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# On the homotopy embeddability of complexes in Euclidean space

## I. The weak thickening theorem

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## 1 Introduction

*1.1 Statement of the main result.* Let $K$ be a finite CW complex. By a *thickening* or *homotopy embedding* of $K$ in the sphere $S^j$, we mean a compact codimension zero submanifold $N$ of $S^j$ together with a homotopy equivalence $h: K \xrightarrow{\simeq} N$. An old result of Stallings ([St]; for a published proof see Wall [$W_1$]) asserts

**Theorem 1.2** (Stallings' embedding theorem) *Suppose that $K$ is homology $k$-dimensional and $r$-connected, with $r \geqq 1$ and $k \leqq j-3$. Then $K$ thickens in $S^j$ if $j \geqq 2k-r$, and moreover, any two such thickenings are unique up to concordance if $j \geqq 2k-r+1$.*

The Stallings/Wall proof is inductive in the sense that the thickening of $K$ is constructed cell-by-cell, at each step appealing to a geometric result of Hudson [Hu], which allows one to embed disks inside the double-point range (= the numerical range where a map in general position has no triple points).

In this paper, a new approach to the thickening problem is presented that avoids the application of Hudson's disk theorem. The idea stems from the observation that a thickening of $K$ in $S^j$ determines

(1) an exterior $C$, and
(2) a boundary $\partial K$ for $K$ as a Poincaré pair of dimension $j$.

Regarding (1), the connectivity assumptions of Stallings' theorem imply that $C$ is determined by the "stable" exterior of $K$ (the $S$-dual), obtained from embedding $K$ in a sphere of large dimension using general position and then deleting a tubular neighborhood. As for (2), the uniqueness part of Stallings' theorem says that the boundary of a tubular neighborhood of the thickening is unique up to diffeomorphism (in the given numerical range). It occurred to the author that there should then be an explicit recipe for constructing this boundary as a Poincaré complex in terms of $K$, $C$, and the Spanier-Whitehead duality between

$K$ and $C$.[1] Once this has been established, the data (1) and (2) amount to having a Poincaré embedding of $(K, \partial K)$ in the sphere. One can then apply the Browder-Casson-Sullivan-Wall theorem [$W_2$, 12.1] to smoothen the Poincaré pair and ultimately obtain an actual thickening.

In this paper we shall furnish an elementary proof of the existence part of Stallings' result under the following extra hypotheses:

(1) $r \geqq 2k-j+1$ (connectivity > generic double-point dimension), and
(2) $k \leqq 2r+1$ ($K$ generically a co-$H$ space).

Our approach is part of a general method which is seen to have the following features:

(1) It is homotopy theoretic in flavor. The only manifold theory we will make use of are the results of simply connected surgery to smooth certain Poincaré complex data. Our method presents the thickening question as a issue in homotopy theory, which we call *the Poincaré thickening problem.*
(2) It holds the promise of settling the thickening problem beyond the known range. For example, in his thesis, Habegger [Ha] exhibited obstructions to thickening (depending on the homotopy type of the space to be embedded) in one dimension better than the double-point range. However, Habegger's obstruction is defined using a local formula obtained from the triangulation of a polyhedron which is homotopy equivalent to the complex $K$. He then shows that the obstruction is independent of a choice the triangulation used. We shall show in a future paper how to define the obstruction without recourse to either a triangulation or a local formula.
(3) The constructions appearing in the paper yield direct applications outside of the double-point range in special circumstances (See § 7).

*1.3 Outline.* The paper is organized as follows: In §2 we define the notion of Poincaré thickening and construct top cell decompositions of closed 1-connected Poincaré complexes. In §3 we show how a diagram whose homotopy pushout is a sphere functorially gives rise to a certain map; this map is an $S$-duality map if and only if the diagram is a Poincaré thickening (3.3). In §4 we define the notion of $^\flat$-duality and show that to every $^\flat$-duality one may construct a Poincaré thickening (4.5). In §5 we prove the weak Poincaré thickening theorem. In §6 we apply the Browder-Casson-Sullivan-Wall theorem to show that the Poincaré theory is equivalent to the smooth theory, provided that the ambient dimension is $\geqq 6$. In §7 we employ the techniques of § 1–6 to give applications to thickenings of suspensions, embeddings of closed Poincaré complexes, and self dual complexes.

*1.4 Conventions.* We work within the category of spaces which are the homotopy type of a finite CW complex. Unless otherwise stated, *all* spaces are assumed to be simply connected. We say that a space is *homology k-dimensional* if its singular homology vanishes above dimension $k$. If $X$ is a space then $\operatorname{cone}(X)=$

[1] It seems that I am not first to have thought of this. Connolly and Williams [C-W] show that the only invariants for thickenings of $K$ in $S^j$ are the exterior and the $S$-duality pairing in the range $\dim(K)-\operatorname{conn}(K) \leqq [j/2]$. However, their methods are almost entirely geometric, relying heavily on PL-handlebody theory. In a future paper, I will show how to deduce this result from an extension of the methods appearing here. I would like to thank the referee for calling to my attention the Connolly-Williams paper

$X \times I/X \times 1$ denotes the unreduced cone on $X$ and $\Sigma X = \mathrm{cone}(X) \cup_{X \times 0} \mathrm{cone}(X)$ denotes the unreduced suspension. For spaces $X$ and $Y$ with basepoints denoted $*$, we let $X \vee Y = X \times * \cup * \times Y \subset X \times Y$ denote the *wedge* of $X$ and $Y$, and the quotient, $X \wedge Y = X \times Y/X \vee Y$ the *smash product* of $X$ and $Y$. If $X$ is pointed, then there are natural equivalences $\Sigma X \simeq X \wedge S^1 \simeq X \times I/\sim$, where the last term is the quotient space of $X \times I$ given by identifying $(x, 0)$ to $(y, 0)$ and $(x, 1)$ to $(y, 1)$ for all $x, y \in X$.

## 2 Poincaré thickenings

*2.1 Poincaré boundaries.* By a *Poincaré boundary* (of dimension $j$) for a space $K$, we mean an oriented $(j-1)$-dimensional Poincaré complex $\Delta$ and a map $i_K\colon \Delta \to K$ such that $(M_{i_K}, \Delta)$ is a Poincaré pair, where $M_{i_K} = K \cup_{i_K} \Delta \times I$ is the mapping cylinder of $i_K$. To keep the notation simple, when $i_K$ is understood we write $(K, \Delta)$ for the triple $(K, \Delta, i_K)$, and abuse terminology by calling $(K, \Delta)$ a "Poincaré pair".

