|D\eta|^2\,dx\,dt$$
$$+\frac{3}{p\sigma_2\tau}\iint_{C^+_{\rho,\tau}} |u_h|^p(t-h,\cdot)\,\eta(\cdot)\,dx\,dt$$

for any $C^+_{\rho,\tau}(t_{n_0},x_0)\subset\tilde{Q}_{h_0}$ and all $2<p\leqq m+2$, as claimed.

3 Bounds for weak solutions

Now we prove boundedness for weak solutions of (1.1). Recall the following Caccioppoli inequality analogous to De Giorgie (see [9]).

Lemma 3.1 *Let u_h be a weak solution of* (1.1). *Then, there exists a positive constant γ depending only on λ, μ and m such that, setting $v_h=\pm u_h$,*

$$\sup_{t_{n_0}-\tau(1-\sigma_2)\leqq t\leqq t_{n_0}} \int_{B_{\rho(1-\sigma_1)}(x_0)} [(v_h-k)^+]^p\,dx + \iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(t_{n_0},x_0)} |D[(v_h-k)^+]^{p/2}|^2\,dx\,dt$$
$$\leqq \gamma((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}) \iint_{C^-_{\rho,\tau}} [(v_h-k)^+]^p\,dx\,dt$$
$$+\frac{1}{p}(\sigma_2\tau)^{-1}\Big(\iint_{C^+_{\rho,\tau}(t_{n_0},x_0)} |v_h|^q\,dx\,dt\Big)^{p/q}\,|C^+_{\rho,\tau}(t_{n_0},x_0)\cap\{v_h>k\}|^{1-p/q} \tag{3.1}$$

holds for any $k\geqq 0$, σ_1, $\sigma_2\in(0,1)$, $C^+_{\rho,\tau}(t_{n_0},x_0)\subset Q$, all $p\leqq q\leqq(m+2)(1+2/m)$ and all $1<p\leqq 2$.

By Lemma 3.1 and an iterative procedure as in [9, p. 105] we obtain the boundedness of weak solutions of (1.1).

Lemma 3.2 *Let u_h be a weak solution of* (1.1). *Then, for any $1<p\leqq 2$, there exists a positive constant γ depending only on λ, μ, m and p such that, setting $v_h=\pm u_h$,*

$$\sup_{C^+_{\rho_0/2,\tau_0/2}(t_{n_0},x_0)} v_h \leqq \gamma\left\{\left(\frac{1}{|C^+_{\rho_0,\tau_0}|}\iint_{C^+_{\rho_0,\tau_0}(t_{n_0},x_0)} (v_h)^p\,dx\,dt\right)^{1/p}(1+\tau_0^{-\frac{1}{2}}\rho_0)^{1/p} + \left(\frac{1}{|C^+_{\rho_0,\tau_0}|}\iint_{C^+_{\rho_0,\tau_0}(t_{n_0},x_0)} (v_h)^q\,dx\,dt\right)^{1/q}\right\} \tag{3.2}$$

holds for any $C^+_{\rho_0,\tau_0}(t_{n_0},x_0)\subset\tilde{Q}_{h_0}$ and all $p\leqq q\leqq(m+2)(1+2/m)$.

4 Estimates for log u_h

Recall the John-Nirenberg estimate [4]:

Lemma 4.1 *Let v be an integrable function in a cube B_1 and assume that there is a positive constant κ such that, for every parallel subcube $B\subset B_1$, we have*

$$\frac{1}{|B|}\int_B |v-\bar{v}_B|\,dx\leqq\kappa.$$

Then there exist positive constants a and α depending only on m such that

$$|\{x\in B_1: |v-\bar{v}_{B_1}|\geqq\sigma\}|\leqq e^{\alpha a}e^{-\alpha\sigma\kappa^{-1}}|B_1| \tag{4.1}$$

holds for every $\sigma>0$.

In the following, let φ be the function defined by $\varphi(s)=\begin{cases}\sqrt{s}, & s>0,\\ 0, & s\leqq 0.\end{cases}$ The following lemma is a variation of the John-Nirenberg estimate for the step function in a time variable.

Lemma 4.2 *Let $v=v_h$ be a time-step function defined as in (1.4) and suppose that v satisfies the following relations in a cube $C_R^+(\bar{t},\bar{x})\subset Q$ with uniform positive constants γ and κ:*

$$\frac{1}{|C_r^+||C_r^-|}\iint_{(t',x')\in C_r^+}\iint_{(t,x)\in C_r^-}\varphi(v(t',x')-v(t,x))\,dx\,dt\,dx'\,dt'\leqq\gamma$$

for all pairs C_r^+ and C_r^- in $C_R^+(\bar{t},\bar{x})$ with $3r^2>h$ and

$$|\{x\in B_r(x_0): |v_n(x)-(\overline{v_n})_{B_r(x_0)}|>\sigma\}|\leqq e^{\alpha a}e^{-\alpha\sigma\kappa^{-1}}|B_r|$$

for any $r\leqq\sqrt{h/3}$, all $x_0\in B_R(\bar{x})$ and any $[(\bar{t}-R^2)/h]\leqq n\leqq[\bar{t}/h]+1$. Then there exists a positive constant ξ independent of h and v such that

$$\frac{1}{|C_{R,R^2/2}^+||C_{R,R^2/2}^-|}\iint_{(t',x')\in C_{R,R^2/2}^+(\bar{t},\bar{x})}\iint_{(t,x)\in C_{R,R^2/2}^-(\bar{t}-2R^2,\bar{x})}\cdot\Psi(v(t',x')-v(t,x))\,dx\,dt\,dx'\,dt'\leqq 1, \tag{4.2}$$

where $\Psi(s)=\gamma^{-1}e^{\xi s}$.

Proof. We proceed similarly as in the proof of the main lemma in [11] and use the notations in [11]. Now we show a variation of Lemma 4 in [11] for a step-function in a time variable:

Take $\varepsilon<\min\{[(\delta-\gamma)/8]^{1+m/2}/2,(1-\delta)/5\}$.

Suppose that there exist constants $s_1(\varepsilon)>0$ and φ_ν for each $C_\nu\subset C_0$, of which the height is greater than h, or which is included in a time-interval, such that

$$\frac{|\{(t,x)\in C_\nu^+: v(t,x)-\varphi_\nu>s\}|}{|C_\nu^+|}+\frac{|\{(t,x)\in C_\nu^-: \varphi_\nu-v(t,x)>s\}|}{|C_\nu^-|}<\varepsilon$$

holds for any $s\geqq s_1$ and all $C_\nu\subset C_0$ as above. Then there exists a positive absolute constant b such that

$$\frac{|\{(t,x)\in D_\nu^+: v(t,x)-\varphi_\nu>s\}|}{|D_\nu^+|}+\frac{|\{(t,x)\in D_\nu^-: \varphi_\nu-v(t,x)>s\}|}{|D_\nu^-|}<e^{-bs/s_1}$$

holds for any $s\geqq s_1$.

We need to change the way of division and selection of the region as follows: We continue the division and the selection of $\dot{D}_\nu^+$ in the same way as in [11] until the height of C_ν corresponding to D_ν^+ becomes equal to or smaller than h. After it, we produce C_ν by dividing a preceding region. Here, if C_ν intersects a lattice of a time-interval, divide the C_ν into two parts, one of which is above the lattice and another is below, and take each part as a new C_ν. Proceed with the selection of $\dot{C}_\nu$ as in the proof of Lemma 4.1 (see [4]). Now we prove

