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On the preservation of k-very ampleness under adjunction

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Contents

Introduction
0 Background material
1 k-very ample line bundles on curves
2 Some special classes and low degree classification
3 Existence theorems for k-reductions
4 k-very ampleness of the adjoint bundle

Introduction

Let L be a line bundle on a smooth, connected, projective surface, S. L is said to be *k-very ample* (see [BS3]) for an integer $k \geqq 0$, if given any 0-dimensional subscheme, $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$, of S with length$(\mathcal{O}_{\mathscr{Z}}) = k + 1$, it follows that the restriction map, $\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_{\mathscr{Z}})$, is onto. Note that L is 0-very ample if and only if L is spanned by global sections, and L is 1-very ample if and only if L is very ample. The notion of k-very ampleness has a classical interpretation in terms of associated secant mappings, and the line bundle, L, is a k-very ample line bundle on S, if and only if an associated line bundle, $L^{\sim}$, on $S^{[k]}$, the Hilbert scheme of 0-dimensional subschemes of S of length k, is very ample (see [BS3] and [CG]). Under this correspondence, of a line bundle $\mathscr{L}$ on S to $\mathscr{L}^{\sim}$ on $S^{[k]}$, $K_S \otimes \mathscr{L}$ goes to $(K_S \otimes \mathscr{L})^{\sim} \cong K_{S^{[k]}} \otimes \mathscr{L}^{\sim}$ (see the appendix to [BS3] for details).

If L is very ample, then either $K_S + L$ is spanned and the mapping, $\phi: S \to \mathbb{P}_{\mathbb{C}}$, associated to $\Gamma(K_S + L)$ has a 2-dimensional image, or (S, L) is on a short list of exceptional surfaces. If $\dim \phi(S) = 2$, then there is a smooth projective surface S', and an ample line bundle, L', on S', such that:

(a) S is the blowup, $r: S \to S'$, of S', along a finite set, B;
(b) $L' := (r_* L)^{**}$, and $K_S + L \cong r^*(K_{S'} + L')$, or equivalently $L \cong r^* L' - r^{-1}(B)$;
(c) $K_{S'} + L'$ is ample.

When it exists, the pair (S', L') is called the *reduction* of (S, L). In [SV] (and independently in [Se]) it is shown that $K_{S'} + L'$ is very ample except for four explicit examples, (S, L), with $L \cdot L \leqq 9$. This result has many applications to projective classification of varieties.

It is very natural to try to generalize the main result of [SV] to the case of k-very ample line bundles. The problem divides naturally into two cases, the "general" case when $L \cdot L \geqq 4k + 5$ and the "special" case when $L \cdot L \leqq 4k + 4$. This breakup stems from the Reider type, existence theorem 0.3.2 for the case $L \cdot L \geqq 4k + 5$. For very ample line bundles this division also occurred; the cases $L \cdot L \leqq 8$ required detailed classification (see [SV] for references to works of Okonek and Ionescu for this low degree classification). We note that the low degree classification ultimately rests on the results on the adjunction mapping in the case $h^1(\mathcal{O}_S) = 0$ (see [So]).

The problem of k-very ampleness of $K_S \otimes L$ was solved for $k = 2$ and the "general case", $L \cdot L \geqq 13$, in [AP]. In this article, the main results of [SV] and [AP] are extended to all positive k in the case when $L \cdot L \geqq 4k + 5$.

It is convenient to deal first with certain special pairs. In what follows if S is a $\mathbb{P}^1$ bundle over a smooth curve C, with invariant e, we let E be a section of minimal self-intersection, $E^2 = -e$, and we let f be a fiber of the bundle. In case C is $\mathbb{P}^1$, we denote the unique $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ with a section, E, of minimal self-intersection, $E^2 = -e$, by $\mathbb{F}_e$. Assume now that L is a k-very ample line bundle on a smooth, complex, projective surface, S. Assume also that $L \cdot L \geqq 4k + 5$ and $k \geqq 2$. The following are worked out in §2.

1. If $S \cong \mathbb{P}^2$ and $K_S \otimes L$ is not k-very ample, then $L \cong \mathcal{O}_{\mathbb{P}^2}(a)$ for $k \leqq a \leqq k + 2$.
2. If $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $K_S \otimes L$ is not k-very ample, then $L \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ for $k \leqq \min\{a, b\} \leqq k + 1$.
3. If $L \cong -tK_S$ for some integer $t > 0$, S is isomorphic to neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, and $K_S \otimes L$ is not k-very ample, then:
   (a) $t = k$ if $K_S^2 \geqq 2$, and
   (b) $t = k + 2$ if $K_S^2 = 1$.
4. If $S \cong \mathbb{F}_1$ with $L \cong aE + bf$, and if $K_S \otimes L$ is not k-very ample, then:
   (a) $b \geqq a + k$ if $k \leqq a \leqq k + 1$, and
   (b) $b = a + k$ if $a \geqq k + 2$.
5. If S is the $\mathbb{P}^1$ bundle over an elliptic curve with invariant $e = -1$, L is numerically equal to $aE + bf$, and $K_S \otimes L$ is not k-very ample, then:
   (a) $b \geqq k + 2 - a$ if $k \leqq a \leqq k + 1$, and
   (b) $a = k + 2$ and $b = 0$ if $a \geqq k + 2$.
6. If S is a $\mathbb{P}^1$ bundle over a smooth curve C, $K_S \otimes L$ is not k-very ample, and if S is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$, or the $\mathbb{P}^1$ bundle with invariant $e = -1$ over an elliptic curve, then $k \leqq L \cdot f \leqq k + 1$.

The above results are worked out in (0.7) and in §2. Proposition 2.5 covers the hardest part of this argument, the case when S is a $\mathbb{P}^1$ bundle.

