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Weights, vertices and a correspondence of characters in groups of odd order

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1 Introduction

If p is a prime and G is a finite group a p-weight of G is a pair (P, γ) where P is a p-subgroup of G and γ is an irreducible complex character of $N_G(P)/P$ with p-defect zero.

In the fundamental paper [1], Alperin conjectured that the number of G-conjugacy classed of p-weights and the number of irreducible p-Brauer characters of the group G coincide. In contrast to the situation for Lie type groups, it is not true that a natural correspondence between Brauer characters and weights exists. It is the aim of this paper to show the following.

Theorem A. *If G is a finite group of odd order, there exists a natural bijection between the G-classes of p-weights of G and the irreducible p-Brauer characters of G.*

In fact, we can prove Theorem A "vertex to vertex".

Theorem B. *If P is a p-subgroup of a group of odd order G, there exists a natural bijection between the irreducible Brauer characters of G with vertex P onto the irreducible Brauer characters of $N_G(P)$ with vertex P.*

In the last few years, Isaacs π-theory has proven to be an important tool for the Character Theory of the Solvable Groups. Even more, for π-separable groups it gives a satisfactory and, we believe, almost complete character theoretic version of the Modular Theory of the p-Solvable Groups. We will use this theory to prove Theorems A and B, and, with the same amount of work, its respective π-versions.

2 π-theory

In [2], Gajendragadkar introduced the π-special characters of a finite π-separable group G for an arbitrary set π of primes. Later, in [8], Isaacs found a

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superset, $B_\pi(G)$, of the Gajendragadkar characters which provide a canonical lift (and a proof of the existence) of the unique basis $I_\pi(G)$ of the complex vector space of class functions defined on the π-elements of G satisfying

(D) If $\chi \in \mathrm{Irr}(G)$, then χ^o is an nonnegative integer linear combination of $I_\pi(G)$ and

(FS) If $\varphi \in I_\pi(G)$ then $\varphi = \chi^o$ for some $\chi \in \mathrm{Irr}(G)$

(where χ^o is the restriction to the π-elements of any class function χ defined on G). When $\pi = p'$, by the Fong-Swan theorem, $I_\pi(G) = IBr(G)$, the set of irreducible Brauer characters of G.

In general, it is hard to compute $B_\pi(G)$ for a π-separable group G. However, if we assume some odd hypothesis, there are some useful tests available. For instance, $B_{p'}(G)$, when p is not 2, is just the set of modularly irreducible p-rational characters of G [6]. More recently, Isaacs proves the following.

(2.1) Theorem. *Let G be a solvable group of odd order and suppose that π is a set of primes. Let ε be a primitive $|G|$th root of unity and let $\sigma \in \mathrm{Gal}(Q(\varepsilon)/Q)$ be such that σ fixes all π-roots of unity and acts like complex conjugation on π'-roots of unity. Then $\chi \in \mathrm{Irr}(G)$ lies $B_\pi(G)$ if and only if $\chi^\sigma = \chi$.*

Proof. See (3.1) of [10].

The important fact that B_π-characters behave well with respect to normal subgroups [8] is heavily used to prove Clifford Theory for I_π-characters in [9] or, for instance, to show the π-version of a well known theorem of Huppert: irreducible Brauer characters of p-solvable groups are induced from p'-degree characters (this follows from (3.4) of [9]). We will use the notation given in Sects. 2 and 3 of [9].

Finally, we need to introduce vertices for sets of primes. This is something nontrivial which has been done in [12]. If $\varphi \in I_\pi(G)$ and $\alpha^G = \varphi$ for some $\alpha \in I_\pi(J)$ with π-degree then the π-complements of J (i.e., the Hall π'-subgroups of J) are uniquely determined up to G-conjugacy by φ and we will refer to them as the set of vertices of φ. Observe that if P is a vertex for φ then $\varphi(1)_{\pi'} = |G:P|_{\pi'}$.

3 A correspondence of characters

By using the deep properties of the odd fully ramified sections, Isaacs showed the existence of a natural bijection from the p'-degree irreducible characters of a group of odd order G onto the p'-degree irreducible characters of the normalizer of a Sylow p-subgroup (10.9 of [4]). This gave a proof of the McKay conjecture for groups of odd order.

In fact, it is possible to replace p' for any set of primes π in (10.9) of [4] and still having a natural bijection $*: \mathrm{Irr}^\pi(G) \to \mathrm{Irr}^\pi(N_G(H))$, where H is a π-complement of a group of odd order G, and $\mathrm{Irr}^\pi(G) = \{\chi \in \mathrm{Irr}(G)$ of π-degree$\}$ (to be more precise it suffices only to assume that $2 \notin \pi$ but our interest here is focused in groups of odd order).

Some of the properties of this correspondence have been recently studied by Williams in [13]. We will use, with his kind permission, some of his results in this section.

As we can see in [4], Isaacs correspondence is constructed by showing the existence of a natural bijection between $\mathrm{Irr}^\pi(G)$ onto $\mathrm{Irr}^\pi(\mathcal{O}^{\pi\pi'}(G)' N_G(H))$. However, to prove most results on the correspondence (as happened in [14]), one immediately needs to relax the algorithm.

(3.1) Theorem (Williams) *Let G be a group of odd order, H a π-complement of G and let $\mathcal{O}^{\pi\pi'}(G)' N_G(H) \subseteq J \subseteq G$. If $\chi \in \mathrm{Irr}^{\pi}(G)$, then there exists a unique irreducible constituent α of χ_J with π-degree and odd multiplicity. Moreover the map $\chi \to \alpha$ is a bijection from $\mathrm{Irr}^{\pi}(G)$ onto $\mathrm{Irr}^{\pi}(J)$, and $\alpha^* = \chi^*$.*

Proof. See [13].