(6.10)–(6.12) in [11] for the above division. Put $\tilde{C}_\nu = \dot{C}_\nu$, of which the height is equal to or smaller than h. The union of the selected rectangles will be denoted by $\dot{D} = \cup \dot{D}_\nu^+ \cup (\cup \tilde{C}_\nu)$. For the proof of (6.10), the left inequality follows from the way of selection. Now note the following holds: If D_ν^+, for which the height of the corresponding C_ν becomes greater than h, has a predecessor $D_{\nu-1}^+$, or C_ν has a predecessor $C_{\nu-1}$, then

$$\varphi_\nu - \varphi_{\nu-1} \leqq 2.$$

The right one in (6.10) follows from the above and the fact that $\dot{D}_\nu^+$ and $\tilde{C}_\nu$ have predecessors $D_{\nu-1}^+$ and, $D_{\nu-1}^+$ or $C_{\nu-1}$, which were not selected so that $\varphi_{\nu-1} \leqq s_0$, respectively. The proof of (6.11) is the same as in [11]. For the proof of (6.12), we add the following argument to the considerations for (6.12) in [11]: Select, out of the $\dot{C}_\nu^-$ and $\tilde{C}_\nu$, a maximal set of nonoverlapping rectangles C_ν^{*-} and $\tilde{C}_\nu^*$ so that every $\dot{C}_\nu^-$ or $\tilde{C}_\nu$ overlaps with some C_λ^{*-} (with $\lambda \leqq \nu$). Here note that $\tilde{C}_\nu$ are disjoint. Fixing C_λ^{*-}, we find from a similar geometrical consideration as in [11] that the $\tilde{C}_\nu$, which overlap with C_λ^{*-}, lie in a rectangle with the same upper or lower base as C_λ^{*-} and of the height is less than $3/(1-\gamma)$ times that of C_λ^{*-}, so that the measure of $(\cup \tilde{C}_\nu)$ is at most $6/(1-\gamma)$ times that of the union of the nonoverlappings C_λ^{*-}. As a result we have (6.12) in [11] with

$$c = 9/(1-\varepsilon)(1-\delta).$$

Lastly, for the deduction of the hypothesis of the above lemma from the hypothesis of the Lemma 4.2, we note only that v_h is a step function in each time-interval. The remainder is completely similar to the proof of the main lemma in [11].

Now we are ready to state the fundamental estimate for $-\log u_n$ $(1 \leqq n \leqq N)$.

Lemma 4.3 *Let u_h be a weak solution of* (1.1) *and us take $B_{2r}(x_0) \subset \Omega$ arbitrarily. Then there exists a positive constant γ independent of h and u_h such that, if u_n, u_{n-1} $(2 \leqq n \leqq N)$ is nonnegative in $B_{2r}(x_0)$ and $v_n = -\log u_n (1 \leqq n \leqq N)$,*

$$\frac{1}{|B_\rho|} \int_{B_\rho(y)} |v_n - \overline{(v_n)}_{B_\rho(y)}| \, dx \leqq \gamma (16\mu^2/\lambda^2 + 2\rho^2/\lambda h)^{1/2} \tag{4.3}$$

holds for any $\rho \leqq r/2$ and $y \in B_r(x_0)$.

Proof. We take $\rho \leqq r/2$, $y \in B_r(x_0)$ arbitrarily and fix them. Now, testing the identity (1.3) with $(u_n)^{-1}\eta^2$ for $\eta \in C_0^\infty(B_{2\rho})$, $\eta = 1$ on B_ρ and $|D\eta|^2 \leqq 4\rho^{-2}$, we have

$$\frac{1}{h} \int_{B_{2\rho}} \left(1 - \frac{u_{n-1}(x)}{u_n(x)}\right) \eta^2 \, dx - \int_{B_{2\rho}} a_n^{\alpha\beta} D_\beta \log u_n D_\alpha \log u_n \eta^2 \, dx + 2 \int_{B_{2\rho}} a_n^{\alpha\beta} D_\beta \log u_n \eta D_\alpha \eta \, dx = 0. \tag{4.4}$$

Noting the nonnegativity of $(1/h) \int_{B_{2\rho}} \{u_{n-1}(x)/u_n(x)\} \eta^2 \, dx$, we have

$$\begin{aligned} \lambda \int_{B_{2\rho}} |D \log u_n|^2 \eta^2 \, dx &\leqq \int_{B_{2\rho}} a_n^{\alpha\beta} D_\beta \log u_n D_\alpha \log u_n \eta^2 \, dx \\ &\leqq \varepsilon\mu \int_{B_{2\rho}} |D \log u_n|^2 \eta^2 \, dx + \frac{\mu}{\varepsilon} \int_{B_{2\rho}} |D\eta|^2 \, dx + \frac{1}{h} \int_{B_{2\rho}} \eta^2 \, dx. \end{aligned} \tag{4.5}$$

From $|D\eta|\leqq 2\rho^{-1}$ and taking $\varepsilon=\lambda/2\mu$ in (4.5), it follows that

$$\frac{1}{|B_{2\rho}|}\int_{B_{2\rho}} |D\log u_n|^2\eta^2\,dx\leqq\frac{2}{\lambda}\left(\frac{8\mu^2}{\lambda\rho^2}+\frac{1}{h}\right).$$

The Hölder and Poincarè inequalities give that

$$\frac{1}{|B_\rho|}\int_{B_\rho} |v_n-\overline{(v_n)}_{B_\rho}|\,dx\leqq\left(\frac{1}{|B_\rho|}\int_{B_\rho}|v_n-\overline{(v_n)}_{B_\rho}|^2\,dx\right)^{1/2}$$
$$\leqq|B_\rho|^{-1/2}\left\{\gamma\rho^2|B_{2\rho}|\frac{2}{\lambda}\left(\frac{8\mu^2}{\lambda\rho^2}+\frac{1}{h}\right)\right\}^{1/2}=\gamma\left\{\frac{2}{\lambda}\left(\frac{8\mu^2}{\lambda}+\frac{\rho^2}{h}\right)\right\}^{1/2}.$$

Therefore we have shown Lemma 4.3.

Remark. Strictly speaking, u_n^{-1} is not admissible as a test function in the identity (1.3). However, testing (1.3) with $(u_n+\varepsilon)^{-1}\eta^2$, calculating as above, and letting ε tend to 0 in the resulting inequality, we obtain (4.3).

Lemma 4.4 *Let u_h be a weak solution of* (1.1) *and us take $B_{2r}(x_0)\subset\Omega$ arbitrarily. Then there exist positive constants a and α depending only on m such that, if u_n, u_{n-1} $(2\leqq n\leqq N)$ is nonnegative in $B_{2r}(x_0)$ and $v_n=-\log u_n$ $(1\leqq n\leqq N)$,*

$$(4.6)\qquad |\{x\in B_r(x_0): |v_n(x)-\overline{(v_n)}_{B_r(x_0)}|>\sigma\}|\leqq e^{\alpha a}e^{-\alpha\sigma\kappa^{-1}}|B_r|.$$

hold, where $\kappa=\kappa(r)=\gamma(16\mu^2/\lambda^2+2r^2/\lambda h)^{1/2}$.

Proof. Immediate from Lemma 4.3 and 4.1.

The following will be shown later.

Lemma 4.5 *Let u_h be a weak solution of* (1.1). *Then there exists a positive constant γ independent of h and u_h such that, if u_h is nonnegative in $C_R^+(\bar{t},\bar{x})\subset Q$ and $u_{[(\bar{t}-R^2)/h]}\geqq 0$ in $B_R(\bar{x})$ then, setting $v=-\log u_h$,*

$$(4.7)\qquad \frac{1}{|C_r^+||C_r^-|}\iint_{(t',x')\in C_r^+}\iint_{(t,x)\in C_r^-}\varphi(v(t',x')-v(t,x))\,dt\,dx\,dt'\,dx'\leqq\gamma$$

holds for all pairs C_r^+ and C_r^- in $C_R^+(\bar{t},\bar{x})$ with $3r^2>h$.

Once we have Lemma 4.4 and 4.5, we conclude from adopting Lemma 4.2 for $v=-\log u_h$ in $C_R^+(\bar{t},\bar{x})$ the following:

Lemma 4.6 *Suppose that the same assumption as Lemma* 4.5 *is satisfied. Then there exist positive constants ξ and γ independent of h and u_h such that*

$$(4.8)\qquad \frac{1}{|C_{R,R^2/2}^+|}\iint_{C_{R,R^2/2}^+(\bar{t},\bar{x})}u^{-\xi}\,dt\,dx\,\frac{1}{|C_{R,R^2/2}^-|}\iint_{C_{R,R^2/2}^-(\bar{t}-2R^2,\bar{x})}u^{\xi}\,dt'\,dx'\leqq\gamma.$$

To prove Lemma 4.5, we need some preliminaries. We assume that $u_h\geqq 0$ in some cube $C_R^+(\bar{t},\bar{x})\subset Q$ (fixed in the sequel) and that $u_{[(\bar{t}-R^2)/h]}\geqq 0$ in $B_R(\bar{x})$. Define

$$v_n=-\log u_n,\quad v(t,\cdot)=v_n(\cdot)\quad (n-1)h<t\leqq nh\quad (1\leqq n\leqq N)\quad\text{on } B_R(\bar{x}).$$