In §4, Proposition 4.1, it is shown that $kK_S + L$ is nef except for pairs (S, L) which are a subset of the above 6 classes. Because the question of the k-very ampleness of $K_S \otimes L$ is worked out for the pairs on the above list, it can be assumed that $kK_S + L$ is nef. If $kK_S + L$ is nef, E is an irreducible curve satisfying $(kK_S + L) \cdot E = 0$, and $E^2 < 0$, then $K_S + L$ is not k-very ample. In this case E is

a smooth rational curve satisfying $E \cdot E = -1$. In the classical case of very ample bundles, the correct move to make is to ask about the very ampleness of $K_{S'} \otimes L'$ on the "minimal reduction" of the pair, (S, L). This can be done also if $k \geqq 2$. In [AP], the 2-reduction, a natural generalization of the minimal reduction, for the case of 2-very ample line bundles is used (see [BS2, (0.9)]). In this article, the *k-reduction* is used for all k.

If $kK_S + L$ is nef then there exists a pair, (S', L'), a *k-reduction* of (S, L), and a morphism $r: S \to S'$, such that

- r expresses S as the blowup of a smooth surface S' at a finite subset, $B \subset S'$;
- $L' := (r_* L)^{**}$ is an ample line bundle on S', and $kK_S + L = r^*(kK_{S'} + L')$;
- $L \cong r^* L' - kr^{-1}(B)$, and there are no smooth, rational, curves, $\ell \subset S'$, satisfying $\ell \cdot \ell = -1$, $L' \cdot \ell \leqq k$.

If $kK_S + L$ is nef and big, then there is only one k-reduction, and it is called the k-reduction of (S, L). The results on k-reductions are contained in (0.4) and §3.

Proposition 4.2 shows that if $S \ncong S'$, then $K_{S'} \otimes L'$ is k-very ample. The main result of this paper, Theorem 4.4, shows that if $S \cong S'$ and (S, L) isn't on the above list of 6 classes where the k-very ampleness is worked out, then $K_S \otimes L$ is k-very ample.

Theorem 4.4 requires most of the work in the paper. By Theorem 0.3.2, there exists a curve, D, obstructing the k-very ampleness of $K_S + L$. The proof of (4.4) splits into the cases when $K_S \cdot D \geqq 0$ and $K_S \cdot D < 0$.

The work for the case, $K_S \cdot D \geqq 0$, is concentrated in a number of useful results, worked out in §1, on the k-very ampleness of line bundles on possibly singular curves. We note that these results are very useful for low degree classification. Using (1.7.3) the pairs (S, L) with $L \cdot L \leqq 2k + 3$ are completely classified in (2.6).

The work for the case, $K_S \cdot D < 0$, takes up most of §4 (see (4.4.5)).

0 Background material

Let S be a smooth connected projective surface defined over the complex field $\mathbb{C}$. We denote its structure sheaf by $\mathcal{O}_S$. For any coherent sheaf $\mathcal{F}$ on S, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(S, \mathcal{F})$ for $i \geqq 0$.

Let L be a line bundle on S. L is said to be *numerically effective* (nef, for short) if $L \cdot C \geqq 0$ for every curve C on X, and in this case L is said to be *big* if $L \cdot L > 0$.

(*0.1*) *Notation.* We fix some more notation.

$\approx$ (respectively $\sim$), the linear (respectively numerical) equivalence of line bundles;
$\chi(L) = \sum_i (-1)^i h^i(L)$, the Euler characteristic of a line bundle L;
$\Gamma(L)$, the space of the global sections of L. We say that L is *spanned* if it is spanned by $\Gamma(L)$;
$|L|$, the complete linear system associated to L;
K_S, the canonical divisor of S;
$q(S) = h^1(\mathcal{O}_S)$, the irregularity of S;
$g(L)$, the *sectional genus* of a line bundle L on S, defined by the equality $2g(L) - 2 = (K_S + L) \cdot L$. It can be easily seen that $g(L)$ is an integer. If $L \cong \mathcal{O}_S(D)$ for a divisor D on S, we also write $g(D)$ for $g(L)$. Furthermore if

D is an irreducible reduced curve on a surface S, $g(D)$ is simply the arithmetic genus, $p_a(D) = 1 - \chi(\mathcal{O}_D)$ of D;

$S^{[r]}$, the Hilbert scheme of 0-dimensional subschemes, $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$, of S, satisfying length$(\mathcal{O}_{\mathscr{Z}}) = r$.

Abuses. Line bundles and divisors are used with little (or no) distinction. Hence we shall freely switch from the multiplicative to the additive notation and vice versa. Sometimes the symbol "$\cdot$" of intersection of cycles is understood.

(0.2) Castelnuovo's bound. Let C be a reduced, irreducible non-degenerate curve of degree d in $\mathbb{P}^N$. Then the arithmetic genus, $g(C)$, of C satisfies (see e.g. [H, Theorem 3.7])

$$g(C) \leqq \left[\frac{d-2}{N-1}\right]\left(d - N - \left(\left[\frac{d-2}{N-1}\right] - 1\right)\frac{N-1}{2}\right),$$

where $[x]$ denotes the greatest integer $\leqq x$.

(0.3) k-very ampleness [BS3, CG]. Though our main focus in this paper is on smooth surfaces, we need to define k-very ampleness for singular varieties, and study it for singular curves. We say that a line bundle L on a reduced, projective variety, X, is *k-very ample* if the restriction map

$$\Gamma(L) \to \Gamma(\mathcal{O}_{\mathscr{Z}}(L))$$

is onto for any 0-dimensional subscheme, $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$, satisfying length$(\mathcal{O}_{\mathscr{Z}}) = k + 1$. Note that L is 0-very ample if and only if L is spanned by its global sections and L is 1-very ample if and only if L is very ample.

Assume now that $X = S$ is a smooth surface. Note that for $k = 2$, L being k-very ample is equivalent to L being *k-spanned* in the sense of [BFS] and [BS2], i.e. $\Gamma(L)$ surjects on $\Gamma(\mathcal{O}_{\mathscr{Z}}(L))$ for any *curvilinear* 0-cycle $\mathscr{Z} \in S^{[k+1]}$, (see [BFS, (0.4), (3.1)]). Furthermore, in view of the main result of [BS3], (Theorem 0.3.2 below) almost all the results of [BS2], stated for k-spannedness, hold for k-very ampleness (see also [BS2, note on p. 51]). The one exception is Lemma 0.5.3 of [BS2], which states that if L_1 and L_2 are k_1-spanned and k_2-spanned respectively, then $L_1 \otimes L_2$ is $(k_1 + k_2)$-spanned. This result is used in [BS2] to show that L^k is k-spanned where L is very ample and specifically to show the k-spanned version of (2.1.1) and (2.1.2) of §2 below. Fortunately the full strength of Lemma 0.5.3 is not needed in [BS2]. Lemma 0.3.5 is a weak analogue of Lemma 0.5.3 of [BS2] which suffices for the k-very ample versions of the results of [BS2].