A trivial consequence of Theorem 3.1 is that χ^* is an irreducible constituent of $\chi_{N_G(H)}$ for $\chi \in \mathrm{Irr}^{\pi}(G)$. Another one is that if y is an automorphism of G fixing H then $(\chi^*)^y = (\chi^y)^*$.

By using Theorem 2.1, we can show that * maps the π-special characters of G onto $\mathrm{Irr}(N_G(H)/H)$. This provides another proof (in the odd case) of a theorem of Isaacs counting the number of π-special characters of a group G (see (1.16) of [15]). Since the π-special characters of $N_G(H)$ are exactly $\mathrm{Irr}(N_G(H)/H)$ (by (2.2), (4.1) and (4.2) of [2]), it will be sufficient to show the following.

(3.2) Lemma. *Let G be a group of odd order and let H be a π-complement of G. Let $\chi \in \mathrm{Irr}(G)$ and let $\alpha \in \mathrm{Irr}^{\pi}(J)$ be such that $[\chi_J, \alpha]$ is odd, where $\mathcal{O}^{\pi\pi'}(G)' N_G(H) \subseteq J \subseteq G$. Then χ is π-special if and only if α is π-special.*

Proof. Since both χ and α have π-degree, by (5.4) of [8]:, it suffices to show that $\chi \in B_{\pi}(G)$ if and only if $\alpha \in B_{\pi}(J)$. By Theorem 2.1, we must show that $\chi^{\sigma} = \chi$ if and only if $\alpha^{\sigma} = \alpha$. Since $[(\chi^{\sigma})_J, \alpha^{\sigma}] = [\chi_J, \alpha]$, the lemma follows from Theorem 3.1.

(3.3) Corollary. *Let G be a group of odd order and let H be a π-complement of G. Let $\chi \in \mathrm{Irr}(G)$. Then χ is π-special if and only if χ^* is π-special. Therefore * maps the π-special characters of G onto $\mathrm{Irr}(N_G(H)/H)$.*

Proof. Apply induction on $|G|$ and (3.2).

(3.4) Corollary. *Let G be a group of odd order and let H be a π-complement of G. Then the map $\varphi \to \varphi^*$ from $\{\varphi \in I_{\pi}(G)$ of π-degree$\}$ onto $\{\varphi \in I_{\pi}(N_G(H))$ of π-degree$\}$ given by $\varphi^* = (\chi^*)^{o}$, where $\chi^{o} = \varphi$ and χ is π-special, is a well defined bijection.*

Proof. By Theorem 9.3 of [8], let $\chi \in B_{\pi}(G)$ such that $\chi^{o} = \varphi$. Then χ has π-degree and thus χ is π-special. Hence, χ^* is π-special and therefore $\chi^{*o} \in I_{\pi}(N_G(H))$ (because by (6.1) of [2], χ^* restricts irreducibly to a Hall π-subgroup of $N_G(H)$). If $\chi^{*o} = \psi^{*o}$ for χ and ψ π-special characters of G, since H is contained in the kernel of both χ^* and ψ^*, we have that $\chi^*(x) = \chi^*(x_{\pi}) = \psi^*(x_{\pi}) = \psi^*(x)$ for all $x \in N_G(H)$. Thus $\chi^* = \psi^*$ and then $\chi = \psi$. Obviously the map is surjective.

The following, which is very much connected with [11], is one of the main results in [13].

(3.5) Theorem (Williams) *Let G be a group of odd order, let H be a π-complement of G and let $H \subseteq J \subseteq G$. Let $\mu \in \mathrm{Irr}^{\pi}(J)$ with $\mu^G \in \mathrm{Irr}(G)$. Then $(\mu^G)^* = (\mu^*)^{N_G(H)}$.*

Proof. See [13].

The core of this section consists in proving two more properties of the * correspondence. If N is a normal subgroup of a group of odd order G with G/N

a π-group and $\chi \in \mathrm{Irr}^{\pi}(G)$ and $\theta \in \mathrm{Irr}^{\pi}(N)$, it is not difficult to see (and it follows directly by applying the algorithm) that χ lies over θ if and only if χ^* lies over θ^*. A proof of this can be found in [13]. We need to relate normal subgroups and correspondents without imposing conditions on the normal subgroups.

We believe the following must be more general, although we only have found a proof of the result in the form going to be needed.

(3.6) Theorem. *Let N be a normal subgroup of a group of odd order G and let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ be π-specials. Let H be a fixed π-complement of G and let $\chi^{(N)} \in \mathrm{Irr}(N_G(H \cap N))$ be such that $(\chi^{(N)})^* = \chi^*$. Then θ is an irreducible constituent of χ_N if and only if θ^* is an irreducible constituent of $(\chi^{(N)})_{N_N(H \cap N)}$.*

Proof. We argue by double induction on $|G|$ and $|G:N|$. By Corollary 3.3, observe that $\chi^{(N)}$ is π-special.

We certainly may assume that $N < G$ and that $N_G(H \cap N) < G$.

Step 1 G/N is cyclic of prime π'-order.