Take n_0, $[(\bar{t}-R^2)/h]+1 \leqq n_0 \leqq [\bar{t}/h]$, arbitrarily and for any $\sigma \geqq 0$, any $B_{2r}(x_0) \subset B_R(\bar{x})$ and any $[(\bar{t}-R^2)/h] \leqq n \leqq [\bar{t}/h]+1$, let

$$\text{(4.9)}\quad \tilde{v}_n = v_n - \int_{B_{2r}(x_0)} v_{n_0}\eta^2\,dy \Big/ \int_{B_{2r}(x_0)} \eta^2\,dy \qquad \text{on } B_{2r}(x_0),$$

$$w_n = \tilde{v}_n - \gamma_2(n-n_0)\,h/r^2, \qquad \gamma_2 = 4\mu^2\,|B_2|/\lambda\,|B_1| \qquad \text{on } B_{2r}(x_0),$$

$$\tilde{v}(t,\cdot) = \tilde{v}_n, \qquad w(t,\cdot) = w_n(\cdot), \qquad (n-1)\,h < t \leqq n\,h,$$

$$\tilde{V}_n = \int_{B_{2r}(x_0)} \tilde{v}_n\,\eta^2\,dy \Big/ \int_{B_{2r}(x_0)} \eta^2\,dy, \qquad W_n = \tilde{V}_n - \gamma_2(n-n_0)\,h/r^2,$$

$$W(t) = W_n, \qquad (n-1)\,h < t \leqq n\,h,$$

$$B_\sigma^n = \{x \in B_r(x_0) : w_n(x) > \sigma\}.$$

Then we have

Lemma 4.7 *There exists a positive constant γ_1 depending only on m, λ such that*

$$\text{(4.10)}\qquad h \sum_{n=n_0+1}^{n_1} |B_\sigma^n| \leqq \gamma_1\, r^2\, |B_r|\, \sigma^{-1}$$

holds for all $n_0+1 \leqq n_1 \leqq [\bar{t}/h]+1$, and any $\sigma > 0$.

Proof of Lemma 4.7. We remark that, since $u_h \geqq 0$ in $C_R^+(\bar{t}, \bar{x})$ and $u_{[(\bar{t}-R^2)/h]} \geqq 0$ in $B_R(\bar{x})$,

$$u_n \geqq 0 \quad \text{in } B_R(\bar{x}) \text{ for all } [(\bar{t}-R^2)/h] \leqq n \leqq [\bar{t}/h]+1.$$

Testing the identity (1.3) by $\varphi = u_n^{-1}\eta^2$, where $\eta \in C_0^\infty(B_{2r})$, $|D\eta| \leqq 2/r$ (see remark after Lemma 4.3), we have (4.4) with $p = r$. Now we make an estimate of each term of (4.4).

For the quotient term of (4.4), remark that

$$1 - u_{n-1}u_n^{-1} \leqq -\log(u_{n-1}u_n^{-1}) = \log u_n - \log u_{n-1},$$

whence

$$\text{(4.11)}\quad \int_{B_{2r}} \frac{1}{h}(1-u_{n-1}u_n^{-1})\,\eta^2\,dx \leqq \int_{B_{2r}} \frac{\log u_n - \log u_{n-1}}{h}\,\eta^2\,dx = \frac{-\int_{B_{2r}} v_n\eta^2\,dx + \int_{B_{2r}} v_{n-1}\eta^2\,dx}{h}.$$

Combining (4.11) with (4.4), we obtain

$$\frac{\int_{B_{2r}} v_n\eta^2\,dx - \int_{B_{2r}} v_{n-1}\eta^2\,dx}{h} + \int_{B_{2r}} a_n^{\alpha\beta} D_\beta v_n D_\alpha v_n\,\eta^2\,dx - 2\int_{B_{2r}} a_n^{\alpha\beta} D_\beta v_n\,\eta\, D_\alpha\eta\,dx \leqq 0.$$

The ellipticity condition (1.2) and Young's inequality gives that for $[(\bar{t}-R^2)/h]+1\leqq n\leqq[\bar{t}/h]+1$,

$$(4.12)\quad \frac{\int_{B_{2r}} v_n\eta^2\,dx-\int_{B_{2r}} v_{n-1}\eta^2\,dx}{h}+\frac{\lambda}{2}\int_{B_{2r}}|Dv_n|^2\eta^2\,dx\leqq\frac{2\mu^2}{\lambda}\int_{B_{2r}}|D\eta|^2\,dx.$$

Recall the following inequality of Poincarè type. (For the proof, we refer to Moser's paper [11].)

Lemma 4.8 *There exists a positive constant* γ *such that*

$$(4.13)\quad \int_{B_{2r}}\Big(v-\int_{B_{2r}} v\eta^2\,dy\Big/\int_{B_{2r}}\eta^2\,dy\Big)^2\eta^2\,dx$$

$$=\min_k\int_{B_{2r}}(v-k)^2\eta^2\,dx\leqq\gamma(4m)^2r^2\int_{B_{2r}}|Dv|^2\eta^2\,dx$$

for $v\in W_2^1(B_{2r})$.

Adopting (4.13) for (4.12), for $[(\bar{t}-R^2)/h]+1\leqq n\leqq[\bar{t}/h]+1$, we have

$$(4.14)\quad \frac{\int_{B_{2r}} v_n\eta^2\,dx-\int_{B_{2r}} v_{n-1}\eta^2\,dx}{h}$$

$$+\frac{\lambda}{2\gamma(4m)^2}r^{-2}\int_{B_{2r}}\Big(v_n-\int_{B_{2r}} v_n\eta^2\,dy\Big/\int_{B_{2r}}\eta^2\,dy\Big)^2\eta^2\,dx$$

$$\leqq\frac{2\mu^2}{\lambda}\int_{B_{2r}}|D\eta|^2\,dx\leqq\frac{4\mu^2}{\lambda}r^{-2}|B_{2r}|.$$

Dividing both sides of (4.14) by $r^{-2}\int_{B_{2r}}\eta^2\,dy$, taking η as $\eta=1$ in $B_r(x_0)$ and noting that $\int_{B_{2r}}\eta^2\,dx\leqq 3^m|B_{2r}|$, we have, for $[(\bar{t}-R^2)/h]+1\leqq n\leqq[\bar{t}/h]+1$,

$$(4.15)\quad \frac{\int_{B_{2r}} v_n\eta^2\,dx\Big/\int_{B_{2r}}\eta^2\,dx-\int_{B_{2r}} v_{n-1}\eta^2\,dx\Big/\int_{B_{2r}}\eta^2\,dx}{h/r^2}$$

$$+\frac{\lambda}{2\gamma(4m)^2}\frac{1}{3^m|B_r|}\int_{B_{2r}}\Big(v_n-\int_{B_{2r}} v_n\eta^2\,dy\Big/\int_{B_{2r}}\eta^2\,dy\Big)^2\eta^2\,dx\leqq\frac{2\mu^2}{\lambda}\frac{|B_{2r}|}{|B_r|}.$$

Now we notice that, from the definition $\tilde{V}_n$ in (4.9)

$$(4.16)\qquad \tilde{V}_{n_0}=0.$$

Taking $\gamma_1=2\gamma 3^m(4m)^2/\lambda$, $\gamma_2=4\mu^2|B_2|/\lambda|B_1|$ and noting that (4.15) remains unchanged if v_l is replaced by $v_l+\text{const.}$, (4.15) are rewritten as follows (Recall the definition (4.9) of $\tilde{V}_{n_0}$ and W_n):

$$
\text{(4.17)} \qquad \begin{cases} \dfrac{\tilde{V}_n-\tilde{V}_{n-1}}{h/r^2}+\gamma_1^{-1}\dfrac{1}{|B_r|}\displaystyle\int_{B_{2r}(x_0)}(\tilde{v}_n-\tilde{V}_n)^2\eta^2\,dx\leqq\gamma_2 \\ (n;\,[(\bar{t}-R^2)/h]+1\leqq n\leqq[\bar{t}/h]+1), \\ \tilde{V}_{n_0}=0. \end{cases}
$$

In terms of W_n, (4.17) becomes

$$
\text{(4.18)} \qquad \begin{cases} \dfrac{W_n-W_{n-1}}{h/r^2}+\gamma_1^{-1}\dfrac{1}{|B_r|}\displaystyle\int_{B_{2r}(x_0)}(w_n-W_n)^2\eta^2\,dx\leqq 0 \\ (n;\,[(\bar{t}-R^2)/h]+1\leqq n\leqq[\bar{t}/h]+1), \\ W_{n_0}=0. \end{cases}
$$