(0.3.1) [BS2, (0.5.1)] If L is k-very ample on S, then $L \cdot C \geqq k$ for every effective curve, C on S, with equality only if $C \cong \mathbb{P}^1$ (see also (1.3.1)).

(0.3.2) Theorem [BS3, 2.1]. *Let L be a nef line bundle on a smooth surface S. Let $L \cdot L \geqq 4k + 5$ for $k \geqq 0$. Then either $K_S + L$ is k-very ample or there exists an effective divisor D such that $L - 2D$ is $\mathbb{Q}$-effective, D contains some 0-dimensional subscheme $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$ of S where the k-very ampleness fails and*

$$L \cdot D - k - 1 \leqq D \cdot D < L \cdot D/2 < k + 1 .$$

The following two facts are useful when applying Theorem 0.3.2.

(0.3.3) Lemma. *Let L be a nef line bundle on a smooth surface S. Let $L \cdot L \geqq 4k + 5$ for $k \geqq 0$. Assume that D is an effective curve on S satisfying*

$$L \cdot D - k - 1 \leqq D \cdot D < L \cdot D/2 < k + 1 .$$

If $k \geqq a^2 + a$ for an integer, $a \geqq 0$, then $L \cdot D \leqq 2k - a + 1$.

Proof. It is convenient to write $L \cdot D = k + \delta + 1$. Then $L \cdot D - k - 1 \leqq D \cdot D$ becomes $\delta \leqq D \cdot D$. Using $L \cdot L \geqq 4k + 5$ and $\delta \leqq D \cdot D$, the Hodge index relation, $(L \cdot L)(D \cdot D) \leqq (L \cdot D)^2$, becomes $(4k + 5)\delta \leqq (k + \delta + 1)^2$. This is equivalent to

$$\delta \leqq (k - \delta + 1)^2 . \tag{0.3.3.1}$$

Assume now that the lemma is false, i.e. assume that $k \geqq a^2 + a$ for an integer, $a \geqq 0$, and $L \cdot D \geqq 2k - a + 2$, i.e. $\delta \geqq k - a + 1$. Then (0.3.3.1) becomes

$$a^2 + 1 \leqq k - a + 1 \leqq \delta \leqq (k - \delta + 1)^2 \leqq a^2 .$$

This absurdity proves the lemma. QED

(0.3.4) Remark. Notation as in (0.3.2). If $L \cdot D \geqq k + 2$, write $L \cdot D = 1 + \delta + k$, $\delta \geqq 1$. Then $D \cdot D \geqq \delta$ and the Hodge index theorem gives $L \cdot L \leqq (1 + \delta + k)^2/\delta$. Note that, for any given $k \geqq 1$, $f(\delta) = (1 + \delta + k)^2/\delta$ is a decreasing function of δ on the interval, $1 \leqq \delta \leqq k + 1$. We will use this simple observation in §4.

(0.3.5) Lemma (see [BS2, (0.1.1)]). *Let $L_1, \ldots, L_k$ be very ample line bundles on a smooth surface, S. Then given any spanned line bundle, $\mathscr{L}$, on S, $L_1 \otimes \cdots \otimes L_k \otimes \mathscr{L}$ is k-very ample.*

Proof. Lemma 0.1.1 of [BS2] shows that $L_1 \otimes \cdots \otimes L_k$ is k-very ample. It is a simple check that the tensor product of a k-very ample line bundle and a spanned line bundle is k-very ample. QED

(0.4) k-reduction. Let L be an ample line bundle on a smooth, projective surface, S. Assume that $kK_S + L$ is nef for some $k > 0$. Note that $\mathscr{L} := (k - 1)K_S + L$ is ample. Let $\mathscr{E}$ be a maximal set of disjoint, smooth, rational curves, $\ell \subset S$, which satisfy $\ell \cdot \ell = -1$, and $\mathscr{L} \cdot \ell = 1$. Let $\pi : S \to S'$ be the reduction morphism with connected fibers obtained by contracting each curve, ℓ, to a point. Note that S' is a smooth projective surface, and $\pi : S \to S'$ is the blowing up of a finite set, B, of distinct points satisfying $\pi^{-1}(B) = \mathscr{E}$. Both $L' := (\pi_* L)^{**}$ and $\mathscr{L}' := (\pi_* \mathscr{L})^{**}$ are ample line bundles since they are ample outside of a finite set of points. We say that the pair, (S', L'), is *a k-reduction* of (S, L).

Let x be a point in B and $\ell = \pi^{-1}(x)$. Then $\ell \cdot \ell = -1$, $-K_S \cdot \ell = \mathscr{L} \cdot \ell = 1$. Hence in particular

$$L \cdot \ell = k . \tag{0.4.1}$$

We claim that

$$L \approx \pi^* L' - k\mathscr{E} . \tag{0.4.2}$$

To see this note that $(L + k\mathscr{E}) \cdot \ell = 0$ for any curve ℓ of $\mathscr{E}$. Then $L + k\mathscr{E} \approx \pi^* M$ for some line bundle M on S'. Therefore, since π is an isomorphism outside of $B = \pi(\mathscr{E})$, we have $M \cong (\pi_* L)^{**}$ on $S' - B$. Since $\operatorname{cod}_{S'} B = 2$ we have $M \cong L'$ on S'.