Suppose that $N < M \lhd G$. Assume first that χ lies over θ. Let $\eta \in \mathrm{Irr}(M)$ be under χ and observe that, because χ is π-special, by (4.1) of [2], η is π-special. By induction, $\eta^{(N)}$ lies over θ^*, and $\chi^{(M)}$ over η^*. Since $N_G(H \cap N) < G$ and $(\chi^{(N)})^{(N_M(H \cap N))} = \chi^{(M)}$, by induction (and using the other direction), it follows that $\chi^{(N)}$ lies over $\eta^{(N)}$, and hence, over θ^*.

Suppose now that $\chi^{(N)}$ lies over θ^* and let $\psi \in \mathrm{Irr}(N_M(H \cap N))$ over θ^* and under $\chi^{(N)}$ (hence ψ is π-special). By induction, ψ^* lies under $\chi^{(M)}$. Now let $\eta \in \mathrm{Irr}(M)$ with $\eta^* = \psi^*$. Since $\psi = \eta^{(N)}$, again by induction, η lies over θ and, therefore χ over θ.

By comments after Theorem 3.5, we may assume that G/N is cyclic of prime π'-order.

Step 2 $\mathcal{O}_{\pi'}(N) = 1$.

Write $U = \mathcal{O}_{\pi'}(N)$. Since θ and χ are π-specials, $U \subseteq \ker \theta \cap \ker \chi$. Also observe that $N_{G/U}(H/U) = N_G(H)/U$ and $N_{G/U}(H/U \cap N/U) = N_G(H \cap N)/U$.

Write $\bar{\chi} \in \mathrm{Irr}(G/U)$ and $\bar{\theta} \in \mathrm{Irr}(N/U)$ for the characters corresponding to χ and θ, respectively. Notice that $\bar{\chi}^* = \overline{\chi^*}$ (to convince yourself of this fact just use Theorem 3.1, Lemma 3.2 and an inductive argument). Hence $(\bar{\chi}^{(N/U)})^* = \bar{\chi}^* = \overline{\chi^*} = \overline{(\chi^{(N)})^*} = (\overline{\chi^{(N)}})^*$ and therefore, $\bar{\chi}^{(N/U)} = \overline{\chi^{(N)}}$. Now, if $|G/U| < |G|$, by induction, it follows that χ lies over θ if and only if $\bar{\chi}$ lies over $\bar{\theta}$ if and only if $\bar{\chi}^{(N/U)}$ lies over $\bar{\theta}^*$ if and only if $\overline{\chi^{(N)}}$ lies over $\bar{\theta}^*$ if and only if $\chi^{(N)}$ lies over θ^*.

Step 3 If M is any normal subgroup of G contained in N, then χ_M is homogeneous.

Since χ has π-degree, all irreducible constituents of χ_M have stabilizers with π-index in G, and thus, it is possible to choose φ an irreducible H-invariant constituent of χ_M. Let $\eta \in \mathrm{Irr}(T|\varphi)$ such that $\eta^G = \chi$, where $T = I_G(\varphi)$. Observe that η is π-special: in the notation of Theorem (2.1), we have that η and η^σ are two characters over $\varphi = \varphi^\sigma$ such that $\eta^G = \chi = \chi^\sigma = (\eta^\sigma)^G$. By uniqueness, $\eta = \eta^\sigma$, and since η has π-degree, by (5.4) of [8], η is π-special.

Suppose that $T < G$. By induction, it follows that $\eta^{(M)}$ lies over φ^*. We claim that $N_T(H \cap M) = I_{N_G(H \cap M)}(\varphi^*)$. Since $N_G(H \cap M)$ acts on M fixing $H \cap M$, it follows that $(\varphi^y)^* = (\varphi^*)^y$, for any $y \in N_G(H \cap M)$, and since $*$ is one to one, the claim is proved. Therefore, $(\eta^{(M)})^{N_G(H \cap M)} \in \mathrm{Irr}(N_G(H \cap M))$. By Theorem 3.5, we have that $((\eta^{(M)})^{N_G(H \cap M)})^* = \eta^{*N_G(H)} = \chi^* = (\chi^{(M)})^*$ and then, $(\eta^{(M)})^{N_G(H \cap M)} = \chi^{(M)}$.

By the same argument, since $\eta_{T\cap N}$ is the Clifford correspondent of χ_N, $(\eta^{(M)}_{T\cap N})^{N_N(H\cap M)}=(\chi_N)^{(M)}$.

Now, since $(\chi^{(M)})^{N_N(H\cap M)}=\chi^{(N)}$ and since $N_G(H\cap M)<G$ (by Step 2), it suffices to show that χ lies over θ if and only if $\chi^{(M)}$ lies over $\theta^{(M)}$ and apply the inductive hypothesis to the proper subgroup $N_G(H\cap M)$ with the normal subgroup $N_N(H\cap M)$ and to the characters $\chi^{(M)}$ and $\theta^{(M)}$.

Since $T<G$, by induction $(\eta^{(T\cap N)})_{N_{T\cap N}(H\cap N)}=(\eta_{T\cap N})^*$.

Now, $N_{T\cap N}(H\cap M)$ is normal in $N_T(H\cap M)$ and $N_{N_T(H\cap M)}(H\cap N_{T\cap N}(H\cap M))=N_T(H\cap N)$. Therefore, $\eta^{(T\cap N)}=(\eta^{(M)})^{N_{T\cap N}(H\cap M))}$. Since $\eta^{(T\cap N)}$ lies over $(\eta_{T\cap N})^*=((\eta_{T\cap N})^{(M)})^*$, we may apply the inductive hypothesis to the proper subgroup of G, $N_T(H\cap M)$ to conclude that $\eta^{(M)}$ lies over $(\eta_{T\cap N})^{(M)}$, and therefore, $(\eta^{(M)})_{N_{T\cap N}(H\cap M)}=(\eta_{T\cap N})^{(M)}$.