From (4.18), that for $n_0\leqq n\leqq[\bar{t}/h]+1$, we obtain

$$
\text{(4.19)} \qquad W_n\leqq W_{n-1}\leqq\cdots\leqq W_{n_0}=0.
$$

Hence, for $\sigma>0$ and $n=n_0,\dots,[\bar{t}/h]+1$,

$$
w_n-W_n\geqq\sigma-W_n>0 \quad \text{in } B_\sigma^n.
$$

Thus, again by (4.18) we have, for any $\sigma>0$ and all $n_0\leqq n\leqq[\bar{t}/h]+1$,

$$
\frac{W_n-W_{n-1}}{h/r^2}+\gamma_1^{-1}\frac{|B_\sigma^n|}{|B_r|}(\sigma-W_n)^2\leqq\frac{W_n-W_{n-1}}{h/r^2}+\gamma_1^{-1}\frac{1}{|B_r|}\int_{B_{2r}(x_0)}(w_n-W_n)^2\eta^2\,dx\leqq 0,
$$

$$
(\sigma-W_n)^{-2}\frac{\sigma-W_n-(\sigma-W_{n-1})}{h/r^2}\geqq\gamma_1^{-1}\frac{|B_\sigma^n|}{|B_r|}.
$$

Noting that, for $a, b\geqq 0$,

$$
-(a^{-1}-b^{-1})\geqq a^{-2}(a-b),
$$

we have, for $n_0\leqq n\leqq[\bar{t}/h]+1$,

$$
\text{(4.20)} \qquad \frac{-((\sigma-W_n)^{-1}-(\sigma-W_{n-1})^{-1})}{h/r^2}\geqq\gamma_1^{-1}\frac{|B_\sigma^n|}{|B_r|}.
$$

Multiplying (4.20) by h/r^2 and summing the resulting inequality from n_0+1 to $n_0+1 \leqq n_1 \leqq [\bar{t}/h]+1$, we obtain

$$\frac{h}{r^2}\sum_{n=n_0+1}^{n_1} \gamma_1^{-1}\frac{|B_\sigma^n|}{|B_r|} \leqq -\sum_{n=n_0+1}^{n_1}\{(\sigma-W_n)^{-1}-(\sigma-W_{n-1})^{-1}\}$$
$$= -(\sigma-W_{n_1})^{-1}+(\sigma-W_{n_0})^{-1} \leqq \sigma^{-1}$$

where we used (4.19) in the last inequality.

We just now are in a position to prove Lemma 4.5.

Proof of Lemma 4.5. Note that

$$(4.21)\quad \frac{1}{|C_r^+||C_r^-|}\iint_{C_r^+(t_0,x_0)}\iint_{C_r^-(t_0,x_0)} \varphi(v(t,x)-v(t',x'))\,dx\,dt\,dx'\,dt'$$
$$\leqq \frac{1}{|C_r^+|}\iint_{C_r^-(t_0,x_0)} \varphi(v(t,x)-\int_{B_{2r}} v_{[t_0/h]+1}\,\eta^2/\int_{B_{2r}}\eta^2)\,dx\,dt$$
$$+\frac{1}{|C_r^-|}\iint_{C_r^+(t_0,x_0)} \varphi(-v(t',x')+\int_{B_{2r}} v_{[t_0/h]+1}\,\eta^2/\int_{B_{2r}}\eta^2)\,dx'\,dt'.$$

We estimate each term of (4.21). Put

$$I_1=\iint_{C_r^-(t_0,x_0)} \varphi(v(t,x)-\int_{B_{2r}} v_{[t_0/h]+1}\,\eta^2/\int_{B_{2r}}\eta^2)\,dx\,dt,$$

$$I_2=\iint_{C_r^+(t_0,x_0)} \varphi(-v(t',x')+\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\,\eta^2\,dy/\int_{B_{2r}(x_0)}\eta^2\,dy)\,dx'\,dt'.$$

Let's set $n_0=[t_0/h]+1$ and define a time-step function g as follows:

$$g(t,\cdot)=\gamma_2(n-n_0)\,h/r^2 \quad \text{for } t\in((n-1)h, nh]$$
$$(n=n_0,\ldots,[(t_0+r^2)/h]+1) \quad \text{in } B_r.$$

By (4.9), we have

$$I_1=\iint_{C_r^-(t_0,x_0)}\varphi(\tilde{v}(t,x))\,dx\,dt=\iint_{C_r^-(t_0,x_0)}\varphi(w(t,x)+g(t,x))\,dx\,dt$$
$$\leqq \iint_{C_r^-(t_0,x_0)}\varphi(w(t,x))\,dx\,dt+\iint_{C_r^-(t_0,x_0)}\varphi(g(t,x))\,dx\,dt$$
$$\leqq \iint_{C_r^-(t_0,x_0)}\varphi(w(t,x))\,dx\,dt+|C_r^-|\sqrt{\gamma_2(1+h/r^2)}$$

where we have used that

$$g(t,\cdot)\leqq\gamma_2(1+h/r^2) \quad \text{for } t\in[[t_0/h]\,h,\ ([(t_0+r^2)/h]+1)\,h].$$

Moreover, since by assumption $h<3r^2$, I_1 is estimated from above by

$$(4.22)\quad \iint_{C_r^-(t_0,x_0)}\varphi(w(t,x))\,dt\,dx+2\sqrt{\gamma_2}\,|C_r^-|.$$

Thus we need to estimate $\iint_{C_r^-(t_0,x_0)} \varphi(w(t,x))\,dx\,dt$. Split

$$(4.23)\qquad \iint_{C_r^-(t_0,x_0)} \varphi(w(t,x))\,dx\,dt = \int_{t_{[t_0/h]+1}}^{t_0+r^2} \int_{B_r(x_0)} \varphi(w(t,x))\,dx\,dt + \int_{t_0}^{t_{[t_0/h]+1}} \int_{B_r(x_0)} \varphi(w(t,x))\,dx\,dt = I_1^1 + I_1^2 .$$

First we estimate I_1^1. We have

$$(4.24)\qquad I_1^1 = \int_{t_{[t_0/h]+1}}^{t_0+r^2} \int_{B_r(x_0)\cap\{w>1\}} \varphi(w(t,x))\,dx\,dt + \int_{t_{[t_0/h]+1}}^{t_0+r^2} \int_{B_r(x_0)\cap\{0<w\leqq 1\}} \varphi(w(t,x))\,dx\,dt \leqq \int_{t_{[t_0/h]+1}}^{t_0+r^2} \int_{B_r(x_0)\cap\{w>1\}} \varphi(w(t,x))\,dx\,dt + |C_r^-| .$$

Note the inclusion: $(t_{[t_0/h]+1}, t_0+r^2) \subset (t_{[t_0/h]+1}, t_{[(t_0+r^2)/h]+1})$. Setting $m(\sigma) = |\{(t,x)\in(t_{[t_0/h]+1}, t_0+r^2)\times B_r(x_0);\, w(t,x)>\sigma\}|$ for $\sigma>0$, we obtain from Lemma 4.7,

$$m(\sigma) \leqq h \sum_{n=[t_0/h]+2}^{[(t_0+r^2)/h]+1} |B_\sigma^n| \leqq \gamma_1 r^2 |B_r| \sigma^{-1}$$

so that, for the first term of the right hand in (4.24), we have

$$(4.25)\qquad \int_{t_{[t_0/h]+1}}^{t_0+r^2} \int_{B_r(x_0)\cap\{w>1\}} \varphi(w(t,x))\,dx\,dt = \int_1^\infty \sqrt{\sigma}(-d\,m(\sigma)) = \int_1^\infty m(\sigma)\,d\sqrt{\sigma} \leqq \int_1^\infty \gamma_1 \sigma^{-1} r^2 |B_r|\,d\sqrt{\sigma} = \gamma_1 r^2 |B_r| .$$

Substituting (4.25) into (4.24), we have

$$(4.26)\qquad I_1^1 \leqq \gamma_1 r^2 |B_r| + |C_r^-| \leqq (\gamma_1+1)|C_r^-| .$$

