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Titel: Traces of pluriharmonic functions on the boundaries of analytic varieties.

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Jahr: 1993

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0213|log19

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Math. Z. 213, 171–177 (1993)

Mathematische Zeitschrift
© Springer-Verlag 1993

Traces of pluriharmonic functions on the boundaries of analytic varieties

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Received 10 July 1992; in final form 2 November 1992

Let Ω be a bounded strictly pseudoconvex domain of class C^∞ in $\mathbb{C}^2$ and E be an analytic variety properly embedded into Ω such that E is of class C^∞ near $\partial\Omega$ and transversal to $\partial\Omega$. Let $PH^\infty(\Omega)$ be the space of all real-valued functions of class C^∞ on $\bar{\Omega}$, pluriharmonic on Ω. Denote by $PH^\infty_{\partial E}(\Omega)$ the space of all traces on ∂E of functions of $PH^\infty(\Omega)$. We prove that $PH^\infty_{\partial E}(\Omega)$ is a closed subspace of finite codimension in the space $C^\infty(\partial E)$ of all smooth real-valued functions on ∂E (in the case when Ω is a ball it was proved in [BO]) and, moreover, we characterize this codimension in terms of geometrical properties of the variety E. This answering a question in [BO].

Let us recall that the Milnor number of the variety $\{f(z_1, z_2)=0\}$ at the singular point P is the first Betti number of the intersection of a small ball $B_\delta(P)$ with center at the point P, and a complex curve without singularities and with equation $f(z_1, z_2)=t$ for sufficiently small t. This number is equal to the multiplicity of the curve $\{f(z_1, z_2)=0\}$ at the point P.

The main result of this paper is

Theorem. *Let Ω be a bounded strictly pseudoconvex simply-connected domain of class C^∞ in $\mathbb{C}^2$ and E be a connected analytic variety properly embedded into Ω such that E is of class C^∞ near $\partial\Omega$ and transversal to $\partial\Omega$. Let $P_1, P_2, \ldots, P_N$ be the singular points of E and $\mu_1, \mu_2, \ldots, \mu_N$ be the Milnor numbers of E at these points. Then $PH^\infty_{\partial E}(\Omega)$ is a closed subspace in $C^\infty(\partial E)$ of codimension $1-\chi(\bar{E})+\sum_{i=1}^{N}\mu_i$, where $\chi(\bar{E})$ is the Euler characteristic of the variety $\bar{E}$.*

Now, let us introduce some notations. Let K be a compact set and $C(K)$ be the algebra of all complex-valued continuous functions on K. Let A be a subalgebra of $C(K)$. Denote by $\operatorname{Re} A$ the algebra of all real parts of functions of A. Further, for each $K_1 \subset K$ denote by A_{K_1} the traces on K_1 of functions of A.

The first statement which we prove is

Lemma. *Let E be a connected compact 1-dimensional complex manifold with the nonempty boundary ∂E of class C^∞. Let $A^\infty(E)$ be the space of all functions holomorphic on E and of class C^∞ on $\bar{E}$. Then the space $\operatorname{Re} A^\infty_{\partial E}(E)$ is a closed subspace in $C^\infty(\partial E)$ of codimension $1-\chi(\bar{E})$.*

Proof. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_S$ be a basis of smooth closed curves in the first homology group $H_1(E)$ of the manifold E. Denote by $\varphi_1, \varphi_2, \ldots, \varphi_S$ the normal parametrization of these curves and by $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_S$ the normal vector fields: $\mathbf{n}_j = -i\dot{\varphi}_j$, $j=1, 2, \ldots, S$. Let f be a function of $C^\infty(\partial E)$. Consider the solution U of the Dirichlet problem for the Laplace-Beltrami operator on the manifold E with the boundary data f. By Cauchy-Riemann conditions, the function U has a single-valued conjugate function V on E if and only if

$$\int_{\Gamma_j} \frac{\partial U}{\partial \mathbf{n}_j} d\varphi_j = \int_{\Gamma_j} \frac{\partial V}{\partial \varphi_j} d\varphi_j = 0 \tag{1}$$

