For any $i<j$, write $\Omega_{i\bar{j}}=\sum_{k,l=1}^{n} c_{k\bar{l}}\psi_k \wedge \bar{\psi}_l$. Then by Lemma 1, $\Omega_{i\bar{j}}=c_{i\bar{j}}\psi_i \wedge \bar{\psi}_j + c_{j\bar{i}}\psi_j \wedge \bar{\psi}_i$, and $c_{i\bar{j}}\cdot c_{j\bar{i}}=0$, $|c_{i\bar{j}}|^2+|c_{j\bar{i}}|^2=1$. By the first Bianchi identity ${}^t\varphi \wedge \Theta=0$, here $\Omega=\Theta$ as e is unitary, we know that $c_{j\bar{i}}$ must vanish. Therefore, $\Omega_{i\bar{j}}=c_{i\bar{j}}\psi_i \wedge \bar{\psi}_j$, $c_{i\bar{i}}=1$, and $|c_{i\bar{j}}|^2=1$.

Let $C=(c_{i\bar{j}})$. Then C is a nowhere zero Hermitian matrix. By the last equality in Lemma 1, $\operatorname{rank}(C)\leqq 1$, hence $C=b\cdot b^*$ for a column vector b. Replace ψ_i by $b_i\psi_i$, we get the desired decomposition of Ω. QED

Proposition 3 *For any $x\in U_g$, and any tangent frame e near x, let ψ be a coframe near x satisfying $\Omega=-\psi\wedge\psi^*$ as in Proposition 2. Then there exists a 1-form λ near x such that $\bar{\lambda}=-\lambda$, $d\psi=\theta\wedge\psi-\lambda\wedge\psi$ and $d\lambda=-\mathrm{Ric}_g$ (θ is the connection matrix under e).*

Proof. Again we may assume that e is unitary. Since $d\varphi=-{}^t\theta\wedge\varphi$, and ψ forms a coframe, one can write $d\psi=\theta\wedge\psi+\xi\wedge\psi$ for some $n\times n$ matrix of 1-forms ξ. Plug it into the second Bianchi identity $d\Theta=\theta\wedge\Theta-\Theta\wedge\theta$, and $\Theta=\Omega=-\psi\wedge\psi^*$, one gets:

$$\xi\wedge\psi\wedge\psi^*+\psi\wedge\psi^*\wedge\xi^*=0.$$

Its (2, 1)-parts gives:

$$\xi^{(1,0)}\wedge\psi\wedge\psi^*+\psi\wedge\psi^*\wedge\xi^{(0,1)*}=0.$$

This implies that

$$\xi^{(0,1)*}=\alpha I$$
$$\xi^{(1,0)}\wedge\psi=-\alpha\wedge\psi.$$

Therefore

$$\xi\wedge\psi=-(\alpha-\bar{\alpha})\wedge\psi.$$

Let $\lambda=\alpha-\bar{\alpha}$, then

$$\bar{\lambda}=-\lambda;\qquad d\psi=\theta\wedge\psi-\lambda\wedge\psi.$$

Differentiate the last equality, one gets $d\lambda=-\mathrm{Ric}_g$. QED

2 Curvature decomposition in the degenerate case

Let V_g be the Zariski open set $\{x\in M:\mathrm{Ric}_g^{n-1}(x)\neq 0\}$. In this section, we shall consider the decomposition of Ω in V_g, since it will be needed later in the proof of Theorem A.

Let us fix a point $x\in V_g\setminus U_g$. Choose an unitary frame e with the dual frame φ such that

$$-\mathrm{Ric}_g=\lambda_1\varphi_1\wedge\bar{\varphi}_1+\ldots+\lambda_n\varphi_n\wedge\bar{\varphi}_n$$

where $\lambda_1\geqq\ldots\geqq\lambda_{n-1}>\lambda_n\geqq 0$ in a neighbourhood V of x. Write $U=V\cap U_g$, then $\lambda_n>0$ in U and $=0$ along $V\setminus U$.