Next we shall deal with I_1^2. Note that

$$(4.27)\qquad I_1^2 = \int_{t_0}^{t_{[t_0/h]+1}} \int_{B_r(x_0)} \varphi(w(t,x))\,dx\,dt = |t_{[t_0/h]+1} - t_0| \int_{B_r(x_0)} \varphi(w_{[t_0/h]+1})\,dx .$$

Recalling the definition of w_n:

$$w_{[t_0/h]+1}=\tilde v_{[t_0/h]+1}=v_{[t_0/h]+1}-\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\,\eta^2\,dx\Big/\int_{B_{2r}(x_0)}\eta^2\,dx$$
$$\leqq |v_{[t_0/h]+1}(x)-\overline{(v_{[t_0/h]+1})_{B_{2r}}}|$$
$$+|\overline{(v_{[t_0/h]+1})_{B_{2r}}}-\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\,\eta^2\,dy\Big/\int_{B_{2r}(x_0)}\eta^2\,dy|$$

we have

$$\int_{B_r(x_0)}\varphi(w_{[t_0/h]+1})\,dx \tag{4.28}$$
$$\leqq \int_{B_{2r}(x_0)}\sqrt{|v_{[t_0/h]+1}(x)-\overline{(v_{[t_0/h]+1})_{B_{2r}}}|}\,dx$$
$$+|B_r|^{1/2}\sqrt{\int_{B_{2r}}|v_{[t_0/h]+1}(x)-\overline{(v_{[t_0/h]+1})_{B_{2r}}}|\,dx}$$
$$\leqq 2|B_{2r}|^{1/2}\Big(\int_{B_{2r}(x_0)}|v_{[t_0/h]+1}(x)-\overline{(v_{[t_0/h]+1})_{B_{2r}}}|\,dx\Big)^{1/2}.$$

Applying Lemma 4.3 to (4.28) gives that

$$\int_{B_r}\varphi(w_{[t_0/h]+1})\,dx\leqq 2\gamma^{1/2}\,(16\mu^2/\lambda^2+2r^2/\lambda h)^{1/4}\,|B_{2r}|. \tag{4.29}$$

Substituting (4.29) into (4.27), noting that $3r^2\geqq h$ and that $|t_{[t_0/h]+1}-t_0|\leqq h$, we have

$$I_1^2\leqq 2\gamma^{1/2}\,h\,|B_{2r}|(r^2/h)^{1/4}(48\mu^2/\lambda^2+2/\lambda)^{1/4} \tag{4.30}$$
$$\leqq 2\gamma^{1/2}\,|B_{2r}|\,h^{3/4}\,r^{1/2}(48\mu^2/\lambda^2+2/\lambda)^{1/4}$$
$$\leqq 2^2\,\gamma^{1/2}\,2^m(24\mu^2/\lambda^2+1/\lambda)^{1/4}\,|C_r^-|.$$

Combining the estimates (4.26) and (4.30) for I_1^1 and I_1^2 with (4.22) gives that

$$I_1=\iint_{C_r^-(t_0,x_0)}\varphi(\bar v)\,dt\,dx\leqq(\gamma_1+1)\,|C_r^-|+\gamma\,|C_r^-|+2\sqrt{\gamma_2}\,|C_r^-|, \tag{4.31}$$

where $\gamma=2^{m+2}\,\gamma^{1/2}(24\mu^2/\lambda^2+1/\lambda)^{1/4}$.

Now we estimate the term

$$I_2=\iint_{C_r^+(t_0,x_0)}\varphi\Big(-v(t',x')+\int_{B_{2r}(x_0)} v_{[t_0/h]+1}\,\eta^2\,dy\Big/\int_{B_{2r}(x_0)}\eta^2\,dy\Big)\,dx'\,dt'.$$

Putting, for $[(t_0-r^2)/h]+1\leqq n\leqq n_0$,

$$\bar v_n(x)=-v_n(x)+\int_{B_{2r}(x_0)} v_{n_0}\,\eta^2\,dy\Big/\int_{B_{2r}(x_0)}\eta^2\,dy \quad \text{on } B_{2r}(x_0),$$

$$\bar w_n(x_0)=\bar v_n(x)-\gamma_2(n_0-n)\,h/r^2 \quad \text{on } B_{2r}(x_0),$$

$$\bar V_n=\int_{B_{2r}(x_0)}\bar v_n\,\eta^2\,dy\Big/\int_{B_{2r}(x_0)}\eta^2\,dy,\qquad \bar W_n=\bar V_n-\gamma_2(n_0-n)\,h/r^2,$$

$$\bar B_\sigma^n=\{x\in B_{2r}(x_0)\colon \bar w_n>\sigma\},$$

similarily to Lemma 4.7, we have

$$h \sum_{n=[(t_0-r^2)/h]+1}^{n_0} |\bar{B}_\sigma^n| \leqq \gamma_1 r^2 |B_r| \sigma^{-1}. \tag{4.32}$$

Exploiting (4.32) with $n_0 = [t_0/h]+1$ similarily to (4.25), we obtain

$$I_2 \leqq \gamma_1 |C_r^+| + \sqrt{3\gamma_2} |C_r^+|. \tag{4.33}$$

Substituting (4.31) and (4.33) into (4.21) gives that

$$\begin{aligned}
&\frac{1}{|C_r^+||C_r^-|} \iint_{C_r^+(t_0,x_0)} \iint_{C_r^-(t_0,x_0)} \varphi(v(t,x)-v(t',x'))\,dt\,dx\,dt'\,dx' \\
&\leqq \frac{1}{|C_r^-|} I_1 + \frac{1}{|C_r^+|} I_2 \\
&\leqq 2\gamma_1 + \gamma + 6\gamma_2,
\end{aligned}$$

which concludes the proof of Lemma 4.5.

5 Proof of theorems

In this section we present the proof of our theorems. Recall the definition (1.9) of $\tilde{n}_r$.

Proof of Theorem 1.1. Since u_h is nonnegative in $C_r^+(t_{n_0}, x_0)$, we find that

$$u_h \geqq 0 \quad \text{in } C^+_{\sqrt{\tilde{n}_r h}}(t_{n_0}, x_0)$$

and that

$$u_{n_0-[\tilde{n}_r/4]-1} \geqq 0 \quad \text{in } B_{\sqrt{\tilde{n}_r h}}(x_0).$$

Thus we can apply Lemma 4.6 to u_h in $C^+_{\sqrt{\tilde{n}_r h}}(t_{n_0}, x_0)$, so that,

$$\begin{aligned}
&\left(\frac{1}{|C^-_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}|} \iint_{C^-_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}(t_{n_0}-\tilde{n}_r, x_0)} (u_h)^\xi\,dx\,dt\right)^{1/\xi} \\
&\leqq \gamma^{1/\xi} \left(\frac{1}{|C^+_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}|} \iint_{C^+_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}(t_{n_0}, x_0)} (u_h)^{-\xi}\,dx\,dt\right)^{-1/\xi}
\end{aligned} \tag{5.1}$$

where γ, ξ are positive constants determined in Lemma 4.6. We can also adopt (2.21) with $p=-\xi$ in Lemma 2.2 for u_h in $C^+_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}(t_{n_0}, x_0)$ to obtain that

$$\left(\frac{1}{|C^+_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}|} \iint_{C^+_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}(t_{n_0}, x_0)} (u_h)^{-\xi}\,dx\,dt\right)^{-1/\xi} \leqq \gamma \inf_{C^+_{\sqrt{\tilde{n}_r h}/2}(t_{n_0}, x_0)} u_h. \tag{5.2}$$

Similarly, exploiting (2.22) in Lemma 2.2 for u_h in $C^-_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}(t_{n_0-\tilde{n}_r}, x_0)$ and noting Hölder inequality, we find

(5.3)
$$\left(\frac{1}{|C^-_{\sqrt{\tilde{n}_r h}/2}|}\iint_{C^-_{\sqrt{\tilde{n}_r h}/2}(t_{\tilde{n}_0-\tilde{n}_r}, x_0)} (u_h)^p\,dx\,dt\right)^{1/p}$$
$$\leqq \gamma\left(\frac{1}{|C^-_{\sqrt{\tilde{n}_r h}, \tilde{n}_r h/2}|}\iint_{C^-_{\sqrt{n_r h}, n_r h/2}(t_{n_0-n_r}, x_0)} (u_h)^\xi\,dx\,dt\right)^{1/\xi}$$

for $0<p<1+2/m$. Combining (5.2) and (5.3) with (5.1), we obtain the assertion (1.10) of Theorem 1.1.