By a *Poincaré thickening* of $K$ in the sphere $S^j$, we mean,

(1) a Poincaré boundary $i_K\colon \Delta \to K$ for $K$ such that $(K, \Delta)$ has dimension $j$;
(2) a cofibration $i_C\colon \Delta \subset C$;
(3) a homotopy equivalence $S^j \xrightarrow{\simeq} C \cup_\Delta K$ which we require to be degree one, in the sense that the composite

$$S^j \xrightarrow{\simeq} C \cup_\Delta K \subset \mathrm{cone}(K) \cup_\Delta \mathrm{cone}(C) \simeq \Sigma \Delta$$

is degree one.

For convenience we refer to the data as a homotopy co-Cartesian diagram

$$(\mathscr{D}) \qquad \begin{tikzcd} \Delta \arrow[r, "i_C"] \arrow[d, "i_K"'] & C \arrow[d] \\ K \arrow[r] & S^j. \end{tikzcd}$$

The definition is symmetric with respect to $K$ and $C$:

**Lemma 2.2** *If $\mathscr{D}$ is a Poincaré thickening of $K$ in $S^j$, then $(C, \Delta)$ is a Poincaré pair of dimension $j$ also.*

*Proof.* This follows from the five lemma by applying the cap product of fundamental classes to Mayer-Vietoris sequence of $(K, C, \Delta)$. □

Two Poincaré thickenings $\mathscr{D}_1$ and $\mathscr{D}_2$ of $K$ are said to be *concordant* if there is a degree one homotopy equivalence mapping the diagram associated with $\mathscr{D}_1$ to the diagram associated with $\mathscr{D}_2$.

*2.3 Top cell decompositions.* Suppose $\Delta$ is a 1-connected, oriented Poincaré complex of dimension $n$. By a *top cell decomposition for $\Delta$*, we mean

(1) a CW complex $A$ of dimension $\leqq n-2$;

(2) a map $\alpha\colon S^{n-1} \to A$;

(3) a degree one homotopy equivalence $h\colon A \cup_\alpha D^n \xrightarrow{\simeq} \Delta$. (Here $D^n$ is provided with a fixed orientation.)

Two top cell decompositions $(A, \alpha, h)$ and $(A', \alpha', h')$ for $\Delta$ are said to be *concordant* if there is a homotopy equivalence $g\colon A \cup_\alpha D^n \xrightarrow{\simeq} A' \cup_{\alpha'} D^n$ such that $h' \circ g$ is homotopic to $h$.

**Proposition 2.4** *If $\Delta$ is a 1-connected oriented Poincaré duality space of dimension $n$, then there exists a top cell decomposition for $\Delta$. Moreover, any two top cell decompositions for $\Delta$ are concordant.*

*Proof.* The uniqueness part of the proposition follows directly from obstruction theory, so it suffices to prove existence. This follows essentially from [$W_2$, 2.9], but since we are assuming that $\Delta$ is 1-connected, we provide a simpler proof.

Without any loss in generality one may assume that $\Delta$ is a CW complex of dimension $n$ (see e.g. [$W_3$]). Let $\Delta^{(i)}$ denote the $i$-skeleton of $\Delta$. Note that by duality, $H_{n-1}(\Delta) \cong H^1(\Delta) = 0$. Consider the homology long exact sequence of the pair $(\Delta, \Delta^{(n-1)})$. In the top dimensions it degenerates into a short exact sequence,

$$0 \to H_n(\Delta) \xrightarrow{\pi_*} H_n(\Delta, \Delta^{(n-1)}) \xrightarrow{\partial} H_{n-1}(\Delta^{(n-1)}) \to 0.$$

Furthermore, $H_{n-1}(\Delta^{(n-1)})$ is a free abelian group because it is the kernel of the $(n-1)$-st boundary operator in the cellular chain complex of $\Delta^{(n-1)}$. It follows that every element of $H_{n-1}(\Delta^{(n-1)})$ lifts up to a representative $(D^n, S^{n-1}) \to (\Delta, \Delta^{(n-1)})$ in $H_n(\Delta, \Delta^{(n-1)})$. Choosing a basis $\{[\chi_\alpha]\}$ for $H_{n-1}(\Delta^{(n-1)})$, we in this way obtain lifted representatives $\chi_\alpha\colon (D^n, S^{n-1}) \to (\Delta, \Delta^{(n-1)})$. Let $X$ be the space which is obtained from $\Delta^{(n-1)}$ by attaching $n$-cells along the maps $\chi_\alpha|_{S^{n-1}}\colon S^{n-1} \to \Delta^{(n-1)}$. Then $\Delta^{(n-1)} \subset X \subset \Delta$, $H^*(X) \cong H^*(\Delta)$ if $* \leqq n-1$, and $H_n(X) = 0$. Let $[\Delta] \in H_n(\Delta)$ be the orientation class. Then $\pi_*([\Delta])$ is represented by a map $\chi_\Delta\colon (D^n, S^{n-1}) \to (\Delta, \Delta^{(n-1)})$. Let $\beta\colon S^{n-1} \to X$ be the composite

$$S^{n-1} \xrightarrow{\chi_\Delta|_{S^{n-1}}} \Delta^{(n-1)} \subset X.$$

Then $X \cup_\beta D^n$ is homotopy equivalent to $\Delta$. To finish the proof, notice that $X$ is 1-connected and $H^*(X) = 0$ for $* \geqq n-1$, so we may replace $X$ by a CW complex $A$ whose dimension is less than or equal to $n-2$ [$W_3$]. $\square$

## 3 The $S$-duality construction

*3.1* Suppose we are given a homotopy co-Cartesian diagram,

$$(\mathscr{D}) \qquad \begin{tikzcd} \Delta \arrow[r, "i_C"] \arrow[d, "i_K"'] & C \arrow[d] \\ K \arrow[r] & S^j \end{tikzcd}$$

in which, by taking mapping cylinders if necessary, we shall always assume that the maps $i_K$ and $i_C$ are cofibrations (we do not assume here that $\mathscr{D}$ is a Poincaré thickening). We will define a map

$$d\colon S^j \to K * C,$$

where $K * C$ is the *topological join* of $K$ and $C$.