Let γ be the number of points blown up under π. Since $\mathscr{E}$ consists of pairwise disjoint curves with self-intersection -1, (0.4.2) gives

$$L \cdot L = L' \cdot L' - k^2 \gamma . \tag{0.4.3}$$

By (0.4.2) we have also that $kK_S + L \approx \pi^*(kK_{S'} + L')$. Furthermore, since $\mathscr{L} = (k-1)K_S + L \approx \pi^*((k-1)K_{S'} + L') - \mathscr{E}$, we have

$$\mathscr{L}' \approx (\pi_*(\pi^*((k-1)K_{S'} + L') - \mathscr{E}))^{**} \approx (k-1)K_{S'} + L' ,$$

and hence that $K_{S'} + \mathscr{L}' \approx kK_{S'} + L'$.

(S', L') has an important minimality property.

(0.4.4) Lemma. *There are no smooth, connected, rational curves, $\ell' \subset S'$, satisfying $\ell' \cdot \ell' = -1$ and $L' \cdot \ell' \leqq k$.*

Proof. Assume that there was such a curve ℓ'. Denote the proper transform of ℓ' under π by $\ell^\wedge$. If $B \cap \ell' \neq \emptyset$, then using (0.4.2), we have $L \cdot \ell^\wedge \leqq L' \cdot \ell' - k \leqq 0$. This contradiction to the ampleness of L shows that $B \cap \ell' = \emptyset$. Thus $\ell^\wedge$ is a smooth rational curve on S, disjoint from $\mathscr{E}$ and satisfying $\ell^\wedge \cdot \ell^\wedge = -1$. From the maximality of the set of curves making up $\mathscr{E}$, we conclude that $\mathscr{L} \cdot \ell^\wedge \neq 1$. This contradicts the ampleness of $\mathscr{L}$ since $\mathscr{L} \cdot \ell^\wedge = ((k-1)K_S + L) \cdot \ell^\wedge = -(k-1) + L \cdot \ell^\wedge = -(k-1) + L' \cdot \ell' \leqq 1$. QED

The above minimality property can be rephrased as saying that *there are no connected, smooth, rational curves, $\ell' \subset S'$, satisfying $\ell' \cdot \ell' = -1$, $\mathscr{L}' \cdot \ell' = 1$.* This follows from $\mathscr{L}' \approx (k-1)K_{S'} + L'$. This has the following important consequence.

(0.4.5) Lemma. *Let k, (S, L), (S', L') be as above. Then either*

(0.4.5.1) *$kK_{S'} + L'$ is ample;*
(0.4.5.2) *S' is a $\mathbb{P}^1$ bundle, $p: S' \to Y$, over a smooth curve, Y, with $kK_{S'} + L' \approx p^*\mathscr{H}$ for an ample line bundle $\mathscr{H}$ on Y; or*
(0.4.5.3) *$L' \approx -kK_{S'}$, and either $(S', L') \approx (\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2k, 2k))$, or $(S', L') \approx (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3k))$.*

Proof. Assume that $K_{S'} + \mathscr{L}' \approx kK_{S'} + L'$ is not ample. Note that by the above, $kK_{S'} + L'$ is nef. It cannot be big. If it was then by [LP, (2.5)] there would be a smooth, rational curve $\ell' \subset S'$, satisfying $\ell' \cdot \ell' = -1$, $\mathscr{L}' \cdot \ell' = 1$. This contradicts the reformulated minimality property. Since $K_{S'} + \mathscr{L}'$ is nef and not big, from [LP, (2.5)] we see that either

1. S' is a conic bundle, $p: S' \to Y$, with $K_{S'} + \mathscr{L}' \approx p^*\mathscr{H}$ for an ample line bundle $\mathscr{H}$ on Y; or
2. $\mathscr{L}' \approx -K_{S'}$.

In case 1 either S' is a $\mathbb{P}^1$ bundle or there exists a fiber $f = f_1 + f_2$ where f_i is a smooth, rational curve with $\mathscr{L}' \cdot f_i = 1$, $f_i \cdot f_i = -1$ for $i = 1, 2$. This would contradict the reformulated minimality property of (S', L'). Therefore case 1 gives case (0.4.5.2).

In case 2 we have that S' is minimal. If not, then there would be a smooth, rational, curve $\ell' \subset S'$ satisfying $\ell' \cdot \ell' = -1$. Thus $\mathscr{L}' \cdot \ell' = -K_{S'} \cdot \ell' = 1$. This again contradicts the reformulated, minimality property. Since the only minimal rational surfaces with ample anti-canonical bundle are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, case 2 gives us case (0.4.5.3). QED

From Lemma 0.4.5 it follows that if $kK_S + L$ is nef and big, then $kK_{S'} + L'$ is ample. From this it follows that (S', L') is the unique k-reduction. In this case we say that (S', L') is *the k-reduction* of (S, L).

(0.5) Special surfaces. Let L be a k-very ample line bundle on S and assume that $\mathscr{L} := (k-1)K_S + L$ is ample. Assume that $(S, \mathscr{L})$ is a *conic bundle* over a smooth curve C, i.e. that there exists a morphism, φ: $S \to C$, with connected fibers such that $K_S + \mathscr{L} \approx \varphi^* H$ for some ample line bundle H on C. For any fiber f of φ we have $\mathscr{L} \cdot f = 2$ and hence $L \cdot f = 2k$. We say that (S, L) is a *k-conic bundle*, or that S is a *k-conic bundle relative to L*. In this case any reducible fiber, f, of φ consists of 2 distinct, irreducible components, f_1, f_2, such that $f_1 \cdot f_2 = 1$. Recalling (0.3.1) and since $L \cdot f = 2k$, we have $L \cdot f_1 = L \cdot f_2 = k$ and $f_1 \cong f_2 \cong \mathbb{P}^1$.

By $\mathbb{F}_r$, with $r \geqq 0$, we denote the r-th Hirzebruch surface. $\mathbb{F}_r$ is the unique holomorphic $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ with a section, E, satisfying $E \cdot E = -r$. In the case $r = 0$, $\mathbb{F}_0$ is simply $\mathbb{P}^1 \times \mathbb{P}^1$. In the case $r \geqq 1$, E is the unique irreducible curve on $\mathbb{F}_r$ with negative self-intersection. □

In the case of Del Pezzo surfaces we need the following result, Corollary 2.6 from [BFS] (see also [BS2, (0.8)]). In [BFS] we proved Corollary 2.6 with the weaker conclusion of k-spannedness. The key point here is that the proof used Theorem 2.1 of [BFS], an earlier version of (0.3.2), which had the same hypotheses, but with the words k-spanned in place of k-very ample in the conclusion. Because of this circumstance, the proof of Corollary 2.6 of [BFS], with (0.3.2) used in place of Theorem 2.1 of [BFS], is the proof of this result.