for $j=1, 2, \ldots, S$. If $f \in \operatorname{Re} A^\infty_{\partial E}(E)$, then $U \in \operatorname{Re} A^\infty(E)$ and so it has a single-valued conjugate function on E. Hence, U satisfies the conditions (1). On the other hand, if U satisfies the conditions (1), then it has a single-valued conjugate function V on E. Since $U \in C^\infty(\bar{E})$, it follows, by Cauchy-Riemann conditions, that $V \in C^\infty(\bar{E})$. Hence, $U + iV \in A^\infty(E)$ and so the trace of U on ∂E belongs to $\operatorname{Re} A^\infty_{\partial E}(E)$. This proves that the function $f \in C^\infty(\partial E)$ belongs to $\operatorname{Re} A^\infty_{\partial E}(E)$ if and only if the corresponding function U satisfies the conditions (1). Those conditions, obviously, given by continuous linearly independent functionals on $C^\infty(\partial E)$. Hence, the space $\operatorname{Re} A^\infty_{\partial E}(E)$ is a closed subspace in $C^\infty(\partial E)$ and its codimension is equal to $\dim H_1(E) = 1 - \chi(\bar{E})$. □

Proof of Theorem. Let $A^\infty(E)$ be the space of all functions on E with the following properties:

1. Each function is holomorphic on the regular part of E.
2. Each function is of class C^∞ to ∂E.
3. Near every singular point P_i of E, each function is the trace on E of a function holomorphic in a neighbourhood of P_i in $\mathbb{C}^2$.

Let us prove that $\operatorname{Re} A^\infty_{\partial E}(E) = PH^\infty_{\partial E}(\Omega)$. Let $f \in PH^\infty_{\partial E}(\Omega)$. Then there is a function $U \in PH^\infty(\Omega)$ such that $U = f$ on ∂E. Since Ω is simply-connected, it follows that U has a single-valued conjugate function V on Ω. Since $U \in C^\infty(\bar{\Omega})$, we conclude, by Cauchy-Riemann conditions, that $V \in C^\infty(\bar{\Omega})$. Hence $U + iV \in A^\infty(\Omega)$ and so the trace of this function on E belongs to $A^\infty(E)$. Therefore, $f \in \operatorname{Re} A^\infty_{\partial E}(E)$. Let us prove now the opposite including. Let $f \in \operatorname{Re} A^\infty_{\partial E}(E)$, this means that f is the trace on ∂E of the real part of a function $F \in A^\infty(E)$. Then, by Amar's theorem [A1], there is an extension $\tilde{F} \in A^\infty(\Omega)$ of the function F. Hence, f is the trace on ∂E of the pluriharmonic function $\operatorname{Re} \tilde{F}$. This proves that $\operatorname{Re} A^\infty_{\partial E}(E) = PH^\infty_{\partial E}(\Omega)$.

Now we prove that the space $\operatorname{Re} A^\infty_{\partial E}(E)$ is a closed subspace in $C^\infty(\partial E)$ of codimension

$$1 - \chi(\bar{E}) + \sum_{i=1}^{N} \mu_i.$$

Consider the normalization E^* of the variety E. This means that we have a smooth complex variety E^*, nonconnected in general, a holomorphic map $\pi: E^* \to E$ and a finite number of points $Q_1^1, Q_1^2, \ldots, Q_1^{m_1}, \ldots, Q_N^1, Q_N^2, \ldots, Q_N^{m_N}$ on E^* such that $\pi(Q_i^j) = P_i$ for $j=1, 2, \ldots, m_i$; $i=1, 2, \ldots, N$ and the map π is a biholomorphism of

$$E^* \setminus \bigcup_{\substack{1 \le j \le m_i \\ 1 \le i \le N}} Q_i^j \quad \text{on} \quad E \setminus \bigcup_{1 \le i \le N} P_i.$$

Let us denote by O_i^j $(j=1, 2, \ldots, m_i;\ i=1, 2, \ldots, N)$ the ring of germs of functions holomorphic in a neighbourhood of the point Q_i^j in the variety E^* and by O_i $(i=1, 2, \ldots, N)$ the ring of germs of traces on the variety E of functions holomorphic in a neighbourhood of the point P_i in $\mathbb{C}^2$. Then the map π gives us the homomorphisms $(\pi_i^j)^*: O_i \to O_i^j$ $(j=1, 2, \ldots, m_i;\ i=1, 2, \ldots, N)$. Denote by π_i^* the homomorphism