First recall the definition of the join $K * C$. This is the space whose points are of the form $sk+(1-s)c$ where $(k, c)\in K \times C$ and $s\in I$. We topologize this as a quotient space of $K \times C \times I$. There are two other well-known models for the join up to homotopy equivalence. The first is

$$\operatorname{hocolim}(K \xleftarrow{p_1} K \times C \xrightarrow{p_2} C) := \operatorname{cone}(K) \times C \cup_{K \times C} K \times \operatorname{cone}(C),$$

and the other is just $\Sigma K \wedge C$ (here we have chosen basepoints).

**Lemma 3.2** *There are natural homotopy equivalences*

$$F\colon K * C \xrightarrow{\simeq} \operatorname{hocolim}(K \leftarrow K \times C \rightarrow C),$$

*and*

$$G\colon K * C \xrightarrow{\simeq} \Sigma K \wedge C.$$

*Proof.* The map $F$ is defined by the formula

$$F(s\,k+(1-s)\,c)=\begin{cases}(k \wedge (1-2s), c) & s\in[0, 1/2]\\ (k, c \wedge (2s-1)) & s\in[1/2, 1].\end{cases}$$

The map $G$ is defined by the formula,

$$G(s\,k+(1-s)\,c)=k \wedge c \wedge s.$$

Let $X_{\leqq 1/2}$ (resp. $X_{\geqq 1/2}$) be the subspace of the join consisting of the points $s\,k+(1-s)\,c$ with $0 \leqq s \leqq 1/2$ (resp. $0 \leqq s \leqq 1/2$). Then

$$X_{\leqq 1/2} \cap X_{\geqq 1/2} = K \times C \times 1/2,$$

and

$$X_{\leqq 1/2} \cup X_{\geqq 1/2} = K * C.$$

Furthermore, the map $F\colon K * C \to \operatorname{cone}(K) \times C \cup_{K \times C} K \times \operatorname{cone}(C)$ when restricted to $X_{\leqq 1/2}$ is a homeomorphism onto $\operatorname{cone}(K) \times C$ and when restricted to $X_{\geqq 1/2}$ is a homeomorphism onto $K \times \operatorname{cone}(C)$. This shows that $F$ is a homeomorphism.

To show that $G\colon K * C \to \Sigma K \wedge C$ is a homotopy equivalence, note that $G$ is homotopic to the composition $H \circ F$ where,

$$H\colon \operatorname{hocolim}(K \leftarrow K \times C \rightarrow C) \rightarrow \operatorname{hocolim}(* \leftarrow K \wedge C \rightarrow *) \simeq \Sigma K \wedge C$$

is the natural map induced by the quotient map $K \times C \to K \wedge C$. Then $H$ fits into the cofibration sequence,

$$\operatorname{hocolim}(K \leftarrow K \vee C \rightarrow C) \rightarrow \operatorname{hocolim}(K \leftarrow K \times C \rightarrow C) \rightarrow \operatorname{hocolim}(* \leftarrow K \wedge C \rightarrow *),$$

in which the first term is obviously contractible, so $H$ is a homotopy equivalence. Consequently, $G \simeq H \circ F$ is an equivalence. □

Now consider the map $a: \Delta \to K \times C$ given by $a(x) = (i_K(x), i_C(x))$. This gives a diagram:

$$\begin{tikzcd} K \arrow[d, equal] & \Delta \arrow[l, "i_K"'] \arrow[r, "i_C"] \arrow[d, "a"] & C \arrow[d, equal] \\ K & K \times C \arrow[l] \arrow[r] & C, \end{tikzcd}$$

and hence a map of homotopy co-limits, $d: S^j \simeq K \cup_\Delta C \to K * C \simeq \Sigma K \wedge C$.

**Proposition 3.3** *If the map $d: S^j \to \Sigma K \wedge C$ is an S-duality, then $(K, \Delta)$ is a Poincaré pair of dimension $j-1$, and therefore the diagram*

$$(\mathscr{D}) \qquad \begin{tikzcd} \Delta \arrow[r, "i_C"] \arrow[d, "i_K"'] & C \arrow[d] \\ K \arrow[r] & S^j \end{tikzcd}$$

*is a Poincaré thickening in $S^j$.*

*Proof.* Let $v: S^j \to K/\Delta$ be the degree one map defined by

$$S^j \simeq K \cup_\Delta C \to (K \cup_\Delta C)/C = K/\Delta.$$

If $\delta: K/\Delta \to K_+ \wedge K/\Delta$ denotes the diagonal map (where $K_+$ is $K$ with a disjoint basepoint added), it will be sufficient to prove that $\delta \circ v: S^j \to K_+ \wedge K/\Delta$ is an $S$-duality (this will establish Poincaré duality).

Let $p: K/\Delta \to \Sigma C$ be the composite

$$K/\Delta = (K \cup_\Delta C)/C \subset (K \cup_\Delta C/C) \cup \mathrm{cone}(K \cup_\Delta C) \simeq \Sigma C,$$

and let $q: K \to K_+$ be the inclusion. Then the following diagram is commutative:

$$\begin{tikzcd} S^j \arrow[r, "\delta \circ v"] \arrow[d, "d"'] & K_+ \wedge K/\Delta \arrow[d, "\mathrm{id} \wedge p"] \\ K \wedge \Sigma C \arrow[r, "q \wedge \mathrm{id}"'] & K_+ \wedge \Sigma C. \end{tikzcd}$$

Hence, applying the slant pairing and cohomology, we get a commutative diagram

$$\begin{tikzcd} H^*(K_+) \arrow[r, "(- \wedge \mathrm{id}_{K/\Delta}) \circ \delta \circ v"] \arrow[d, "q^*"'] & \tilde{H}_{j-*}(K/\Delta) \arrow[d, "p_*"] \\ H^*(K) \arrow[r, "(- \wedge \mathrm{id}_{\Sigma C}) \circ d"'] & \tilde{H}_{j-*}(\Sigma C). \end{tikzcd}$$