Theorem 1.1 implies a Hölder estimate for a weak solution u_h of (1.1), which has been derived in [6], but our proof is entirely different from that in [6]. Also recall the definition (1.8) of δ.

Lemma 5.1 *Let u_h be a weak solution of (1.1) and $(\bar{t}, \bar{x})$ be taken arbitrarily in $\tilde{Q}_{h_0}$ with*

$$d=\tfrac{1}{4}\min(|\bar{t}-N_0 h_0|^{1/2}, \operatorname{dist}(\bar{x}, \partial\Omega)).$$

Assume that $\iint_{C_d^+(\bar{t}, \bar{x})} |u_h|^2\,dt\,dx \leqq \gamma_1$ with a uniform constant γ_1. Then there exist positive constants γ and $\alpha<1$, depending only on λ, μ, m, d, and γ_1, such that

(5.4)
$$|u_h(t_{n'}, x')-u_h(t_n, x)| \leqq \gamma(\delta((t_n, x), (t_{n'}, x')))^\alpha$$

holds for any $(t_{n'}, x')$, $(t_n, x) \in C_d^+(\bar{t}, \bar{x})$ with $\delta((t_{n'}, x'), (t_n, x)) \geqq \sqrt{h}$.

Proof. Notice that by Lemma 3.2 a weak solution u_h is uniformly bounded in $C_d^+(\bar{t}, \bar{x})$. Take $(t_{n'}, x')$, $(t_n, x) \in C_d^+(\bar{t}, \bar{x})$ satisfying $\delta((t_{n'}, x'), (t_n, x)) \geqq \sqrt{h}$, arbitrarily. Introduce

(5.6)
$$M=\sup_{C_d^+(t_n, x)} u_h, \qquad m=\inf_{C_d^+(t_n, x)} u_h,$$

$$\tilde{n}_d = \text{the greatest number satisfying } n<d^2/h.$$

Since $M-u_h$, u_h-m are weak solutions of (1.1) and nonnegative in $C_d^+(t_n, x)$, we can apply Theorem 1.1 for $M-u_h$, u_h-m in $C_d^+(t_n, x)$ to obtain

(5.7)
$$\frac{1}{|C^-_{\sqrt{\tilde{n}_d h}/2}|}\iint_{C^-_{\sqrt{\tilde{n}_d h}/2}(t_{n-\tilde{n}_d}, x)} (M-u_h)\,dx\,dt \leqq \gamma \inf_{C^+_{\sqrt{\tilde{n}_d h}/2}(t_n, x)} (M-u_h),$$

(5.8)
$$\frac{1}{|C^-_{\sqrt{\tilde{n}_d h}/2}|}\iint_{C^-_{\sqrt{\tilde{n}_d h}/2}(t_{n-\tilde{n}_d}, x)} (u_h-m)\,dx\,dt \leqq \gamma \inf_{C^+_{\sqrt{\tilde{n}_d h}/2}(t_n, x)} (u_h-m).$$

Adding (5.7) and (5.8), and replacing $\max(\gamma, 1)$ by γ, we have

(5.9)
$$\operatorname*{osc}_{C^+_{\sqrt{\tilde{n}_d h}/2}(t_n, x)} u_h \leqq (1-\gamma^{-1}) \operatorname*{osc}_{C_d^+(t_n, x)} u_h.$$

By iteration we have, for any positive integer ν,

$$(5.10)\qquad \omega_{d_\nu} \leqq \theta^\nu \omega_{d_{\nu-1}}$$

where

$$\theta = 1-\gamma^{-1}, \qquad \omega_{d_\nu} = \operatorname*{osc}_{C^+_{d_\nu}} u_h$$

and

$$d_0 = d, \quad d_\nu = \left(d^2/2^\nu - \sum_{k=1}^{\nu} \delta_k/2^{\nu-k+1}\right)^{1/2}, \quad \delta_k = d^2_{k-1} - \tilde{n}_{d_{k-1}} h \quad (k=1, 2, \ldots, \nu).$$

If

$$\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \sqrt{h} < d/2,$$

then there exists a positive integer ν such that

$$d/2^{\nu+1} < \max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \sqrt{h} \leqq d/2^\nu.$$

Now, adopting (5.10) successively gives that

$$\begin{aligned}(5.11)\qquad |u_n(x) - u_{n'}(x')| &\leqq \omega_{d_\nu} \leqq \theta\, \omega_{d_{\nu-1}} \\ &\leqq \ldots \leqq \theta^\nu \omega_d = (1/2)^{\alpha\nu} \omega_d \\ &\leqq ((2/d) \max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + (2/d)\sqrt{h})^\alpha \omega_d \\ &\leqq (2\,\delta((t_{n'}, x'), (t_n, x))/d + 2\sqrt{h}/d)^\alpha \omega_d,\end{aligned}$$

where $\alpha = -\log\theta/\log 2$. Since $\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} \geqq \sqrt{h}$, it follows from (5.11) that

$$(5.12)\qquad |u_n(x) - u_{n'}(x')| \leqq (4/d)^\alpha \delta^\alpha((t_{n'}, x'), (t_n, x))\, \omega_d.$$

If

$$\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \sqrt{h} \geqq d/2,$$

we obtain, from the boundedness of u_h in $C^+_d(\bar{t}, \bar{x})$

$$\begin{aligned}(5.13)\qquad |u_n(x) - u_{n'}(x')| &\leqq 2U \\ &\leqq 2U((2/d)(\max\{|t_n - t_{n'}|^{1/2}, |x-x'|\} + \sqrt{h}))^\alpha \\ &= 2U(4/d)^\alpha \delta^\alpha((t_{n'}, x'), (t_n, x)).\end{aligned}$$

Proof of Theorem 1.2. Fix $B_{2r} = B_{2r}(x_0) \subset \tilde{\Omega}_{h_0}$ with $r^2 \leqq h$. Suppose that $u_n(N_0 + 1 \leqq n \leqq N)$ is nonnegative in B_{2r}. Scaling $x = x_0 + r\,y$ and letting

$$\tilde{a}^{\alpha\beta}_n(y) = a^{\alpha\beta}_n(x_0 + r\,y), \qquad \tilde{u}_n(y) = u_n(x_0 + r\,y), \qquad \tilde{u}_{n-1}(y) = u_{n-1}(x_0 + r\,y) \quad \text{on } B_1,$$

from (1.3) we obtain

$$(5.14)\qquad \int_{B_1} \tilde{a}^{\alpha\beta}_n D_\beta \tilde{u}_n D_\alpha \varphi\, dy + \int_{B_1} \frac{\tilde{u}_n - \tilde{u}_{n-1}}{h/r^2} \varphi\, dy = 0 \qquad \text{for any } \varphi = (\varphi^i) \in W^1_{2,0}(B_1).$$

Applying Harnack's theorem for elliptic equations (see [3, Theorem 8.18, p. 194]) to $\tilde{u}_n$ in B_1, it follows from (5.14) that, for any $1 \leqq p < m/(m-2)$, and $q > m$, there exists a positive constant γ depending only on m, q, p, λ and μ such that

$$(5.15) \qquad \left(\frac{1}{|B_1|} \int_{B_1} (\tilde{u}_n)^p \, d y\right)^{1/p} \leqq \gamma \left\{\inf_{B_1} \tilde{u}_n + \left(\frac{1}{|B_1|} \int_{B_1} g^q \, d y\right)^{1/q}\right\},$$

where

$$g = \frac{\tilde{u}_n - \tilde{u}_{n-1}}{h/r^2}.$$

Now, applying Lemma 5.1 to u_h in $C_d^+(T, x_0)$ with $d = (1/4) \min\{|T - N_0 h_0|^{1/2}, \operatorname{dist}(x_0, \partial\Omega)\}$, we have that

$$(5.16) \qquad |g| \leqq \gamma \, h^{\alpha/2-1} \, r^2$$

with positive constants γ, $\alpha < 1$, independent of h, v_n, which were determined in Lemma 5.1. Thus, since $r^2 \leqq h$ implies

$$h^{\alpha/2-1} \, r^2 \leqq r^{\alpha-2+2} = r^\alpha,$$

from (5.15) and (5.16) we obtain the assertion in Theorem 1.2.