$$\bigoplus_{j=1}^{m_i} (\pi_i^j)^*: O_i \to \bigoplus_{j=1}^{m_i} O_i^j.$$

By Gusein-Zade's theorem [G], the real codimension of

$$\pi_i^*(O_i) \quad \text{in} \quad \bigoplus_{j=1}^{m_i} O_i^j$$

is equal to $\mu_i + (m_i - 1)$. By Theorem 4F of [W], there exist some function $u_i \in O_i$ (it calls a universal denominator) such that for each function h, holomorphic in the regular part of E and bounded near P_i, one has $u_i h \in O_i$. Since the function $\pi_i^*(u_i)$ has only a finite number of zero derivatives at the points $Q_i^1, Q_i^2, \ldots, Q_i^{m_i}$, we conclude that there is a natural number p_i such that each holomorphic on E^* function, which has zero derivatives of all degrees less then p_i at each point Q_i^j $(j=1, 2, \ldots, m_i)$, belongs to $\pi_i^*(O_i)$. Hence, in a local coordinates on E^* near the points Q_i^j $(j=1, 2, \ldots, m_i)$, the subspace $\pi_i^*(O_i)$ of the space $\bigoplus_{j=1}^{m_i} O_i^j$ can be defined by $\frac{1}{2}(\mu_i + (m_i - 1))$ independent complex linear equations on the first $m_i\, p_i$ Taylor coefficients, p_i coefficients at each point Q_i^j $(j=1, 2, \ldots, m_i)$.

Let $A^\infty(E^*)$ be the space of all functions holomorphic on E^* and of class C^∞ to ∂E^*. Since the linear equations on the Taylor coefficients are, obviously, independent for different numbers i, we conclude that $\pi^*(A^\infty(E))$ is a closed subspace in $A^\infty(E^*)$ and its codimension is equal to

$$\sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} (m_i - 1).$$

Let $E_1^*, E_2^*, \ldots, E_K^*$ be the connected components of the set E^*. Since the variety E is connected, it follows that $A^\infty(E) \cong \operatorname{Re} A^\infty(E) \oplus \mathbb{R}$, i.e. the space $A^\infty(E)$ is naturally isomorphic to the space $\operatorname{Re} A^\infty(E) \oplus \mathbb{R}$. Since $\operatorname{Re} A^\infty(E) \cong \operatorname{Re} A_{\partial E}^\infty(E)$, we conclude that $A^\infty(E) \cong \operatorname{Re} A_{\partial E}^\infty(E) \oplus \mathbb{R}$. By the same arguments,

$$A^\infty(E^*) \cong \bigoplus_{j=1}^{K} \operatorname{Re} A_{\partial E_j^*}^\infty(E_j^*) \bigoplus \mathbb{R}^K.$$

Since $\pi^*(A^\infty(E))$ is a closed subspace in $A^\infty(E^*)$ of codimension

$$\sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} (m_i - 1),$$

it follows that $\pi^*(\operatorname{Re} A^\infty_{\partial E}(E))$ is a closed subspace in

$$\bigoplus_{j=1}^{K} \operatorname{Re} A^\infty_{\partial E_j^*}(E_j^*)$$

of codimension

$$\sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} (m_i - 1) - (K-1).$$

By Lemma, $\operatorname{Re} A^\infty_{\partial E_j^*}(E_j^*)$ is a closed subspace in $C^\infty(\partial E_j^*)$ of codimension $1-\chi(\bar{E}_j^*)$, for each $j=1, 2, \ldots, K$. Hence, $\pi^*(\operatorname{Re} A^\infty_{\partial E}(E))$ is a closed subspace in

$$\bigoplus_{j=1}^{K} C^\infty(\partial E_j^*)$$

of codimension

$$\sum_{j=1}^{K} (1-\chi(\bar{E}_j^*)) + \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} (m_i - 1) - (K-1).$$

Since $\bigoplus_{j=1}^{K} C^\infty(\partial E_j^*) = \pi^*(C^\infty(\partial E))$, we conclude that $\operatorname{Re} A^\infty_{\partial E}(E)$ is a closed subspace in $C^\infty(\partial E)$ of codimension

$$\sum_{j=1}^{K} (1-\chi(\bar{E}_j^*)) + \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} (m_i - 1) - (K-1).$$