By hypothesis, the map $d\colon S^j \to \Sigma K \wedge C$ is an $S$-duality; implying that the bottom arrow of this diagram is an isomorphism in all dimensions. Since $q\colon K \to K_+$ induces an isomorphism on cohomology in positive dimensions, and since $p$ induces an isomorphism on homology below dimension $j$, it follows that the top arrow of the diagram is an isomorphism for $* \geqq 0$. But when $* = 0$, the top arrow is an isomorphism because $v\colon S^j \to K/\Delta$ is degree one. Consequently,

$$H^*(K_+) \xrightarrow{(-\wedge \mathrm{id}_{K/\Delta})\circ\delta\circ v} \tilde{H}_{j-*}(K/\Delta)$$

is an isomorphism in every dimension. Hence, $\delta \circ v\colon S^j \to K_+ \wedge K/\Delta$ is a duality map and we infer that $(K, \Delta)$ is a Poincaré pair of dimension $j$. □

## 4 $^b$-duality

*4.1 The $^b$-product.* Suppose that $K$ and $C$ are pointed spaces. We first recall the computation of the homotopy fibre of the inclusion $K \vee C \subset K \times C$. In the literature the fibre is denoted as $K^b C$ and is called the *$^b$-product* of $K$ and $C$ (cf. [$G_1$, Hi]). $K^b C$ is naturally equivalent to the space $\Omega K * \Omega C$ (the symbol $\Omega$ means the space of pointed loops), and we shall equate the two spaces whenever necessary. The inclusion of the fibre $i\colon K^b C \to K \vee C$ can be described as follows: Let $PK$, $PC$ be the based path spaces of $K$, $C$ respectively, and let $K^I$, $C^I$ be the corresponding spaces of unbased paths. There is then a natural commutative diagram of inclusions

$$\begin{tikzcd}
PK \times \Omega C \arrow[d, "\cap"'] & \Omega K \times \Omega C \arrow[l, "\supset"'] \arrow[r, "\subset"] \arrow[d, "\cap"'] & \Omega K \times PC \arrow[d, "\cap"] \\
K^I \times PC \arrow[r, "\supset"] & PK \times PC \arrow[r, "\subset"] & PK \times C^I.
\end{tikzcd}$$

The pushout of the top line of the diagram is naturally equivalent to $\Omega K * \Omega C$ by 3.2, the pushout of the bottom line is naturally homotopy equivalent to $K \vee C$, and the map $i\colon \Omega K * \Omega C \to K \vee C$ is defined to be the induced map of pushouts.

There is also a natural map $F\colon \Omega K * \Omega C \to \Omega(K \wedge C)$ defined by

$$F(s\theta + (1-s)\,y) = t \mapsto \begin{cases} \theta(2st) \wedge \psi(t), & t \in [0, 1/2] \\ \theta(t) \wedge \psi(2(1-s)\,t), & t \in [1/2, 1]. \end{cases}$$

We will make use of the following:

**Lemma 4.2** *Suppose that $K$ is $r$-connected and $C$ is $s$-connected. Then the connectivity of the map $F\colon K^b C \to \Omega(K \wedge C)$ is given by*

$$\operatorname{conn}(F) = \min(r, s) + r + s + 1.$$

*Proof.* The cofibration

$$K \vee C \to K \times C \to K \wedge C$$

is also a fibration in the range $\leqq \operatorname{conn}(K \vee C)+\operatorname{conn}(K \wedge C)=\min(r,s)+r+s+1$ by the Blakers-Massey excision theorem. Furthermore, $F\colon K^\flat C \to \Omega(K \wedge C)$ is the homotopy transgression of this cofibration, so the lemma follows. $\square$

Suppose we are given a CW complex $\Delta$ of the form $A \cup_\alpha D^{j-1}$ where $A$ is a 1-connected CW complex of dimension $\leqq n-2$. Suppose further that we are given a homotopy co-Cartesian diagram

$$(\mathscr{D}) \qquad \begin{array}{ccc} \Delta & \xrightarrow{i_C} & C \\ {\scriptstyle i_K}\downarrow & & \downarrow \\ K & \longrightarrow & S^j \end{array}$$

where $K$ and $C$ have homology dimension $\leqq j-2$, and where $i_K$ and $i_C$ are cofibrations. Let $a\colon \Delta \to K \times C$ be the map $(i_K, i_C)$. Let $F$ be the homotopy fibre of the restriction of $a$ to $A$. As $a|_A \circ \alpha\colon S^{j-2} \to K \times C$ has a preferred null homotopy, it follows that there is a canonically defined map

$$D\colon S^{j-2} \to F$$

such that the composite

$$S^{j-2} \xrightarrow{D} F \subset A$$

is homotopic to $\alpha$.

Let us apply the construction in the special case $A = K \vee C$, $\Delta = (K \vee C) \cup_\alpha D^{j-1}$, and where it is assumed that the composite

$$S^{j-2} \xrightarrow{\alpha} K \vee C \subseteq K \times C$$

is null homotopic. In this case, a choice of null homotopy determines maps

$$i_K\colon \Delta \to K \quad \text{and} \quad i_C\colon \Delta \to C$$

extending the projections $K \vee C \to K$ and $K \vee C \to C$. Clearly, the homotopy pushout of $i_K$ with $i_C$ is homotopy equivalent to the sphere $S^j$. Applying the above construction, we obtain a map

$$D\colon S^{j-2} \to K^\flat C.$$

**Proposition 4.3** *Suppose $\mathscr{D}$ is a Poincaré thickening in $S^j$, with $\Delta = (K \vee C) \cup_\alpha D^{j-1}$ as above. Then the composite*

$$S^{j-2} \xrightarrow{D} K^\flat C \xrightarrow{\Phi} \Omega(K \wedge C)$$

*is adjoint to an S-duality. In fact,* $\operatorname{adj}(\Phi \circ D)\colon \Sigma S^{j-2} \to K \wedge C$ *is a desuspension of the duality map $d\colon S^j \to \Sigma K \wedge C$ of* §3.