Now, since $\chi^{(M)}$ has π-degree and it is induced from $\eta^{(M)}$, it follows that $|N_G(H\cap M):N_T(H\cap M)|$ is a π-number. Since, by step 1, G/N is a π'-group, we have that $N_G(H\cap M)=N_T(H\cap M)\,N_N(H\cap M)$. By Mackey, $(\chi^{(M)})_{N_N(H\cap M)}$ $=((\eta^{(M)})^{N_G(H\cap M)})_{N_N(H\cap M)}=((\eta_{T\cap N})^{(M)})^{N_N(H\cap M)}=(\chi_N)^{(M)}$.

Now, if $\chi_N=\theta$, certainly $\chi^{(M)}$ lies over $\theta^{(M)}$. Conversely, if $\chi^{(M)}$ lies over $\theta^{(M)}$, $(\chi_N)^{(M)}=\theta^{(M)}$, and thus $\chi_N=\theta$ by uniqueness. This proves Step. 3.

Final Step

Let $K=\mathcal{O}^{\pi\pi'}(G)$, $L=K'$, $J=LN_G(H)$ and observe that $\mathcal{O}^{\pi\pi'}(N)\subseteq K\subseteq N$, $KJ=G$ and (by an standard argument) that $K\cap J=L$.

First we claim that if $L\subseteq Y\subseteq K$ for a normal subgroup Y of G, then all complements of K/Y in G are G-conjugate. Let $Y_o=YN_G(H)$. We know that Y_o is a complement of K/Y in G. If Y_1 is another such complement, we may assume that $H\subseteq Y_1$. Then, since KH is normal in G, $KH\cap Y_1=HY\lhd Y_1$, and thus $Y_1\subseteq N_G(HY)=N_G(H)\,Y=Y_o$. By order considerations, $Y_1=Y_o$.

We choose now K/L_o a chief factor of G and let $J_o=JL_o=L_o\,N_G(H)$. Since $K\subseteq N$, it follows by the previous step that χ_K and χ_{L_o} are homogeneous. Write $\chi_K=e\xi$, where $\xi\in\mathrm{Irr}(K)$. By the going down Theorem 6.18 of [5], ξ_{L_o} is irreducible or fully ramified over K/L_o. Therefore, by Corollary 4.2 of [8], the previous claim and Theorem 9.1 of [4], we can write

$$\chi_{J_o}=\beta+2\Delta_1,$$

where β is an irreducible π-special character of J_o (by two applications of Theorem 2.1), Δ_1 is a character of J_o or zero, and $\beta^*=\chi^*$ (by Theorem 3.1).

If $\mathcal{O}^{\pi\pi'}(N)=1$, by step 2, N would be a π-group. But in this case, $H\cap N=1$ and the theorem is true. Now let $\mathcal{O}^{\pi\pi'}(N)/Y$ be a chief factor of G and let $X_o=YX$, where $X=\mathcal{O}^{\pi\pi'}(N)'\,N_G(H\cap N)$. Since $NN_G(H\cap N)=G$, it follows that $\mathcal{O}^{\pi\pi'}(N)\,X=G$. Also, $\mathcal{O}^{\pi\pi'}(N)\cap X=\mathcal{O}^{\pi\pi'}(N)\cap N_N(H\cap N)\,\mathcal{O}^{\pi\pi'}(N)'=\mathcal{O}^{\pi\pi'}(N)'$ and therefore, X_o is a proper subgroup of G.

We claim that all complements of $\mathcal{O}^{\pi\pi'}(N)/Y$ in G are G-conjugate to X_o. Since $N\cap X_o$ is not normal in G (because $|N:N\cap X_o|$ is a π-number and N $=(N\cap X_o)\,\mathcal{O}^{\pi\pi'}(N))$ and X_o is maximal in G, it follows that $X_o=N_G(N\cap X_o)$. Since complements of $\mathcal{O}^{\pi\pi'}(N)/Y$ in N are conjugate in N by the previous claim, this claim follows.

Now, by the same argument as before, we may write

$$\chi_{X_{\mathcal{O}}} = \tau + 2\Delta_2,$$

where τ is π-special (by two applications of Theorem 2.1) and Δ_2 is a character of $X_{\mathcal{O}}$ or zero. Then $\chi_{N \cap X_{\mathcal{O}}} = \tau_{N \cap X_{\mathcal{O}}} + 2\Delta_{2N \cap X_{\mathcal{O}}}$ and by Theorem 3.1,

$$(\tau_{N \cap X_{\mathcal{O}}})^* = (\chi_N)^*.$$

Now let $\chi_{\mathcal{O}} \in \mathrm{Irr}(X_{\mathcal{O}})$ and $\beta_{\mathcal{O}} \in \mathrm{Irr}(X_{\mathcal{O}} \cap J_{\mathcal{O}})$ be the π-specials characters with $\chi_{\mathcal{O}}^* = \chi^* = \beta_{\mathcal{O}}^*$.

Since $\mathcal{O}^{\pi\pi'}(X_{\mathcal{O}})' N_{X_{\mathcal{O}}}(H) \subseteq X_{\mathcal{O}} \cap J_{\mathcal{O}}$, we can write

$$\tau_{X_{\mathcal{O}} \cap J_{\mathcal{O}}} = \tau_{\mathcal{O}} + 2A_1 + B_1,$$

where all irreducible constituents of B_1 do not have π-degree, $\tau_{\mathcal{O}}$ is π-special and $\tau^* = \tau_{\mathcal{O}}^*$.