Since the map π "glue" m_i points $Q_i^1, Q_i^2, \ldots, Q_i^{m_i}$ at one point P_i for each $i=1, 2, \ldots, N$, it follows that

$$\sum_{j=1}^{K} \chi(\bar{E}_j^*) - \sum_{i=1}^{N} (m_i - 1) = \chi(\bar{E}).$$

Therefore, the codimension of $\operatorname{Re} A^\infty_{\partial E}(E)$ in $C^\infty(\partial E)$ is equal to

$$K - \sum_{j=1}^{K} \chi(\bar{E}_j^*) + \sum_{i=1}^{N} (m_i - 1) + \sum_{i=1}^{N} \mu_i - (K-1) = 1 - \chi(\bar{E}) + \sum_{i=1}^{N} \mu_i.$$

Since, as we proved above, $\operatorname{Re} A^\infty_{\partial E}(E) = PH^\infty_{\partial E}(\Omega)$, we conclude that $PH^\infty_{\partial E}(\Omega)$ is a closed subspace in $C^\infty(\partial E)$ of codimension

$$1 - \chi(\bar{E}) + \sum_{i=1}^{N} \mu_i.$$

This proves the theorem. $\square$

Remark 1 Using the same arguments one can prove that in the case when the variety E is not connected and has p connected components, the space $PH^{\infty}_{\partial E}(\Omega)$ is also a closed subspace in $C^{\infty}(\partial E)$ and its codimension is equal to

$$p-\chi(\bar{E})+\sum_{i=1}^{N}\mu_i.$$

Now let us define the Bennequin number of a link. Recall that a link in the boundary ∂B of the unit ball B in $\mathbb{C}^2$ is a finite set of embedded oriented circles that do not intersect in ∂B. Let γ be a link in ∂B which is transversal to the distribution $\{T^{\mathbb{C}}_P(\partial B)\}$ of complex tangent spaces to ∂B. Let us define an orientation on ∂B by the natural orientation on the complex tangent space $T^{\mathbb{C}}_{(z_1,z_2)}(\partial B)$ and by the vector (iz_1, iz_2) of oriented normal to $T^{\mathbb{C}}_{(z_1,z_2)}(\partial B)$ in $T_{(z_1,z_2)}(\partial B)$. Consider the vector field $v(z_1,z_2)=(i\bar{z}_2, -i\bar{z}_1)$ on ∂B. Since the vector $v(z_1,z_2)$ is orthogonal to (z_1,z_2) and (iz_1, iz_2), it lies in the space $T^{\mathbb{C}}_{(z_1,z_2)}(\partial B)$ at the point $(z_1,z_2)\in\partial B$. Since the link γ is transversal to the distribution $\{T^{\mathbb{C}}_p(\partial B)\}$, we can consider the link γ' obtained by a small displacement of γ in ∂B along the vector field equal to $v(z_1,z_2)$ at the point (z_1,z_2). By Seifert's theorem (see [Rol]), each link in ∂B is the oriented boundary of a compact oriented surface embedded into ∂B. Hence, one can define the link number $\ell(\gamma)$ of the links γ and γ' as the algebraic sum of the intersections of γ' with the Seifert surface of the link γ. Note, that this number does not depend of the choice of the complex tangential vector field v and of the choice of the Seifert surface. The number $\ell(\gamma)$ is called the Bennequin number of the curve γ. More complete information about the Bennequin number can be found in [B].

Proposition. *Let E be a connected analytic variety properly embedded into the unit ball B in $\mathbb{C}^2$. Suppose that E is of class C^{∞} near ∂B and transversal to ∂B. Let $P_1, P_2, \ldots, P_N$ be the singular points of E and $\mu_1, \mu_2, \ldots, \mu_N$ be the Milnor numbers of E at these points. Take the natural orientation on the boundary ∂E of the variety E. Then*

$$\ell(\partial E)=-\chi(\bar{E})+\sum_{i=1}^{N}\mu_i.$$

Proof. Using Amar's theorem [A2], we can construct a function $\Phi\in A^{\infty}(B)$ such that $E=\{z\in\bar{B}: \Phi(z)=0\}$. Then, for some sequence $\varepsilon_n\to 0$ the sets $E_n=\{z\in\bar{B}: \Phi(z)=\varepsilon_n\}$ are smooth complex curves in B. Since, by Proposition 4 of paragraph II of [B], $\ell(\partial E_n)=-\chi(\bar{E}_n)$ for smooth complex curves E_n, and since, by stability of link number under small perturbations of links, $\ell(\partial E)=\ell(\partial E_n)$ for n big enough, it follows that $\ell(\partial E)=-\chi(\bar{E}_n)$ for n big enough. Since, by the definition of the Milnor numbers,