*Proof.* We first construct a desuspension of $d$. We have a commutative diagram

$$\begin{tikzcd} K \vee C \arrow[r, equal] \arrow[d] & K \vee C \arrow[d] \\ \Delta \arrow[r, "(i_K, i_C)"] & K \times C \end{tikzcd}$$

where the vertical arrows are the natural inclusions. Taking vertical cofibres, we obtain a map

$$S^{j-1} = \Delta/(K \vee C) \to K \times C/K \vee C = K \wedge C,$$

which desuspends $d$.

On the other hand, it is not difficult to see that the above map by definition is the same as the adjoint to $\Phi \circ D$. □

Now let $K$ and $C$ be arbitrary spaces.

**Definition 4.4** Any map $D$: $S^{j-2} \to K^\flat C$ satisfying the conclusion of 4.3 (i.e., $\mathrm{adj}(\Phi \circ D)$ is an $S$-duality) will be called a *$^\flat$-duality map.*

The utility of this definition is illustrated by the following.

**Theorem 4.5** *If $D$: $S^{j-2} \to K^\flat C$ is a $^\flat$-duality map, then there exists a Poincaré thickening of $K$ in the sphere $S^j$ with complement $C$ and boundary $\Delta = (K \vee C) \cup_\alpha D^{j-1}$.*

*Proof.* The Poincaré thickening is constructed as follows. Let $\Delta = (K \vee C) \cup_\alpha D^{j-1}$, where

$$\alpha\colon\ S^{j-2} \to K \vee C$$

is the composite

$$S^{j-2} \xrightarrow{D} K^\flat C \xrightarrow{i} K \vee C.$$

The maps $i_K$: $\Delta \to K$ and $i_C$: $\Delta \to C$ will be defined so as to extend the projections of $K \vee C$ to $K$ and $C$. The extension is given by the canonical null homotopy of the composite,

$$K^\flat C \xrightarrow{i} K \vee C \subset K \times C.$$

The homotopy pushout of $i_K$ with $i_C$ is clearly homotopy equivalent to the sphere $S^j$, and the fact that $(K, \Delta)$ satisfies Poincaré duality results from 3.3 and 4.3. □

## 5 The weak thickening theorem

*5.1* Suppose $K$ is a homology $k$-dimensional space which is $r$-connected. We may assume that $K$ is a CW complex of dimension $\leq k$.

By transversality, we embed $K$ in $S^m$ for $m$ large, ($m \gg 2k+1$). This defines a stable exterior $W = S^m \setminus n(K)$, where $n(K)$ is a regular neighborhood of $K$. By Alexander duality, $W$ is homology $(m-r-2)$-dimensional and $(m-k-2)$-connected. We need a lemma which shows that $W$ desuspends.

**Lemma 5.2** (Desuspension lemma) *Suppose $Y$ is an $s$-connected space ($s \geqq 1$) of homology dimension $t$. Then*

$$Y \simeq \Sigma^u X \qquad \text{if } u \leqq 2s - t + 2,$$

*and moreover, $X$ is unique up to equivalence if $u \leqq 2s - t + 1$.*

*Proof.* We can assume that $Y$ is a $t$-dimensional CW complex [$W_3$]. By iteration, it is sufficient to prove this when $u = 1$, since each desuspension reduces the connectivity and dimension by one. But when $u = 1$, the lemma follows by desuspending the attaching maps for the cells of $Y$ using the Freudenthal suspension theorem [S, p. 461]. □

We are now in a position to prove the weak Poincaré thickening theorem.

**Theorem 5.3** (Weak Poincaré thickening) *Let $K$ be $r$-connected and homology $k$-dimensional. Then $K$ Poincaré thickens in the sphere $S^j$ provided that*

$$k \leqq 2r + 1,$$
$$2k - r + 1 \leqq j.$$

*Moreover, any two Poincaré thickenings are concordant when strict inequality is satisfied.*

*Proof (Existence).* Applying the desuspension lemma (5.2) to the stable complement $W$, we see that

$$W \simeq \Sigma^u C, \qquad \text{if } u \leqq 2(m-k-2) - (m-r-2) + 2,$$
$$= m - 2k + r.$$

Set $j = m - u$. Then the inequality becomes

$$2k - r \leqq j,$$

and this is guaranteed by the hypotheses.

Next, let

$$d^s\colon S^{m-1} \to K \wedge W = K \wedge \Sigma^{m-j} C$$

be the canonical Spanier-Whitehead duality map (as constructed e.g. in the proof of 4.3). Since $K$ is $r$-connected and $C$ is $(j-k-2)$-connected, it follows that $K \wedge C$ is $(j-k+r-1)$-connected, and so by the Freudenthal suspension theorem, there is a map

$$d\colon S^{j-1} \to K \wedge C$$

which is a desuspension of $d^s$ provided that $j - 1 \leqq 2(j-k+r-1)+1$, or equivalently,

$$2k - 2r \leqq j.$$

Furthermore, if strict inequality holds then the desuspension is unique up to homotopy. Since we are assuming already that $2k-r+1 \leqq j$, we see that $d$ exists and is unique up to homotopy.

By 4.5, it will be sufficient to construct a $^{\flat}$-duality map $D\colon S^{j-2} \to K^{\flat}C$. As $K$ is $r$-connected and $C$ is $(j-k-2)$-connected, it follows by 4.2 that

$$\Phi\colon K^{\flat}C \to \Omega(K \wedge C)$$

is $\min(r, j-k-2)+r+(j-k-2)+1$-connected. Consequently, if

$$j-2 \leqq \min(r, j-k-2)+r+(j-k-2)+1,$$

then the adjoint of the $S$-duality, $\mathrm{adj}(d)\colon S^{j-2} \to \Omega(K \wedge C)$, factors up to homotopy through a $^{\flat}$-duality map $D\colon S^{j-2} \to K^{\flat}C$.

Now, the last inequality is satisfied in particular when

$$k \leqq 2r+1, \quad \text{and}$$
$$2k-r+1 \leqq j.$$

Consequently, $K$ Poincaré thickens in $S^j$ when the above inequalities hold.