Also, since $\mathcal{O}^{\pi\pi'}(N \cap J_{\mathcal{O}})' N_{N \cap J_{\mathcal{O}}}(H \cap N) \subseteq N \cap X_{\mathcal{O}} \cap J_{\mathcal{O}}$, we can write

$$\beta_{N \cap X_{\mathcal{O}} \cap J_{\mathcal{O}}} = \varepsilon + 2A_2 + B_2,$$

where all irreducible constituents of B_2 do not have π-degree, ε is π-special and $\varepsilon^* = (\beta_{N \cap J_{\mathcal{O}}})^*$. Since $J_{\mathcal{O}} < G$, by induction, $(\beta^{(N \cap J_{\mathcal{O}})})_{N_{N \cap J_{\mathcal{O}}}(H \cap N)} = (\beta_{N \cap J_{\mathcal{O}}})^*$.

Now, observe that $(\beta_{\mathcal{O}})^{(N \cap X_{\mathcal{O}} \cap J_{\mathcal{O}})}$ is a character of $N_{X_{\mathcal{O}} \cap J_{\mathcal{O}}}(H \cap N) = N_{J_{\mathcal{O}}}(H \cap N)$ (because $X_{\mathcal{O}}$ contains $N_G(H \cap N)$). Also, $\beta^{(N \cap J_{\mathcal{O}})}$ is a character of $N_{J_{\mathcal{O}}}(H \cap N)$. Now, we have that $(\beta_{\mathcal{O}})^{(N \cap X_{\mathcal{O}} \cap J_{\mathcal{O}})} = \beta^{(N \cap J_{\mathcal{O}})}$ (because their $*$ is the same, χ^*), and therefore it follows that $(\beta_{\mathcal{O}})^{(N \cap X_{\mathcal{O}} \cap J_{\mathcal{O}})}$ lies over $(\beta_{N \cap J_{\mathcal{O}}})^* = \varepsilon^*$. By induction, $\beta_{\mathcal{O}}$ lies over ε and hence $\beta_{\mathcal{O} N \cap X_{\mathcal{O}} \cap J_{\mathcal{O}}} = \varepsilon$.

We have that $\chi_{N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} = (\chi_{J_{\mathcal{O}}})_{N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} = \beta_{N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} + 2\Delta_{1 N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} = \varepsilon + 2A_2 + B_2 + 2\Delta_{1 N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}}$. Therefore $[\chi_{N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}}, \varepsilon] \equiv 1 \bmod 2$. Also, $\chi_{N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} = \tau_{N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} + 2\Delta_{2 N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} = \tau_{\mathcal{O} N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} + 2A_{1 N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} + B_{1 N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} + 2\Delta_{2 N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}}$.

Since ε is $X_{\mathcal{O}} \cap J_{\mathcal{O}}$-invariant, $X_{\mathcal{O}} \cap J_{\mathcal{O}} / X_{\mathcal{O}} \cap J_{\mathcal{O}} \cap N$ is cyclic and all the irreducible constituents of B_1 have not π-degree, we have $[B_{1 N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}}, \varepsilon] = 0$. Then $1 \equiv [\chi_{N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}}, \varepsilon] \equiv [\tau_{\mathcal{O} N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}}, \varepsilon] \bmod 2$ and thus $\tau_{\mathcal{O} N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}} = \varepsilon$.

Now, $\tau_{\mathcal{O}}$ and $\beta_{\mathcal{O}}$ are two π-special extensions of ε. Since $X_{\mathcal{O}} \cap J_{\mathcal{O}} / N \cap J_{\mathcal{O}} \cap X_{\mathcal{O}}$ is a π'-group, by (6.1) of [2], $\tau_{\mathcal{O}} = \beta_{\mathcal{O}}$. Then $\tau^* = \tau_{\mathcal{O}}^* = \beta_{\mathcal{O}}^* = \chi_{\mathcal{O}}^*$ and thus $\tau = \chi_{\mathcal{O}}$. Also, $(\tau^{(X_{\mathcal{O}} \cap N)})^* = \tau^* = \tau_{\mathcal{O}}^* = \beta_{\mathcal{O}}^* = \chi^* = (\chi^{(N)})^*$ and hence $\tau^{(X_{\mathcal{O}} \cap N)} = \chi^{(N)}$. By the inductive hypothesis, $(\tau^{(X_{\mathcal{O}} \cap N)})_{N_N(H \cap N)} = (\tau_{X_{\mathcal{O}} \cap N})^*$.

Now $\chi_N = \theta$ if and only if $(\tau_{X_{\mathcal{O}} \cap N})^* = \theta^*$ if and only if $(\tau^{(X_{\mathcal{O}} \cap N)})_{N_N(H \cap N)} = \theta^*$ if and only if $(\chi^{(N)})_{N_N(H \cap N)} = \theta^*$.

When the group of odd order G happens to have a normal Hall π-subgroup, say $\mathcal{O}$, then the normalizer of a π-complement H of G is $H \times C_{\mathcal{O}}(H)$ and, of course, the Isaacs correspondent χ^* of some $\chi \in \mathrm{Irr}^{\pi}(G)$ is very much related to the Glauberman-Isaacs correspondent $(\chi_{\mathcal{O}})^* \in \mathrm{Irr}_H(\mathcal{O})$. It is an easy exercise to check that $\chi^* = 1_H \times (\chi_{\mathcal{O}})^*$.