(0.6) Proposition [BFS, (2.6)]. *Let S be a Del Pezzo surface. Then K_S^t is k-very ample for $k \geqq 0$ and a positive integer, t, if and only if:*

(0.6.1) $t \geqq k/3$ *if* $S = \mathbb{P}^2$;
(0.6.2) $t \geqq k/2$ *if* $S = \mathbb{P}^1 \times \mathbb{P}^1$;
(0.6.3) $t \geqq k+2$ *if* $K_S \cdot K_S = 1$;
(0.6.4) $t \geqq k$ *if* $K_S \cdot K_S \geqq 3$ *or* $K_S \cdot K_S = 2$ *and* $k \neq 1$.

The following is an immediate consequence of Proposition 0.6.

(0.7) Corollary. *Let S be a Del Pezzo surface. Assume that $L = -tK_S$ is k-very ample for some $k \geqq 2$ and a positive integer, t. Then $K_S + L$ is k-very ample unless*

(0.7.1) $k/3 \leqq t < k/3 + 1$ *if* $S = \mathbb{P}^2$;
(0.7.2) $k/2 \leqq t < k/2 + 1$ *if* $S = \mathbb{P}^1 \times \mathbb{P}^1$;
(0.7.3) $t = k+2$ *if* $K_S \cdot K_S = 1$;
(0.7.4) $t = k$ *if* $K_S \cdot K_S \geqq 2$.

1 k-very ample line bundles on curves

Let C be a reduced, pure dimensional, projective curve. Given a k-very ample line bundle, L, on C we have

$$h^0(L) \geqq k+1 \,. \tag{1.0}$$

To see this, take a 0-cycle $\mathscr{Z}$ consisting of $k+1$ distinct points on C. Then length$(\mathscr{O}_{\mathscr{Z}}) = k+1$. Since L is k-very ample, the restriction map, $\Gamma(L) \to \Gamma(\mathscr{O}_{\mathscr{Z}}(L))$, is surjective, and (1.0) follows.

The following lemma is very useful in the study of k-very ampleness on curves.

(1.1) Lemma. *Let L be a k-very ample line bundle on a reduced, pure dimensional, projective curve, C. If $\{x_1, \ldots, x_r\} \subset \mathrm{reg}(C)$ are distinct points with $1 \leqq r \leqq k$, then $\mathscr{L} := L \otimes [x_1]^{-1} \otimes \cdots \otimes [x_r]^{-1}$ is $(k-r)$-very ample. Moreover $\deg \mathscr{L} = \deg L - r$, and $h^0(\mathscr{L}) = h^0(L) - r$.*

Proof. It is enough to show the lemma for $r = 1$, i.e. for a single point, $x \in \mathrm{reg}(C)$. Thus we must show that given any 0-dimensional subscheme, $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$, satisfying $\mathrm{length}(\mathcal{O}_{\mathscr{Z}}) = k$, the restriction map, $\Gamma(\mathscr{L}) \to \Gamma(\mathcal{O}_{\mathscr{Z}}(\mathscr{L}))$, is onto. Let $\mathfrak{I}$ be the ideal sheaf such that $\mathcal{O}_C/\mathfrak{I} \cong \mathcal{O}_{\mathscr{Z}}$. Let $\mathfrak{m}_x$ denote the maximal ideal sheaf of the point $x \in C$. Let $\mathfrak{K} := \mathfrak{I} \otimes \mathfrak{m}_x$, and let $(\mathscr{W}, \mathcal{O}_{\mathscr{W}})$ be the 0-dimensional subscheme with $\mathcal{O}_{\mathscr{W}} := \mathcal{O}_C/\mathfrak{K}$. Since the point x is a smooth point of C and since the local rings, $\mathcal{O}_{C,y}$, are principal for points, $y \in \mathrm{reg}(C)$, we conclude that $\mathrm{length}(\mathcal{O}_{\mathscr{W}}) = k + 1$. Thus by the k-very ampleness of L, we conclude that the restriction map $\Gamma(L) \to \Gamma(\mathcal{O}_{\mathscr{W}}(L))$ is onto.

Let t be a section of $[x]$ vanishing only at x. Tensoring with t gives a map from $\mathscr{L}$ to L. Under this sheaf map, $\mathscr{L}$ goes isomorphically to $L \otimes \mathfrak{m}_x$, $\mathscr{L} \otimes \mathfrak{I}$ goes isomorphically to $L \otimes \mathfrak{K}$, and $\mathcal{O}_{\mathscr{Z}} \otimes \mathscr{L}$ goes isomorphically to $\mathcal{O}_{\mathscr{W}} \otimes L \otimes \mathfrak{m}_x$. We thus have the following exact, commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L} \otimes \mathfrak{I} & \to & \mathscr{L} & \to & \mathcal{O}_{\mathscr{Z}} \otimes \mathscr{L} & \to & 0 \\
 & & \downarrow \otimes t & & \downarrow \otimes t & & \downarrow \otimes t & & \\
0 & \to & L \otimes \mathfrak{K} & \to & L & \to & \mathcal{O}_{\mathscr{W}} \otimes L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \mathbb{C}_x & \to & \mathbb{C}_x & \to & 0\,. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

A straightforward diagram chase shows that $\Gamma(L) \to \Gamma(\mathcal{O}_{\mathscr{W}}(L))$ being onto implies that $\Gamma(\mathscr{L}) \to \Gamma(\mathcal{O}_{\mathscr{Z}}(\mathscr{L}))$ is onto.