*(Uniqueness)*. Suppose that

$$(\mathscr{D}) \qquad \begin{array}{ccc} \Delta & \xrightarrow{i_C} & C \\ {\scriptstyle i_K}\downarrow & & \downarrow \\ K & \longrightarrow & S^j \end{array}$$

is a Poincaré thickening in the sphere $S^j$, and that $K$ is $r$-connected and homology $k$-dimensional, and

$$k < 2r+1,$$
$$2k-j+1 < r.$$

By 2.4, we may assume that $\Delta = A \cup_{\alpha} D^{j-1}$ where $A$ is a CW complex of dimension $\leqq j-3$, and $\tilde{i}_K$ and $\tilde{i}_C$ are the restrictions of $i_K$ and $i_C$ to $A$. It follows that the homotopy colimit

$$\mathrm{hocolim}(K \xleftarrow{\tilde{i}_K} A \xrightarrow{\tilde{i}_C} C)$$

is contractible. By the Blakers-Massey excision theorem, the map

$$A \xrightarrow{(\tilde{i}_K, \tilde{i}_C)} K \times C$$

factors uniquely up to homotopy through $K \vee C$, provided that $\dim(A) < \mathrm{conn}(K) + \mathrm{conn}(C)$, or equivalently, if

$$\max(k, j-r-2) < r+(j-k-2)+1.$$

The last inequality, in turn, is implied by the pair inequalities

$$2k-r+1<j, \quad \text{and}$$
$$k<2r+1.$$

Hence, if these conditions are satisfied, there is a unique homotopy class of map $A \to K \wedge C$ for which the composition with the inclusion $K \vee C \to K \times C$ gives $(\tilde{i}_K, \tilde{i}_C)$ up to homotopy. It then follows by an easy homology argument that the map $A \to K \vee C$ is a homotopy equivalence. Consequently, $\Delta$ has a decomposition of the form $(K \vee C) \cup_\alpha D^{j-1}$, in such a way that with respect to this identification, $i_K$: $\Delta \to K$ and $i_C$: $\Delta \to C$ extend the projections of $K \vee C$ onto each factor.

The attaching map $\alpha$: $S^{j-2} \to K \vee C$ for the top cell of $\Delta$ then clearly factors up to homotopy as

$$S^{j-2} \xrightarrow{D} K^\flat C \xrightarrow{i} K \vee C,$$

for some $^\flat$-duality map $D$ (4.3). Since

$$\Phi\colon\ K^\flat C \to \Omega(K \wedge C)$$

is $\min(r, j-k-2)+r+(j-k-2)+1$ connected, it again follows immediately from the connectivity assumptions that $D$ is unique up to homotopy. Consequently, the Poincaré thickening above is unique up to concordance. This completes the proof of 5.3. □

## 6 The weak thickening theorem in the smooth category

This spaces in this section are not a priori assumed to be 1-connected. Let

$$(\mathscr{D}) \qquad \begin{array}{ccc} \Delta & \xrightarrow{i_C} & C \\ {\scriptstyle i_K}\downarrow & & \downarrow \\ K & \longrightarrow & S^j \end{array}$$

be a Poincaré thickening in the $j$-sphere. By a *smoothing* $\mathscr{S}$ the $\mathscr{D}$, we mean a codimension zero compact submanifold $(V, \partial V)$ of $S^j$ together with orientation preserving homotopy equivalences $e_0\colon (V, \partial V) \xrightarrow{\simeq} (K, \Delta)$ and $e_1\colon (C_V, \partial V) \xrightarrow{\simeq} (C, \Delta)$, with $C_V = \mathrm{cl}(S^j \setminus V)$, such that $e_0$ coincides with $e_1$ on $\partial V$. This amounts to a having a degree one homotopy equivalence of triads

$$e\colon\ (V, C_V, \partial V) \xrightarrow{\simeq} (K, C, \Delta).$$

In particular, a smoothing determines a thickening of the complex $K$ in the usual sense. Two smoothings $\mathscr{S}_0 = (V, e)$ and $\mathscr{S}_1 = (W, e')$ of $K$ in $S^j$ are said to be *concordant* if there exists an $h$-cobordism $Z \subset S^j \times I$ with $Z_0 = V$ and $Z_1 = W$, together with a map of triads $e\colon (Z, C_Z, \partial Z) \xrightarrow{\simeq} (K, C, \Delta)$ which is a homotopy between $e$ and $e'$. The next result is a restatement of the Browder-Casson-Sullivan-Wall theorem [W$_2$, 12.1].

**Theorem 6.1** *If $\mathscr{D}$ is a Poincaré thickening of $K$ in $S^j$, $j \geqq 6$, with $\Delta$ and $K$ 1-connected, and if the homology dimension of $K$ is $\leqq j-3$, then there exists a smoothing $\mathscr{S}$ of $\mathscr{D}$. Furthermore, $\mathscr{S}$ is unique up to concordance.*

For a complex $K$ of homology dimension $k \leqq j-3$, let $h\,\mathrm{Emb}(K, S^j)$ denote the set of (smooth) concordance classes thickenings of $K$ in $S^j$. Similarly, let $p\,\mathrm{Emb}(K, S^j)$ denote the set of concordance classes of Poincaré thickenings of $K$ in $S^j$. Let $p\,\mathrm{Emb}_1(K, S^j) \subset p\,\mathrm{Emb}(K, S^j)$ be the subset consisting of Poincaré thickenings $\mathscr{D} = (K, C, \Delta)$ such that $\Delta$ is 1-connected.

**Corollary 6.2** *If $j \geqq 6$ then the forgetful map*

$$h\,\mathrm{Emb}(K, S^j) \to p\,\mathrm{Emb}(K, S^j)$$

*is one-to-one with image $p\,\mathrm{Emb}_1(K, S^j)$.*

*Proof.* Let $(V, \partial V) \subset S^j$ be a thickening of $K$ in $S^j$. Since $K$ is 1-connected and $\dim(K) \leqq j-3$, there exists a handlebody decomposition

$$\mathscr{H} = \partial V \cup (\bigcup h_\alpha)$$

of $V \operatorname{rel} \partial V$ whose handles $h_\alpha$ satisfy $2 \leqq \operatorname{index}(h_\alpha) \leqq j-3$. Let $\mathscr{S}$ denote the spine of $\mathscr{H}$. Then $\dim \mathscr{S} \leqq j-3$, and so by general position, any map $\gamma\colon (D^2, S^1) \to (V, \partial V)$ can be deformed $\operatorname{rel} S^1$ so as not to intersect $\mathscr{S}$. Hence, the perturbed image of $\gamma$ will lie in a collar of $\partial V$, so $\gamma|_{S^1}\colon S^1 \to \partial V$ is null homotopic. Consequently, $\partial V$ is 1-connected since $V$ is 1-connected. This shows that the image of the forgetful map $h\,\mathrm{Emb}(K, S^j) \to p\,\mathrm{Emb}(K, S^j)$ is contained in $p\,\mathrm{Emb}_1(K, S^j)$.