To end this section we need the following.

(3.7) Theorem. *Let N be a normal π-subgroup of G, where G is a group of odd order. Let $\chi \in \mathrm{Irr}^{\pi}(G)$ and let $\theta \in \mathrm{Irr}_H(N)$ be under χ, where H is a fixed π-complement of G. Then the Glauberman-Isaacs correspondent $\theta^* \in \mathrm{Irr}(C_H(N))$ lies under the Isaacs correspondent $\chi^* \in \mathrm{Irr}(N_G(H))$.*

Proof. First we claim that if M is any normal subgroup of G and $\eta \in \mathrm{Irr}^{\pi}(MN_G(H))$ is such that $\eta^* = \chi^*$, then η is an irreducible constituent of $\chi_{MN_G(H)}$.

We prove the claim by induction on $|G|$. Certainly, we may assume that $MN_G(H) < G$. Let $\bar{G} = G/M$, $\bar{K} = \mathcal{O}^{\pi\pi'}(\bar{G})$ and let $\bar{L} = \bar{K}'$. Since $N_{\bar{G}}(\bar{H}) < \bar{G}$, then $\bar{L}N_{\bar{G}}(\bar{H}) < \bar{G}$, and thus $LN_G(H) < G$. Now, $\mathcal{O}^{\pi\pi'}(G)' N_G(H) \subseteq LN_G(H) < G$ and therefore by Theorem 3.1, there exists a π-degree irreducible constituent ψ of $\chi_{LN_G(H)}$ such that $\psi^* = \chi^*$. By induction, η is a constituent of $\psi_{MN_G(H)}$ and therefore of $\chi_{MN_G(H)}$.

Now that the claim is proved, let $\eta \in \mathrm{Irr}^{\pi}(NN_G(H))$ be such that $\eta^* = \chi^*$. Because the stabilizers of the irreducible constituents of η_N have π-index, we may choose γ an irreducible H-invariant constituent of η_N. By the claim, θ and γ are two H-invariant irreducible constituents of χ_N and then, by (2.10) of [7], for instance, θ and γ are $N_G(H)$-conjugate. Therefore, θ is an irreducible constituent of γ_N and by induction, we may assume that $NN_G(H) = G$. Now NH is a normal subgroup of G and we choose $\xi \in \mathrm{Irr}^{\pi}(NH)$ over θ and under χ. Since by comments after Theorem 3.5, ξ^* lies under χ^*, by induction, we may assume that G has a normal Hall π-subgroup. In this case, $\chi^* = 1_H \times \theta^*$ which certainly lies over θ^*.

4 Main results

Before entering into the details of the proofs of Theorems A and B, let us outline how the natural correspondence (in the classical case $\pi = p'$) is constructed.

If P is a p-subgroup of a finite group G let us denote by $IBr(G|P)$ the set of all irreducible Brauer characters of G with vertex P. We have shown in Sect. 3, that if G is a group of odd order and P is a Sylow p-subgroup of G, then there exists a natural correspondence $*$: $IBr(G|P) \to IBr(N_G(P)|P)$. Now we want to remove the hypothesis of P being a Sylow p-subgroup of G. So let us assume that P is any p-subgroup of G and let $\varphi \in IBr(G|P)$. By a well known theorem of Huppert [3], we may write $\varphi = \gamma^G$, where $\gamma \in IBr(W)$ and γ has p'-degree. Since the Sylow p-subgroups of W are vertices for φ, we may assume that $P \in Syl_p(W)$ and therefore we have defined some Brauer character $\gamma^* \in IBr(N_W(P))$.

We will show that $\gamma^{*N_G(P)} \in IBr(N_G(P))$, that $\gamma^{*N_G(P)}$ only depends on φ and that the map $\varphi \to \gamma^{*N_G(P)}$ is a well defined bijection from $IBr(G|P)$ onto $IBr(N_G(P)|P)$.

The following theorem is a key result for proving the existence of vertices for sets of primes as well as for proving our main results.

(4.1) Theorem. *Let $\varphi \in I_\pi(G)$, let N be a normal subgroup of G* where *G is π-separable and assume that $\varphi = \gamma^G$ for some $\gamma \in I_\pi(W)$. If all irreducible constituents of φ_N have π-degree then $|W: W \cap N|$ is a π-number.*

Proof. See [12].

If Q is a π'-subgroup of a π-separable group G, we denote by $I_\pi(G|Q)$ the elements of $I_\pi(G)$ with vertex Q. Next one is one of the main results in [12].

(4.2) Theorem. *If G is solvable with nilpotent π-complement, then $|I_\pi(G|Q)| = |I_\pi(N_G(Q)|Q)|$.*

Proof. See [12].

Since (P, γ) is a p-weight of G if and only if $\gamma \in IBr(N_G(P)|P)$, Theorems A and B will be proved once we show the following.