Proof of Theorem 1.3 Fix a cube $C_d^+(\bar{t}, \bar{x}) \subset Q$ with $d = (1/4) \min\{|\bar{t} - N_0 h_0|^{1/2}, \operatorname{dist}(\bar{x}, \partial\Omega)\}$. We use the notation: $u = u_h$, $v = u_h^\pm$. Now we shall improve a Caccioppoli type inequality Lemma 2.3. Firstly we consider the case $1 < p \leqq 2$. If $\sigma_2 \tau > 3h$, we have (2.27) and (2.28). If $\sigma_2 \tau \leqq 3h$, we remark that, adopting Hölder estimate (Lemma 5.1) for u_h in $C_d^+(\bar{t}, \bar{x})$ yields the following calculations: For all $C_{\rho,\tau}^+(t_{n_0}, x_0) \subset C_d^+(\bar{t}, \bar{x})$ and for any $\varepsilon > 0$

$$\begin{aligned}(5.17) \qquad & \iint_{C_{\rho,\tau}^+} \frac{\pm u(t,\cdot) + \varepsilon - (\pm u(t-h,\cdot) + \varepsilon)}{h} (v(t,\cdot) + \varepsilon)^{p-1} \eta^2(\cdot) \, d x \, d t \\ & \geqq -\gamma h^{\alpha/2} \, 3(\sigma_2 \tau)^{-1} \iint_{C_{\rho,\tau}^+} |v(t,\cdot) + \varepsilon|^{p-1} \eta^2(\cdot) \, d x \, d t \\ & \geqq -(\sigma_2 \tau)^{-1} \iint_{C_{\rho,\tau}^+} |v(t,\cdot) + \varepsilon|^p \, d y \, d s - 2^p \gamma \cdot (\sigma_2 \tau)^{-1} h^{p\alpha/2} |C_{\rho,\tau}^+|,\end{aligned}$$

where $\gamma = \gamma(d)$ is determined in Lemma 5.1. In the last inequality we used Young's inequality. Thus, calculating similarly as in (2.29), we have, for all $C_{\rho,\tau}^+(t_{n_0}, x_0) \subset C_d^+(\bar{t}, \bar{x})$,

$$\begin{aligned}(5.18) \qquad & \iint_{C_{\rho,\tau}^+} |D(v+\varepsilon)^{p/2}|^2 \eta^2 \sigma \, d x \, d t \\ & + \frac{p^2}{\lambda(p-1)} \varepsilon^{p-1} \iint_{C_{\rho,\tau}^+ \cap \{\pm u \leqq 0\}} a^{\alpha\beta} D_\beta(v(t,\cdot)) \eta D_\alpha \eta \, d x \, d t \\ & \leqq \frac{\mu^2 p^2}{\lambda^2 (p-1)^2} ((\sigma_2 \tau)^{-1} + (\sigma_1 \rho)^{-2}) \iint_{C_{\rho,\tau}^+} (v+\varepsilon)^p \, d x \, d t \\ & + \frac{2^{p-1} 3 p}{\lambda(p-1)} (\sigma_2 \tau)^{-1} h^{p\alpha/2} |C_{\rho,\tau}^+|.\end{aligned}$$

We also remark that (2.19) holds for $v+\varepsilon=u_h^{\pm}+\varepsilon>0$ in this case. Therefore we have, for all $C^+_{\rho,\tau}(t_{n_0}, x_0)\subset C^+_d(\bar{t}, \bar{x})$ and any $1<p\leqq 2$,

$$(5.19)\quad \sup_{t_{n_0}-\tau(1-\sigma_2)\leqq t\leqq t_{n_0}} \int_{B_{\rho(1-\sigma_1)}} (v+\varepsilon)^p(t,\cdot)\,dx+\iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}} |D(v+\varepsilon)^{p/2}|^2\,dx\,dt$$
$$+\frac{p^2}{\lambda(p-1)}\varepsilon^{p-1}\iint_{C^+_{\rho,\tau}\cap\{\pm u\leqq 0\}} a^{\alpha\beta}D_\beta v(t,\cdot)\eta D_\alpha\eta\,dx\,dt$$
$$\leqq\gamma\frac{p(2p-1)}{(p-1)^2}\{((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1})\iint_{C^+_{\rho,\tau}}(v+\varepsilon)^p\,dx\,dt+2^p(\sigma_2\tau)^{-1}h^{p\alpha/2}|C^+_{\rho,\tau}|\}.$$

Since each term of the right hand of (5.19) is finite for $1<p\leqq 2$, we are able to pass ε to 0 in (5.19), so that

$$(5.20)\quad \sup_{t_{n_0}-\tau(1-\sigma_2)\leqq t\leqq t_{n_0}} \int_{B_{\rho(1-\sigma_1)}} v^p(t,\cdot)\,dx+\iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}} |Dv^{p/2}|^2\,dx\,dt$$
$$\leqq\gamma\frac{p(2p-1)}{(p-1)^2}\{((\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1})\iint_{C^+_{\rho,\tau}}v^p\,dx\,dt+2^p(\sigma_2\tau)^{-1}h^{p\alpha/2}|C^+_{\rho,\tau}|\}.$$

Next we consider the case $p>2$. If $\sigma_2\tau>3h$, then we have (2.30). If $\sigma_2\tau\leqq 3h$, using Hölder estimate (Lemma 5.1) and calculating similarly as in (5.17) yields that, for any $C^+_{\rho,\tau}(t_{n_0}, x_0)\subset C^+_d(\bar{t}, \bar{x})$

$$\iint_{C^+_{\rho,\tau}}\frac{\pm u(t,\cdot)-\pm u(t-h,\cdot)}{h}(v^{(M)})^{p-1}(t,\cdot)\,\eta^2(\cdot)\,dx\,dt$$
$$\geqq-(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}\lambda v^{(M)})^p(t,\cdot)\,dy\,ds-2^p\gamma\cdot(\sigma_2\tau)^{-1}h^{p\alpha/2}|C^+_{\rho,\tau}|.$$

By calculating similarly as in (2.31), we have, for all $C^+_{\rho,\tau}(t_{n_0}, x_0)\subset C^+_d(\bar{t}, \bar{x})$ and for any $t\in[t_{n_0}-\tau(1-\sigma_2), t_{n_0}]$

$$(5.21)\quad 0\geqq\int_{B_\rho}v^{(M)}(t,\cdot)\eta^2\,dx-4(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}(v^{(M)})^p\eta^2\,dx\,dt-2^p\gamma\cdot(\sigma_2\tau)^{-1}h^{p\alpha/2}|C^+_{\rho,\tau}|$$
$$+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}a^{\alpha\beta}(t,\cdot)D_\beta(v^{(M)})^{p/2}D_\alpha(v^{(M)})^{p/2}\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$+2\iint_{C^+_{\rho,\tau}}a^{\alpha\beta}(t,\cdot)D_\beta v(v^{(M)})^{p-1}\eta D_\alpha\eta(\cdot)\,\sigma(t)\,dx\,dt.$$

Noting the boundedness of v in $C^+_d(\bar{t}, \bar{x})$ from Lemma 3.2, we can pass M to the limit in (5.21) for $p>2$ to obtain, for all $C^+_{\rho,\tau}\subset C^+_d(\bar{t}, \bar{x})$ and any $t\in[t_{n_0}-\tau, t_{n_0}]$

$$(5.22)\quad 0\geqq\int_{B_\rho}v(t,\cdot)\cdot\eta^2\,dx-4(\sigma_2\tau)^{-1}\iint_{C^+_{\rho,\tau}}v^p\eta^2\,dx\,dt-2^p\gamma\cdot(\sigma_2\tau)^{-1}h^{p\alpha/2}|C^+_{\rho,\tau}|$$
$$+\frac{4(p-1)}{p^2}\iint_{C^+_{\rho,\tau}}a^{\alpha\beta}(t,\cdot)D_\beta v^{p/2}D_\alpha v^{p/2}\eta^2(\cdot)\,\sigma(t)\,dx\,dt$$
$$+2\iint_{C^+_{\rho,\tau}}a^{\alpha\beta}(t,\cdot)D_\beta v\,v^{p-1}\eta D_\alpha\eta(\cdot)\,\sigma(t)\,dx\,dt.$$

By Young's inequality, we have (5.20) for all $C^+_{\rho,\tau}\subset C^+_d(\bar{t},\bar{x})$ and for any $p>2$. As a result we have obtained (5.20) for all $C^+_{\rho,\tau}\subset C^+_d(\bar{t},\bar{x})$ and for any $p>1$.