The proof that the forgetful map $h\,\mathrm{Emb}(K, S^j) \to p\,\mathrm{Emb}_1(K, S^j)$ is bijective now follows immediately from 6.1. $\square$

## 7 Applications

*7.1 Thickenings of suspensions.* Let $X$ and $Y$ be spaces. We define a map

$$\rho\colon\ \Sigma X \wedge Y \to (\Sigma X)\,\flat\,(\Sigma Y) \simeq \Sigma(\Omega \Sigma X) \wedge (\Omega \Sigma Y)$$

by the formula

$$\rho(x \wedge y \wedge t) = (s \mapsto x \wedge s) \wedge (s \mapsto y \wedge s) \wedge t.$$

*Remark.* Let $i\colon (\Sigma X)\,\flat\,(\Sigma Y) \to \Sigma X \vee \Sigma Y$ be the map given by the homotopy fibre of the inclusion $\Sigma X \vee \Sigma Y \subset \Sigma X \times \Sigma Y$. Then

$$i \circ \rho\colon\ \Sigma X \wedge Y \to \Sigma X \vee \Sigma Y$$

is the *generalized Whitehead Product* map [G$_2$, § 5].

Let $\Phi\colon (\Sigma X)\,\flat\,(\Sigma Y) \to \Omega(\Sigma X \wedge \Sigma Y)$ be the map defined in 4.1. Then by direct calculation,

$$\Phi \circ \rho\colon\ \Sigma X \wedge Y \to \Omega(\Sigma X \wedge \Sigma Y),$$

is given by the rule

$$(x \wedge y \wedge t) \mapsto u \mapsto \begin{cases} (2ut \wedge x) \wedge (y \wedge t), & \text{if } u \in [0, 1/2] \\ (t \wedge x) \wedge (y \wedge 2(1-u)t), & \text{if } u \in [1/2, 1]. \end{cases}$$

Consequently, the adjoint $\operatorname{adj}(\Phi \circ \rho)$: $\Sigma^2 X \wedge Y \to \Sigma^2 X \wedge Y$ is given by formula

$$\operatorname{adj}(\Phi \circ \rho)(x \wedge y \wedge (t, u)) = \begin{cases} (x \wedge y \wedge (2ut \wedge t), & \text{if } u \in [0, 1/2] \\ (x \wedge y \wedge (t \wedge 2(1-u)t), & \text{if } u \in [1/2, 1]. \end{cases}$$

On the two suspension coordinates, this map is induced by the self-map of the unit square $I^2$ which converts a horizontal cross section into a diagonal cross section:

$$(t, u) \mapsto \begin{cases} (2ut, t), & \text{if } u \in [0, 1/2] \\ (t, 2(1-u)t), & \text{if } u \in [1/2, 1]. \end{cases}$$

As this map rel $\partial I^2$ has degree one, it follows that

**Lemma 7.2** $\operatorname{adj}(\Phi \circ \rho)$: $\Sigma^2 X \wedge Y \to \Sigma^2 X \wedge Y$ *is homotopic to the identity.*

We are now in a position to state a result concerning thickenings of suspensions.

**Theorem 7.3** *Let $K \simeq \Sigma X$ and $C \simeq \Sigma Y$ be spaces with suspension structures, and suppose that there exists a Spanier-Whitehead duality map,*

$$d\colon S^{j-2} \to \Sigma^{-1} K \wedge C \simeq \Sigma X \wedge Y.$$

*Then there exists a Poincaré thickening $\mathscr{D}$ of $K$ in the sphere $S^j$ whose exterior is homotopy equivalent to $C$, and moreover, with respect to this equivalence the S-duality map of $\mathscr{D}$ is homotopic to $d$.*

*Remark.* Note that no connectivity restrictions appear in 7.3.

*Proof of 7.3* Let $\rho$ be as in 7.1. Then the map $\rho \circ d$: $S^{j-2} \to K^\flat C$ is a $^\flat$-duality map by 7.2. Applying 4.5 yields the desired Poincaré thickening. □

We will derive two corollaries of 7.3. The first is a weakening of the classification theorem of [C-W] in the case of complexes which desuspend.

For a space $K$ and a positive integer $j$, consider the set of pairs $(C, d)$ such that $C$ is a space and $d$: $S^j \to \Sigma K \wedge C$ is a Spanier-Whitehead duality. Define an equivalence relation on this set as follows: $(C_0, d_0) \sim (C_1, d_1)$ if there is a homotopy equivalence $h$: $C_0 \to C_1$ such that $\Sigma(\mathrm{id}_K \wedge h) \circ d_0 \simeq d_1$. Let $SW_j(K)$ denote the resulting set of equivalence classes, and let $SW_j^\Sigma(K) \subset SW_j(K)$ be the equivalence classes of pairs $(C, d)$ such that $C$ desuspends.

The $S$-duality construction (§ 3) defines a natural map

$$\theta_j\colon p\,\mathrm{Emb}(K, S^j) \to SW_j(K).$$

We then have the following.

**Corollary 7.4** (compare [C-W]) *Suppose $K \simeq \Sigma X$ for some space $X$. Let $k$ be the homology dimension of $K$ and $r$ the connectivity of $K$. Then $SW_j^{\Sigma}(K)$ is contained in the image of $\theta_j$ provided that*

$$k-r \leqq (j-1)/2.$$

*Proof.* Let $d$: $S^j \to \Sigma K \wedge C$ be an element of $SW_j^{\Sigma}(K)$. Then $C \simeq \Sigma Y$ for some space $Y$, and we may rewrite $d$ as a map $S^j \to \Sigma(\Sigma X) \wedge (\Sigma Y)$. By $S$-duality, the target of $d$ is $r+(j-k-2)+1$ connected, so by the Freudenthal theorem and the numerical hypothesis, there is a two-fold desuspension $\Sigma^{-2} d$: $S^{j-2} \to \Sigma X \wedge Y$. Applying 7.3 then completes the proof. □

We now show how 7.3 can be used to eliminate the extra connectivity hypothesis on $K$ in the weak Poincaré thickening theorem (5.3) under the assumption that $K$ has a suspension structure.