(4.3) Theorem. *Suppose that G is a group of odd order and let $\varphi \in I_\pi(G|Q)$.*

(a) *If $\alpha^G = \varphi$, where $\alpha \in I_\pi(W)$ has π-degree and Q is a π-complement of W, then $\alpha^* \in I_\pi(N_W(Q))$ induces irreducibly to $N_G(Q)$.*

(b) *The map $I_\pi(G|Q) \to I_\pi(N_G(Q)|Q)$ given by $\varphi \to \alpha^{*N_G(Q)}$ is a well defined natural injection.*

(c) *If a π-complement of G is nilpotent, then above map is a bijection.*

Proof. First of all, if $\varphi \in I_\pi(G|Q)$, let us show that there exists some $\alpha \in I_\pi(W)$ as in (a). Certainly, if φ has π-degree, Q is a π-complement of G and we may choose $\alpha = \varphi$. If φ has not π-degree, by repeated applications of (3.4) of [9] and the Clifford Correspondence ((3.2) of [9]), we may write $\varphi = \alpha^G$, for some $\alpha \in I_\pi(W)$ with $\alpha(1)$ a π-number. If Q_o is a π-complement of W, Q_o is a vertex for φ and thus Q_o and Q are G-conjugate. By conjugating by some appropriate element, we may assume that Q is a π-complement of W, as wanted.

(a) We argue by induction on $|G|$. Suppose first φ has π-degree. Let $\mu \in \mathrm{Irr}(W)$ be π-special such that $\mu^o = \alpha$. Then, since $(\mu^G)^o = (\mu^o)^G = \alpha^G = \varphi$, $\mu^G \in \mathrm{Irr}(G)$ and by an application of Theorem 2.1, μ^G is π-special. Then, applying Corollary 3.4 and Theorem 3.5, $\alpha^{*N_G(Q)} = (\mu^{*N_G(Q)})^o = \varphi^*$ is irreducible.

Suppose now that φ has not π-degree, and let $N \lhd G$ be such that all irreducible constituents of φ_N have π-degree and are not G-invariant. Then, by Theorem 4.1, $|WN : W|$ is a π-number and thus Q is a π-complement of WN. Consider $\alpha^{WN} \in I_\pi(WN)$, and notice that α^{WN} has π-degree. Thus, because the stabilizers of the irreducible constituents of $(\alpha^{WN})_N$ have π-index we may choose a Q-invariant irreducible constituent $\theta \in I_p(N)$ of $(\alpha^{WN})_N$.

By (3.2) of [9], let $\beta \in I_\pi(J|\theta)$ be such that $\beta^{WN} = \alpha^{WN}$, where $J = T \cap WN$ and $T = I_G(\theta)$. Then β has π-degree, Q is a π-complement of J and $\beta^G = \varphi$. Therefore, $\beta^T \in I_\pi(T|Q)$ and by induction $\beta^* \in I_\pi(N_J(Q))$ is such that $\beta^{N_T(Q)} \in I_\pi(N_T(Q))$. Also, by the first paragraph of the proof of (a), $(\alpha^*)^{N_{WN}(Q)} = (\alpha^{WN})^*$ and $(\beta^*)^{N_{WN}(Q)} = (\alpha^{WN})^*$. Therefore, $(\alpha^*)^{N_G(Q)} = ((\alpha^{WN})^*)^{N_G(Q)} = (\beta^*)^{N_G(Q)}$.

Now let $\nu \in \mathrm{Irr}(J)$ be π-special with $\nu^o = \beta$ and write $\nu_N = e\omega$, where $\omega \in \mathrm{Irr}(N)$ is π-special and $\omega^o = \theta$. Since the $\mathbf{B}_\pi$-lifting is unique it follows that $I_G(\omega) = T$. Let $\nu^{(N)} \in \mathrm{Irr}(N_J(Q \cap N))$ be such that $(\nu^{(N)})^* = \nu^*$. By Theorem 3.6, $\nu^{(N)}$ lies over ω^*. Since both characters are π-specials we may considered them as $\nu^{(N)} \in \mathrm{Irr}(N_J(Q \cap N)/Q \cap N)$ and $\omega^* \in \mathrm{Irr}(N_N(Q \cap N)/Q \cap N)$. Now ω^* is Q-invariant (because ω is), $N_N(Q \cap N)/Q \cap N)$ is a normal π-subgroup of $N_J(Q \cap N)/Q \cap N$ and by Theorem 3.7, $\omega^{**} \in \mathrm{Irr}(N_N(Q)/Q \cap N)$ (the Glauberman-Isaacs correspondent of ω^*) lies over $(\nu^{(N)})^* = \nu^* \in \mathrm{Irr}(N_J(Q)/Q \cap N)$. Since $(\omega^x)^{**} = (\omega^{**})^x$ for $x \in N_G(Q)$, we have that $N_T(Q) = I_{N_N(Q)}(\omega^{**}) = I_{N_N(Q)}((\omega^{**})^o)$. Now $(\nu^*)^o = \beta^*$ lies over $(\omega^{**})^o$ and so does $\beta^{*N_T(Q)} \in I_\pi(N_T(Q)|(\omega^{**})^o)$. Then, by (3.2) of [9], $\alpha^{*N_G(Q)} = \beta^{*N_G(Q)} \in I_\pi(N_G(Q))$, as wanted.

(b) In the notation of (a), since α^* has π-degree and Q is a π-complement of $N_W(Q)$, obviously, $(\alpha^*)^{N_G(Q)} \in I_\pi(N_G(Q)|Q)$.

Suppose now that $\alpha^G = \varphi = \beta^G$, where $\alpha \in I_\pi(W|Q)$ and $\beta \in I_\pi(V|Q)$ have π-degree. We show that $(\alpha^*)^{N_G(Q)} = (\beta^*)^{N_G(Q)}$ by induction on $|G|$.