The degree assertion of the lemma is clear. The assertion $h^0(\mathscr{L}) = h^0(L) - r$ follows since the k-very ampleness of L implies that specification of values of a section of L at up to $k+1$ distinct points are independent conditions. QED

(1.2) Proposition. *Let L be a k-very ample line bundle on a reduced, pure dimensional, projective curve C. Then if $k \geqq 1$, $h^0(L) \geqq k + 1$ with equality only if $C \cong \mathbb{P}^1$ and $\deg L = k$.*

Proof. Choose $k-1$ distinct, smooth points $\{x_1, \ldots, x_{k-1}\}$ of C. By Lemma 1.1 above, we know that $\mathscr{L} := L \otimes [x_1]^{-1} \otimes \cdots \otimes [x_{k-1}]^{-1}$ is very ample and $h^0(\mathscr{L}) = h^0(L) - (k-1)$. If $h^0(L) = k+1$, then $h^0(\mathscr{L}) = 2$. In this case $\Gamma(\mathscr{L})$ must give an embedding of C into $\mathbb{P}^1$, and $\mathscr{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, and so $\deg L = k$. QED

(1.3) Corollary. *Let L be a k-very ample line bundle on a reduced, pure dimensional, projective curve C. Assume that $k \geqq 1$.*

(1.3.1) *Then $\deg L \geqq k$ with equality only if $C \cong \mathbb{P}^1$.*
(1.3.2) *If $\deg L \geqq k+1$ then $h^0(L) \geqq k+2$.*

Proof. Note that from the exact sequence

$$0 \to \mathcal{O}_C \to L \to \mathcal{S} \to 0\,,$$

where $\mathcal{S}$ is a skyscraper sheaf supported on the zero set of a section of L, we get

$$h^0(L) \leqq 1 + \deg L\,. \tag{1.3.3}$$

Since $h^0(L) \geqq k+1$ by (1.0), $\deg L \geqq k$. If $\deg L = k$, we find $h^0(L) = k+1$, so that $C \cong \mathbb{P}^1$ by Proposition 1.2 and (1.3.1) is proved. Note that (1.3.2) is an immediate consequence of (1.2). QED

The following result takes care of the cases when the degree of L is either $k+1$ or $k+2$.

(1.4) Proposition. *Let L be a k-very ample line bundle on a reduced, pure dimensional, projective curve C.*

(1.4.1) *If* $\deg L = k+1$, *and* $k \geqq 2$, *then* $C \cong \mathbb{P}^1$.
(1.4.2) *If* $\deg L = k+2$, $k \geqq 1$, *and C is irreducible, then either* $h^0(L) = k+3$ *and C is isomorphic to* $\mathbb{P}^1$, *or* $h^0(L) = k+2$ *and C is isomorphic to a degree* 3 *curve in* $\mathbb{P}^2$.

Proof. Let $\deg L = k+1$. Then by combining (1.3.2) and the general relation (1.3.3) we have $h^0(L) = k+2$. Let $\mathcal{L} := L \otimes [x_1]^{-1} \otimes \cdots \otimes [x_{k-1}]^{-1}$ for $k-1$ distinct, smooth points $x_1, \ldots, x_{k-1}$ of C. By Lemma 1.1, $\mathcal{L}$ is very ample, $\deg \mathcal{L} = 2$, and $h^0(\mathcal{L}) = 3$. Thus $\Gamma(\mathcal{L})$ gives an isomorphism of C with a degree 2 curve of $\mathbb{P}^2$. Note that C is irreducible since $k \geqq 2$ and $k+1 = \deg L = \deg L_{C_1} + \cdots + \deg L_{C_t} \geqq tk$ where t is the number of irreducible components of C. This proves (1.4.1).

Now assume that $\deg L = k+2$. By (1.3.3) we have that $h^0(L) \leqq k+3$. By (1.3.2) we can assume that $h^0(L) \geqq k+2$. Let $\mathcal{L} := L \otimes [x_1]^{-1} \otimes \cdots \otimes [x_{k-1}]^{-1}$ for $k-1$ distinct, smooth points $x_1, \ldots, x_{k-1}$ of C. If $h^0(L) = k+3$ then, by Lemma 1.1, $h^0(\mathcal{L}) = 4$, and $\Gamma(\mathcal{L})$ gives an isomorphism of C with a degree 3 curve in $\mathbb{P}^3$. By Castelnuovo's inequality, (0.2), we conclude that the arithmetic genus of C is zero and therefore that $C \cong \mathbb{P}^1$. Now assume that $h^0(L) = k+2$. Then $\Gamma(\mathcal{L})$ gives an isomorphism of C with a degree 3 curve in $\mathbb{P}^2$. This shows (1.4.2). QED

(1.5) Proposition. *Let L be a k-very ample line bundle on a reduced, pure dimensional, projective curve C. Let* $d := \deg L$. *If* $h^0(L) = k+2$, *then C is isomorphic to a curve in* $\mathbb{P}^2$ *with degree equal to* $d-(k-1)$. *In particular the arithmetic genus of C is* $(d-k)(d-k-1)/2$. *If C is irreducible then either* $k=1$, *or* $(C, L) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k+1))$, *or C is isomorphic to a degree* 3 *plane curve with* $d = k+2$.

Proof. Let $\mathcal{L} := L \otimes [x_1]^{-1} \otimes \cdots \otimes [x_{k-1}]^{-1}$ for $k-1$ distinct, smooth points $x_1, \ldots, x_{k-1}$ of C. Then by Lemma 1.1, $\Gamma(\mathcal{L})$ gives an embedding of C into $\mathbb{P}^2$. Since $\deg \mathcal{L} = d-(k-1)$ we conclude from the adjunction formula that

$$2g-2 = (d-(k-1))(d-(k-1)-3) = (d-k+1)(d-k-2)\,,$$

where g denotes the arithmetic genus of C. Solving for g we get $g = (d-k)(d-k-1)/2$.