From now on by Moser's iterative procedure we shall consider the boundedness of a weak solution u_h of (1.1). Now we fix $(t_{n_0}, x_0)\in C^+_{d/2}(\bar{t},\bar{x})$ and ρ_0, τ_0, $0<\rho_0$, $\tau_0<d/2$. We proceed our inductive calculation similarly as in the proof of Lemma 2.2. Noting a scaling transform (2.22), we obtain, from (5.20), that, putting $\tilde{v}(s,y)=v(t_{n_0}+\rho_0^2 s, x_0+\rho_0 y)$

$$\sup_{0\geqq t\geqq\tilde{\tau}(1-\sigma_2)}\int_{B_{\rho(1-\sigma_1)}(0)}(\tilde{v}^{p/2})^2(t,\cdot)\,dy+\iint_{C^+_{\rho(1-\sigma_1),\tau(1-\sigma_2)}(0)}|D\tilde{v}^{p/2}|^2\,dy\,ds \tag{5.23}$$

$$\leqq\gamma\frac{p(2p-1)}{(p-1)^2}\{((\sigma_1\tilde{\rho})^{-2}+(\sigma_2\tilde{\tau})^{-1})\iint_{C^+_{\tilde{\rho},\tau}(0)}(\tilde{v}^{p/2})^2\,dy\,ds+2^p h^{p\alpha/2}(\sigma_2\tilde{\tau})^{-1}|C^+_{\tilde{\rho},\tilde{\tau}}|\}$$

for $0<\tilde{\rho}<1$, $0<\tilde{\tau}<\theta=\rho_0^{-2}\tau_0$, σ_1, $\sigma_2\in(0,1)$ and any $p>1$, where γ depends only on λ, μ, m and d.

Let's take sequences p_ν, ρ_ν and τ_ν as follows: For $\nu=0,1,\dots$,

$$p_\nu=p(1+2/m)^\nu,\qquad \rho_\nu=1/2+(1/2)^{\nu+1}\quad\text{and}\quad \tau_\nu=\theta(1/2+(1/2)^{\nu+1}).$$

Noting that

$$\frac{p_\nu}{p_\nu-1}<\frac{p}{p-1}$$

and exploiting Sobolev's type inequality (see [9, p. 76]) and (5.23) successively, we have that

$$\iint_{C^+_{\rho_{\nu+1},\tau_{\nu+1}}}(\tilde{v}^{p_\nu/2})^{2(1+2/m)}\,dy\,ds \tag{5.24}$$

$$\leqq\left(\frac{p}{p-1}\right)^2 2^{2(1+2/m)}\beta^{2(1+2/m)}\{\gamma[2^{3(\nu+3)}+\theta^{-1}2^{\nu+2}]$$

$$\cdot\iint_{C^+_{\rho_\nu,\tau_\nu}}(\tilde{v}^{p_\nu/2})^2\,dy\,ds+\theta^{-1}2^{\nu+2}h^{\alpha p_\nu/2}|C^+_{\rho_\nu,\tau_\nu}|$$

$$+2^{\nu+4}\iint_{C^+_{\rho_\nu,\tau_\nu}}(\tilde{v}^{p_\nu/2})^2\,dy\,ds\}^{1+2/m}$$

$$\leqq\beta^{2(1+2/m)}(\max(\gamma,1))^{1+2/m}2^{2(\nu+4)(1+2/m)}\{[1+\theta^{-1}]\iint_{C^+_{\rho_\nu,\tau_\nu}}(\tilde{v}^{p_\nu/2})^2\,dy\,ds$$

$$+\theta^{-1}h^{\alpha p_\nu/2}\iint_{C^+_{\rho_\nu,\tau_\nu}}1\,dy\,ds+\iint_{C^+_{\rho_\nu,\tau_\nu}}(\tilde{v}^{p_\nu/2})^2\,dy\,ds\}^{1+2/m}$$

$$\leqq\beta^{2(1+2/m)}(\max(\gamma,1))^{1+2/m}2^{2(\nu+4)(1+2/m)}(1+\theta^{-1})^{1+2/m}$$

$$\cdot\Big(\iint_{C^+_{\rho_\nu,\tau_\nu}}((\tilde{v}^{p_\nu/2})^2+h^{\alpha p_\nu/2})\,dy\,ds\Big)^{1+2/m}.$$

Since $h<1$, we obtain from (5.24) that, for any $\nu=0, 1, \ldots,$

(5.25)
$$\iint_{C^+_{\rho_{\nu+1},\tau_{\nu+1}}} ((\tilde{v}^{p_\nu})^{2(1+2/m)}+h^{p\alpha/2})\,dy\,ds$$
$$\leqq \beta^{2(1+2/m)}(\max(\gamma, 1))^{1+2/m}\, 2^{2(\nu+4)(1+2/m)}(1+\theta^{-1})^{1+2/m}$$
$$\cdot\Big(\iint_{C^+_{\rho_{\nu+1},\tau_{\nu+1}}} (\tilde{v}^{p_\nu}+h^{p\alpha/2})\,dy\,ds\Big)^{1+2/m}.$$

Dividing the both side of (5.25) by $\rho^m_{\nu+1}\tau_{\nu+1}$, and taking the power of order $1/p_{\nu+1}$ in the resulting inequality, we have, for any $\nu=0, 1, \ldots,$

$$\Big(\rho^{-m}_{\nu+1}\tau^{-1}_{\nu+1}\iint_{C^+_{\rho_{\nu+1},\tau_{\nu+1}}} (\tilde{v}^{p_{\nu+1}}+h^{p\alpha/2})\,dy\,ds\Big)^{1/p_{\nu+1}}$$
$$\leqq [\beta^2\{\max(\gamma, 1)\}\, 2^8]^{p^{-1}(1+2/m)^{-\nu}}\, 4^{p^{-1}\nu(1+2/m)^{-\nu}}(1+\theta)^{p^{-1}(1+2/m)^{-\nu}}$$
$$\cdot(\theta^{2/m})^{p^{-1}(1+2/m)^{-\nu-1}}\Big(\rho_\nu^{-m}\tau_\nu^{-1}\iint_{C^+_{\rho_\nu,\tau_\nu}} (\tilde{v}^{p_\nu}+h^{p\alpha/2})\,dy\,ds\Big)^{1/p_\nu}.$$

By iterating the above inequality infinitely with starting from $\nu=0$ we have

(5.26)
$$\sup_{C^+_{1/2,\theta/2}} v\leqq [\beta^2\{\max(\gamma, 1)\}\, 2^8]^{p^{-1}(1+m/2)}\, 4^{p^{-1}}$$
$$\cdot\sum_{j=0}^{\infty} j\cdot(1+2/m)^{-j}(1+\theta)^{p^{-1}(1+m/2)}\,\theta^p\Big(\rho_0^{-m}\tau_0^{-1}\iint_{C^+_{\rho_0,\tau_0}} (v^p+h^{p\alpha/2})\,dy\,ds\Big)^{1/p}.$$

Now we are in a position to show (1.12). Let's classify our proof into two cases.

Case 1 $r^2>h$. Then, by taking $\rho_0^2=\tau_0=r^2$ and using $r^2>h$ in (5.26), we have (1.12).

Case 2 $r^2\leqq h$. We deduce from applying Harnack theorem on elliptic equations (see [3, Theorem 8.17, p. 194]) for $\tilde{u}_n$ in B_1 that, for any $p>1$ and $q>m$, there exists a positive constant γ depending only on m, q, p, λ and μ such that, setting

$$\tilde{v}_n=\max\{\pm\tilde{u}_n, 0\}\qquad (n=1, 2, \ldots, N),$$

(5.27)
$$\sup_{B_1}\tilde{v}_n\leqq\gamma\left\{\left(\frac{1}{|B_1|}\int_{B_1}(\tilde{v})^p\right)^{1/p}+\left(\frac{1}{|B_1|}\int_{B_1} g^q\right)^{1/q}\right\}$$

holds for $1\leqq n\leqq N$, where

$$g=\frac{\tilde{u}_n-\tilde{u}_{n-1}}{h/r^2}.$$

Thus, noting (5.16) and that $r^2\leqq h$, (1.12) is obtained.

Acknowledgement. I would like to thank Professor K. Kikuchi for drawing my attention to this problem and for his encouragement. His paper [6], which has been known to him for ten years, is the starting point of this paper. I'm also grateful to Professor M. Struwe for his interest and suggestions. Due to them, the original lengthy proof was shortened, the notation simplified.

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