Since $\Omega_{i\bar{i}}\leqq 0$, and $\operatorname{tr}_\omega\Omega_{i\bar{i}}(x)=\mathrm{Ric}(e_i,\bar{e}_i)|_x=-\lambda_i(x)$, hence $\Omega_{i\bar{i}}(x)\neq 0$ for $1\leqq i\leqq n-1$ and $\Omega_{n\bar{n}}(x)=0$.

Therefore, there exist (1, 0)-forms $\psi_1,\ldots,\psi_{n-1}$ in V such that $\Omega_{i\bar{i}}=-\psi_i\wedge\bar{\psi}_i$ for each $i\leqq n-1$, and $\psi_1\wedge\ldots\wedge\psi_{n-1}\neq 0$ in V.

Write $\psi_i = \sum_{j=1}^{n-1} a_{ij}\varphi_j + b_i\varphi_n$, and $A=(a_{ij})$. Then $\Omega_{n\bar{n}}(x)=0$ gives ${}^t A\bar{A}(x) = \operatorname{diag}(\lambda_1(x),\dots,\lambda_{n-1}(x))>0$, hence $\det A(x)\neq 0$. Thus by shrinking V if necessary, we have $\psi_1\wedge\dots\wedge\psi_{n-1}\wedge\varphi_n\neq 0$ in V.

For any $y\in U$, Proposition 2 gives that $\Omega=-\psi'\wedge{}^t\bar{\psi}'$ for some coframe $\psi'=B\varphi$ near y. Then for $1\leqq i\leqq n-1$, $\psi'_i=\alpha_i\psi_i$ near y for some $|\alpha_i|=1$. By the first Bianchi identity: ${}^t\varphi\wedge\Theta=0$, hence ${}^tB=B$.

Write:

$$B=\begin{pmatrix} H & b \\ {}^t b & c\end{pmatrix}.$$

Since

$$\operatorname{Ric}_g=\operatorname{tr}(\Omega)=-{}^t\psi'\wedge\bar{\psi}'=-{}^t\varphi(B\bar{B})\bar{\varphi}$$

we have

$$H\bar{H}+b\,{}^t\bar{b}=\operatorname{diag}(\lambda_1,\dots,\lambda_{n-1})$$
$$H\bar{b}+b\bar{c}=0$$
$${}^t b\bar{b}+c\bar{c}=\lambda_n$$

therefore

$$\sum_{i=1}^{n-1}\lambda_i|b_i|^2=\lambda_n\sum_{i=1}^{n-1}|b_i|^2.$$

This together with the fact that $\lambda_1\geqq\dots\geqq\lambda_{n-1}>\lambda_n\geqq 0$ implies that near y:

$$b=0;\qquad \Omega_{n\bar{n}}=-c\bar{c}\varphi_n\wedge\bar{\varphi}_n.$$

Now if we write $\Omega_{n\bar{n}}=-{}^t\varphi E\bar{\varphi}$ in V, where

$$E=\begin{pmatrix} F & h \\ {}^t h & a\end{pmatrix}\geqq 0.$$

Then $h=0$ in U, hence in V. Since $\operatorname{rank}(E)\leqq 1$, while in U, $a=|c|^2>0$, therefore $F=0$ in U, hence in V. Namely we have

$$\Omega_{n\bar{n}}=-\psi_n\wedge\bar{\psi}_n;\qquad \psi_n=\tau\varphi_n$$

in the whole neighbourhood V.

Use the denseness of U and $\{\psi_1,\dots,\psi_{n-1},\varphi_n\}$ as the coframe, a little modification of the proofs of Propositions 2 and 3 gives the following:

Proposition 4 *For any $x\in V_g$ and any frame e near x, there exist $(1,0)$-forms $\psi_1,\dots,\psi_n$ and 1-form λ in a neighbourhood of with $\bar{\lambda}=-\lambda$ such that*

$$\Omega=-\psi\wedge{}^t\bar{\psi};\qquad d\psi=\theta\wedge\psi-\lambda\wedge 4;\qquad d\lambda=-\operatorname{Ric}_g.$$