Now assume that C is irreducible. Assume moreover that $k \geqq 2$ and $C \ncong \mathbb{P}^1$, i.e. $g \geqq 1$. Note that if $C \cong \mathbb{P}^1$, then $L \cong \mathcal{O}_{\mathbb{P}^1}(k+1)$ since $h^0(L) = k+2$. By (1.4.1) we

have that $d = \deg L \geqq k+2$. By Castelnuovo's inequality, (0.2), we have that

$$g \leqq \left[\frac{d-2}{k}\right]\left(d-k-1-\left(\left[\frac{d-2}{k}\right]-1\right)\frac{k}{2}\right).$$

Combined with $g = (d-k)(d-k-1)/2$ we have

$$\begin{aligned}(d-k)(d-k-1) &\leqq \frac{d-2}{k}\left(2d-2k-2-\left(\left[\frac{d-2}{k}\right]-1\right)k\right)\\ &= \frac{d-2}{k}\left(d-k+\left(\frac{d-2}{k}-\left[\frac{d-2}{k}\right]\right)k\right).\end{aligned}$$

Using the fact that given two non-negative integers a, b with $b>0$, $a/b-[a/b] \leqq (b-1)/b$, we conclude that

$$((d-1)-(k-1))((d-2)-(k-1)) \leqq \frac{d-2}{k}(d-k+(k-1)) = \frac{(d-1)(d-2)}{k}$$

which gives

$$(k-1)(d-1)(d-2)-(k-1)((d-1)+(d-2))k+(k-1)^2 k \leqq 0\,.$$

Dividing by $k-1$ we conclude that

$$(d-1)(d-2)-((d-1)+(d-2))k+(k-1)k \leqq 0\,,$$

and thus that $(d-1-k)(d-2-k) \leqq k$. This gives that $d-2 \leqq k+\sqrt{k}$. Going back to Castelnuovo's inequality with this bound for d gives $g \leqq d-k-1$. Using $g=(d-k)(d-k-1)/2$ we conclude that $d-k \leqq 2$. Since $d \geqq k+2$ we conclude that $d=k+2$ and therefore that $g=1$. This proves the proposition. QED

(1.6) Corollary. *Let L be a k-very ample line bundle on a reduced, irreducible curve, C. Let $d=\deg L$. If $d \leqq 2k+3$ then $h^1(L)=0$. In particular either $d \geqq g(C)+k+2$, or C is isomorphic to $\mathbb{P}^1$, or to a degree 3 curve in $\mathbb{P}^2$.*

Proof. By Corollary 1.3 and Proposition 1.5, either C is isomorphic to $\mathbb{P}^1$, or to a degree 3 curve in $\mathbb{P}^2$, or $h^0(L) \geqq k+3$. In the first two cases $h^1(L)=0$. In the last case Castelnuovo's bound (0.2) gives $g(C) \leqq d-h^0(L)+1$. Therefore by the Riemann–Roch theorem, $h^1(L)=h^0(L)-d+g(C)-1 \leqq 0$, i.e. $h^1(L)=0$.

Thus $h^0(L)=d-g(C)+1$. If C is neither $\mathbb{P}^1$ nor isomorphic to a degree 3 curve in $\mathbb{P}^2$, then $d-g(C)+1=h^0(L) \geqq k+3$, i.e. $d \geqq g(C)+k+2$. QED

The following result plays a key role in this paper.

(1.7) Theorem. *Let L be a k-very ample line bundle on a reduced, irreducible, Gorenstein curve, C. Let $d:=\deg L$ and $k \geqq 2$. If $d \leqq 2g(C)+k-1$, then $g(C) \geqq k+2$.*

Proof. By (1.3.1) and (1.4.1), we can assume without loss of generality that $g(C) \geqq 2$. Therefore by (1.3) and (1.5), we also have $h^0(L) \geqq k+3$. If $h^1(L)>0$, then $d \leqq 2g(C)-2$. Since L is spanned, we have Clifford's inequality, $h^0(L) \leqq d/2+1$. (We don't know if Clifford's theorem holds in the singular case without spannedness, but with spannedness of L, the usual argument works with no difficulties.) Thus $k+3 \leqq d/2+1 \leqq g(C)$ giving the desired conclusion. Therefore we can assume without loss of generality than $h^1(L)=0$, and $g(C) \leqq k+1$.

(1.7.1) Claim. Given an integer $e \in \{0, \ldots, 2g(C) - 2\}$, there exists a line bundle $\mathscr{E}$ on C such that $\deg \mathscr{E} = e$, $h^0(\mathscr{E}) > 0$ and $h^1(\mathscr{E}) > 0$.

Proof. If $e = 0$, then take $\mathscr{E} = \mathscr{O}_C$. If $1 \leqq e \leqq g(C) - 1$, take any set of e distinct, smooth, points $\{x_1, \ldots, x_e\} \subset C$. Let $\mathscr{E} = [x_1] \otimes \cdots \otimes [x_e]$. Clearly $h^0(\mathscr{E}) \geqq 1$. By the Riemann–Roch theorem, $h^1(\mathscr{E}) = h^0(\mathscr{E}) - \deg \mathscr{E} + g(C) - 1 \geqq 1 - e + g(C) - 1 = g(C) - e \geqq 1$.

Serre duality immediately reduces the problem of finding $\mathscr{E}$ satisfying the claim with $g(C) \leqq e \leqq 2g(C) - 2$ to the above case with $0 \leqq e \leqq g(C) - 2$. □

(1.7.2) Claim. We can choose a line bundle $\mathscr{E}$ on C with $h^0(\mathscr{E}) > 0$, $h^1(\mathscr{E}) > 0$, and

$$\deg L - k - 1 \leqq \deg \mathscr{E} \leqq \deg L - g(C) .$$

Proof. Note that the assumption, $g(C) \leqq k + 1$, insures that $d - g(C) \geqq d - k - 1$. Let $e = d - k - 1$. Note also that by assumption, $d - k - 1 \leqq 2g(C) - 2$. Thus we can choose the desired line bundle $\mathscr{E}$ with $e = \deg \mathscr{E}$. □