$$-\chi(\bar{E})+\sum_{i=1}^{N}\mu_i=-\chi(\bar{E}_n),$$

we conclude that

$$\ell(\partial E)=-\chi(\bar{E})+\sum_{i=1}^{N}\mu_i. \quad \square$$

Now let us consider a finite set γ of circles embedded into ∂B, which do not intersect in ∂B and such that the polynomial hull $\hat{\gamma}$ of γ is connected. Then, by Stolzenberg's theorem [S], $\hat{\gamma}\setminus\gamma$ is a 1-dimensional variety. By Chirka's theorem [C] (see also [F] and [Ros]), $\hat{\gamma}$ is smooth near ∂B and transversal to ∂B. Hence, γ is transversal to the distribution $\{T_P^{\mathbb{C}}(\partial B)\}$ of complex tangent spaces to ∂B. Therefore, if we take an orientation on γ as on the boundary of the variety $\hat{\gamma}\setminus\gamma$, one can define the Bennequin number $\ell(\gamma)$ of the link γ. Then, by Theorem and Proposition, we have

Corollary 1 *Let B be the unit ball in $\mathbb{C}^2$. Let γ be a finite set of circles embedded into ∂B of class C^∞ that do not intersect in ∂B and such that the polynomial hull $\hat{\gamma}$ of γ is nontrivial and connected. Then the space $PH_\gamma^\infty(B)$ is a closed subspace in $C^\infty(\gamma)$ of codimension $1+\ell(\gamma)$.*

Remark 2 Eroshkin [E] found the necessary topological condition on a compact set $K\subset\partial B\subset\mathbb{C}^2$ to have the connected polynomial hull $\hat{K}$:
if $\hat{K}$ is connected, then $\pi_2(\partial B\setminus K)=0$.

Remark 3 Since irreducible analytic varieties are connected, one can check the existence of the connected polynomial hull $\hat{\gamma}$ of γ using the moment conditions:

$\hat{\gamma}\setminus\gamma$ is a nonempty irreducible analytic variety if and only if for each polynomials $P(z_1,z_2)$ and $Q(z_1,z_2)$ we have

$$\int_\gamma (P(z_1,z_2)\,dz_1+Q(z_1,z_2)\,dz_2)=0$$

and these conditions do not hold for any proper subset of smooth closed curves γ' in γ.

Since the polynomial hull $\hat{\gamma}$ is connected for a connected curve γ, one has:

Corollary 2 *Let γ be a simple closed curve of class C^∞ on the boundary ∂B of the unit ball B in $\mathbb{C}^2$. If the curve γ is not polynomially convex, then the space $PH_\gamma^\infty(B)$ is a closed subspace in $C^\infty(\gamma)$ and its codimension is equal to $1+\ell(\gamma)$.*

Let $PH^C(B)$ be the space of all functions continuous on the closed unit ball $\bar{B}$ in $\mathbb{C}^2$ and pluriharmonic on the open ball B. Bruna and Ortega [BO] proved that for a smooth simple closed curve $\gamma\subset\partial B$ transversal to $\{T_P^{\mathbb{C}}(\partial B)\}$, the space $PH_\gamma^C(B)$ is a closed subspace in $C(\gamma)$ of finite codimension (see also [BB]). They also asked the question of a characterization of this codimension in terms of the curve γ (see [BO, p. 547]). Since $PH_\gamma^\infty(B)$ is everywhere dense in $PH_\gamma^C(B)$, and $C^\infty(\gamma)$ is everywhere dense in $C(\gamma)$, we can give an answer on this question:

Corollary 3 *Let γ be a simple closed curve of class C^∞ on the boundary ∂B of the unit ball B in $\mathbb{C}^2$. Then, the codimension of $PH_\gamma^C(B)$ in $C(\gamma)$ is equal to zero, if γ is polynomially convex, and is equal to $1+\ell(\gamma)$, if γ is not polynomially convex.*

Acknowledgments. Part of this paper was done while I visited the Centre de Recerca Matematica (Barcelona) and it was finished while I visited the Université de Provence (Marseille). I want to thank the host institutions for their hospitality and support. I want to thank also Professor J. Bruna, who proposed to me this problem, and Professors J. Detraz and B. Coupet for all their help.

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