**Corollary 7.5** *If $K \simeq \Sigma X$ is $r$-connected and has homology dimension $k$, then $K$ Poincaré thickens in $S^j$ provided that*

$$2k-r+1 \leqq j.$$

*Proof.* Applying 5.2 we see that the stable complement $W \subset S^m$ desuspends $u$-times, where $u=m-2k+r$. Set $j=m-u+1$. Let $C$ be the $(u-1)$-st desuspension of $W$. Then $C$ desuspends once more, $C \simeq \Sigma Y$. Moreover, the Freudenthal suspension theorem implies that Spanier-Whitehead duality map

$$d^s\colon\ S^{m-1} \to K \wedge W \simeq K \wedge \Sigma^{u-1} C \simeq \Sigma X \wedge \Sigma^u Y$$

desuspends $u$ times to a map $d$: $S^{j-2} \to \Sigma^{-1} K \wedge C \simeq \Sigma X \wedge Y$. One may then apply 7.3 to obtain the thickening of $K$. □

*7.6 Closed Poincaré complexes.* Suppose $X$ is an $r$-connected, closed Poincaré duality space of dimension $n$. Recall the definition of a Poincaré embedding of $X$ in $S^j$ (cf. [$W_2$, §11]). This amounts to the existence of a homotopy co-Cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & C \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \longrightarrow & S^j, \end{array}$$

such that $i$ is a cofibration and $p$: $E \to X$ is equivalent over $X$ to an oriented spherical fibration with fibre $S^{j-n-1}$, or equivalently, if $M_p$ is the mapping cylinder of $p$, then $(M_p, E)$ is a $j$-dimensional Poincaré pair (cf. [B]).

**Theorem 7.7** *$X$ Poincaré embeds in $S^j$ under the assumptions*

$$n \leqq 3r+2,$$
$$2n-3r-1 \leqq j,$$
$$n+r+3 \leqq j.$$

*Proof.* By 2.4, we may suppose that $X = K \cup_\beta D^n$, where $K$ i a CW complex of dimension $\leqq n-2$, and $\beta$: $S^{n-1} \to K$ is the attaching map for the top cell of $X$. By duality, $K$ has homology dimension $\leqq n-r-1$ because $X$ is $r$-connected. Let us apply the weak thickening theorem (5.3) to thicken $K$ in $S^{j-1}$. Under the connectivity assumptions of 5.3, $K$ will thicken in $S^{j-1}$ if

$$n-r-1 \leqq 2r+1, \quad \text{and}$$
$$2(n-r-1)-r \leqq j-1.$$

As these inequalities are implied by the assumptions, a Poincaré thickening of $K$ exists. Consequently, there is a co-Cartesian diagram

$$(\mathscr{D}) \qquad \begin{tikzcd} \Delta \arrow[r, "i_C"] \arrow[d, "i_K"'] & C \arrow[d] \\ K \arrow[r] & S^{j-1} \end{tikzcd}$$

where $(K, \Delta)$ is a Poincaré pair. By the third assumption in the statement of 7.7, we have

$$(j-1)-(n-r-1) = j-n-r \geqq 3,$$

and so we may apply the smoothing theorem (6.1) and henceforth assume that $(K, \Delta)$ is a smooth codimension zero submanifold of $S^{j-1}$.

Thinking of $S^{j-1}$ as the boundary of $D^j$, we may define a map

$$k\colon\ K \cup_\beta D^n \to D^j \subset S^j,$$

which extends the inclusion of $K$ by the cone construction:

$$k(r \cdot x) := r \cdot \beta(x), \qquad \text{where } x \in S^{j-1}, \quad r \in I.$$

Let $N$ be a closed regular neighborhood of $K$ in $D^j$ such that $\partial N$ is transverse to $k(D^n)$. Then

$$P := N \cup k(D^n)$$

is a subpolyhedron of $S^j$ which is homotopy equivalent to $X$. Let $n(P)$ be a regular neighborhood of $P$ and let $C$ denote its complement. Then $X \simeq n(P)$, and

$$\begin{tikzcd} \partial n(P) \arrow[r] \arrow[d, "\cap"'] & C \arrow[d] \\ n(P) \arrow[r] & S^j \end{tikzcd}$$

is the desired Poincaré embedding of $X$ in $S^j$. $\square$

**Corollary 7.8** *If $X$ is an $r$-connected Poincaré duality space of dimension $n \leqq 3r+2$, then $X$ Poincaré embeds in $S^{n+r+3}$.*

*Proof.* Set $j = n+r+3$. Then $j$ satisfies the criteria of 7.7, so $X$ Poincaré embeds in $S^j$. $\square$

*7.9 Self-dual complexes.* A space $K$ is said to be *self n-dual* if there exists an $S$-duality

$$S^{n-1} \rightsquigarrow K \wedge K.$$

(The symbol $\rightsquigarrow$ denotes an arrow in the stable category.)

*Examples 7.10* (1) The sphere $S^k$ is self $(2k+1)$-dual since it is homologically equivalent to its own exterior in $S^{2k+1}$.
(2) Let $V^{n+1} \subset S^{n+2}$ be a Seifert surface for a knot $S^n \subset S^{n+2}$. Then $V$ is self $(n+2)$-dual. This is an elementary exercise in the Mayer-Vietoris sequence.

**Theorem 7.11** (Self-dual embedding theorem) *Let $K$ be an $r$-connected space which is self $n$-dual, $n \geqq 6$. Then $K$ thickens in $S^n$ provided*

$$n \leqq 3r+3.$$

*Proof.* The assumption that $K$ is self $n$-dual and $r$-connected implies by $S$-duality that $K$ is homology $(n-r-2)$-dimensional.

By the weak thickening theorem (5.3) and the smoothing theorem (6.1), K will thicken smoothly in $S^n$ provided that

$$(n-r-2) \leqq 2r+1,$$
$$2(n-r-2)-r+1 \leqq n.$$

But the above inequalities reduce to the condition that $n \leqq 3r+3$. □

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