Suppose first that φ has π-degree. Then Q is a π-complement of G and if α_o is the π-special lifting of α, by Theorem 3.3, $\varphi^* = (\alpha^G)^* = (\alpha_o^G)^{*o} = ((\alpha_o^*)^{N_G(Q)})^o = \alpha^{*N_G(Q)}$ and by the same reason $\varphi^* = \beta^{*N_G(Q)}$.

So we may assume that φ has not π-degree. In this case, by (3.4) of [9], let N be a normal subgroup of G such that the irreducible constituents of φ_N have π-degree and are not G-invariant. By Theorem 4.1, $|W: W \cap N|$ and $|V: V \cap N|$ are π-numbers. Let θ and η be Q-invariant irreducible constituents of $(\alpha^{WN})_N$ and $(\beta^{VN})_N$, respectively and let $\delta \in I_\pi(T \cap WN)$ and $\tau \in I_\pi(I \cap VN)$ be such that $\delta^{WN} = \alpha^{WN}$ and $\tau^{VN} = \beta^{VN}$, where $T = I_G(\theta)$ and $I = I_G(\eta)$.

Since α^{WN} and β^{VN} are irreducible constituents of φ_{WN} and φ_{VN}, respectively, it follows that $\theta = \eta^g$ for some $g \in G$, and thus, $I^g = T$ and $(\tau^I)^g = \delta^T$ (because both are the Clifford correspondents of φ over θ). Now, since $\tau \in I_\pi(I \cap VN|Q)$, then $\tau^g \in I_\pi((I \cap VN)^g|Q^g)$ and thus $\delta^T = (\tau^g)^T \in I_\pi(T|Q^g) \cap I_\pi T|Q)$. Since vertices in T are T-conjugate, we have that $Q^g = Q^t$ for some $t \in T$. Then $gt^{-1} \in N_G(Q)$, $\eta^{gt^{-1}} = \theta^{t^{-1}} = \theta^{t^{-1}} = \theta$ and hence η and θ are $N_G(Q)$-conjugate. Then, certainly we may assume that $\theta = \eta$ and $\delta^T = \tau^I$. Since $T < G$, by induction, we have that $(\tau^*)^{N_T(Q)} = (\delta^*)^{N_T(Q)}$. Therefore, $(\alpha^*)^{N_G(Q)} = (\delta^*)^{N_G(Q)} = (\tau^*)^{N_G(Q)} = (\beta^*)^{N_G(Q)}$.

Finally, suppose that $(\alpha^*)^{N_G(Q)} = (\beta^*)^{N_G(Q)}$, where $\alpha \in I_\pi(W|Q)$ and $\beta \in I_\pi(V|Q)$ have π-degree and α^G and β^G are irreducible. We prove that $\alpha^G = \beta^G$ by induction on $|G|$.

If α^G has π-degree, then Q is a π-complement of G, β^G has π-degree and since $(\alpha^G)^* = (\alpha^*)^{N_G(Q)} = (\beta^*)^{N_G(Q)} = (\beta^G)^*$, it follows by Corollary 3.4, that $\alpha^G = \beta^G$. Again, we may assume that α^G has not π-degree and we may choose $N \lhd G$ such that the irreducible constituents of $(\alpha^G)_N$ have π-degree and are not G-invariant.

Since by (4.1), $|WN: W|$ is a π-number, we certainly may choose $\theta \in I_\pi(N)$ a Q-invariant irreducible constituent of $(\alpha^{WN})_N$. Although at this moment we do not know whether $|VN: V|$ is a π-number or not, still it is possible to choose a Q-invariant irreducible constituent η of $(\beta^{VN})_N$: since $\beta^{VN} \in I_\pi(VN|Q)$, if ε is an irreducible constituent of $(\beta^{VN})_N$, then $\beta^{VN} = \xi^{VN}$, for some $\xi \in I_\pi(VN \cap I_{VN}(\varepsilon)|\varepsilon)$. Then, if P is a vertex for ξ, then P is a vertex for β^{VN} and thus $P = Q^x$ for some $x \in VN$. Therefore $\eta = \varepsilon^{x^{-1}}$ is a Q-invariant irreducible constituent of $(\beta^{VN})_N$. Let us denote by θ^{**} the character $\omega^{**\rho}$ of part (a), and recall that $I_{N_G(Q)}(\theta^{**}) = N_T(Q)$, where $T = I_G(\theta)$. Now let $\delta \in I_\pi(T \cap WN|\theta)$ be such that $\delta^{WN} = \alpha^{WN}$, let $I = I_G(\eta)$ and let $\tau \in I_\pi(I \cap VN|\eta)$ be such that $\tau^{VN} = \beta^{VN}$. Recall that θ^{**} and η^{**} lie under δ^* and τ^*, respectively. Now since $\tau^{*N_G(Q)} = \delta^{*N_G(Q)}$, we have that θ^{**} and η^{**} are $N_G(Q)$-conjugate. By replacing β by some $N_G(Q)$-conjugate, we may assume that $\theta^{**} = \eta^{**}$. Then $\theta = \eta$, because both $*$ are one to one maps. Now, $I = T$, $N_I(Q) = N_T(Q) = I_{N_G(Q)}(\theta^{**})$ and since $((\tau^*)^{N_T(Q)})^{N_G(Q)} = ((\delta^*)^{N_T(Q)})^{N_G(Q)}$, by uniqueness in the Clifford Correspondence, $(\tau^*)^{N_T(Q)} = (\delta^*)^{N_T(Q)})$. By induction, $\delta^T = \tau^T$, and then $\alpha^G = \delta^G = \tau^G = \beta^G$.

(c) Apply (b) and Theorem 4.2.

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