Choose a line bundle, $\mathscr{E}$, according to Claim 1.7.2, and let $e = \deg \mathscr{E}$. Consider the exact sequence

$$0 \to L - \mathscr{E} \to L \to \mathscr{S} \to 0 ,$$

where $\mathscr{S}$ is a torsion sheaf of length e. Note that $h^0(L - \mathscr{E}) \geqq h^0(L) - \text{length}(\mathscr{S}) = h^0(L) - e \geqq d - g(C) + 1 - e$. Note that by the conditions in Claim 1.7.2, $d - g(C) + 1 - e \geqq 1$. Thus we have a not identically zero section, $t_{\mathscr{R}}$, of $\mathscr{R} := L - \mathscr{E}$. Consider the exact sequence

$$0 \to \mathscr{E} \to L \to \mathscr{G} \to 0$$

obtained by tensoring $\mathscr{E}$ with $t_{\mathscr{R}}$. By claim 1.7.2, $\text{length}(\mathscr{G}) = d - e \leqq k + 1$. Since L is k-very ample, and since $\text{length}(\mathscr{G}) \leqq k + 1$, we conclude that the restriction map, $\Gamma(L) \to \Gamma(\mathscr{G})$, is onto. Thus $h^1(\mathscr{E}) = h^1(L) = 0$. This contradicts the fact that by the above claim, $h^1(\mathscr{E}) > 0$. QED

(1.7.3) Corollary. *Let L be a k-very ample line bundle on a reduced, irreducible, Gorenstein curve, C. Let $d = \deg L$, $k \geqq 2$, and assume that $g(C) \geqq 2$. If $d \leqq 2k + 3$, then $d \geqq 2g(C) + k$.*

Proof. Use Corollary 1.6. Since $d \leqq 2k + 3$, we have $h^1(L) = 0$, and since $g(C) \geqq 2$, we have $d \geqq g(C) + k + 2$. If $d < 2g(C) + k$, then by (1.7) it would follow that $g(C) \geqq k + 2$, giving the contradiction, $2k + 3 \geqq d \geqq g(C) + k + 2 \geqq 2k + 4$. QED

(1.7.4) Corollary. *Let L be a k-very ample line bundle on a reduced, irreducible, Gorenstein curve, C. Let $d = \deg L$ and assume that $k \geqq 2$. If $d \geqq 3g(C) - 2$, then $d \geqq 2g(C) + k$.*

Proof. For $g(C) = 0$, the result is simply (1.3.1). For $g(C) = 1$, the result is obtained by combining (1.3.1) and (1.4.1). Therefore we can assume that $g(C) \geqq 2$. If $d < 2g(C) + k$, then by Theorem 1.7, we have $g(C) \geqq k + 2$. This gives the contradiction $d \geqq 3g(C) - 2 \geqq 2g(C) - 2 + k + 2 = 2g(C) + k$. QED

2 Some special classes and low degree classification

Let S be a smooth surface and L a k-very ample line bundle on S. In this section we consider a few special classes (S, L) where the k-very ampleness of the adjoint line bundle $K_S + L$ can be easily studied.

The following result follows by combining the results of [BS2, §3], the main theorem of [BS3], and Lemma 0.3.5 (see in particular the last paragraph of (0.3)).

(*2.1*) *The cases when* $S = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ *or* $\mathbb{F}_r$, $r \geqq 1$.

(*2.1.1*) Let $S = \mathbb{P}^2$. Then $L = \mathcal{O}_{\mathbb{P}^2}(n)$ is k-very ample if and only if $n \geqq k$ and $K_S + L = \mathcal{O}_{\mathbb{P}^2}(n-3)$ is k-very ample if and only if $n \geqq 3 + k$.

Similarly if $S = \mathbb{P}^1 \times \mathbb{P}^1$, then $L = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ is k-very ample if and only if $a \geqq k, b \geqq k$ and $K_S + L = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a-2, b-2)$ is k-very ample if and only if $a \geqq 2 + k$ and $b \geqq 2 + k$.

(*2.1.2*) Let $S = \mathbb{F}_r$, $r \geqq 1$. Let E, f denote a section of minimal self-intersection $E^2 = -r$ and a fiber of the ruling. Then by [BS2, (3.1)], with Lemma 0.3.5 used in place of [BS2, (0.5.3)], $L \sim aE + bf$ is k-very ample if and only if $a \geqq k$ and $b \geqq ar + k$. Therefore $K_S + L \sim (a-2)E + (b-2-r)f$ is k-very ample if and only if $a \geqq k + 2$ and $b \geqq (a-1)r + k + 2$. By using the previous relations we have e.g. the following.

Let $S = \mathbb{F}_r$, $r \geqq 1$. *Let* $L \sim aE + bf$ *be* k*-very ample. Then* $K_S + L$ *is* k*-very ample unless either* $a = k, k+1$, *or* $r = 1$ *and* $b = a + k$.

Indeed, if $K_S + L$ is not k-very ample, we have either $a = k$, $k+1$ or $ar + k \leqq b < (a-1)r + k + 2$, which gives $r = 1$ and $b = a + k$. □

The special case when S is a $\mathbb{P}^1$ bundle over an elliptic curve of invariant $e = -1$ will occur in the sequel. We treat this case separately.

(2.2) Proposition. *Let* S *be a* $\mathbb{P}^1$ *bundle,* $p: S \to C$, *over a smooth elliptic curve* C *of invariant* $e = -1$. *Let* E, f *denote a section of minimal self-intersection* $E^2 = 1$ *and a fiber of the ruling. Let* $L \sim aE + bf$ *be a line bundle on* S. *Then* L *is* k*-very ample if and only if*

$$a \geqq k; \quad a + b \geqq k + 2; \quad a + 2b \geqq k + 2 .$$

Proof. Let $\mathscr{E} := E - (1/2)f$. Then it is a well known fact that $\mathscr{E}$ is nef and $2\mathscr{E}$ is numerically equivalent to a smooth effective elliptic curve (see e.g. [BS1, (0.15)]). Note that $\mathscr{E} \cdot \mathscr{E} = f \cdot f = 0, \mathscr{E} \cdot E = 1/2, f \cdot E = f \cdot \mathscr{E} = 1$ and take $\mathscr{E}, f$ as a basis for the second rational homology $H_2(S, \mathbb{Q})$ of S. Then $L \sim \alpha\mathscr{E} + \beta f$ for some rational numbers α, β. Write

$$L \sim \alpha\mathscr{E} + \beta f = \alpha E + \left(\beta - \frac{\alpha}{2}\right) f .$$

Then $a = \alpha$, $b = \beta - \dfrac{\alpha}{2}$ so that the above numerical conditions becomes respectively

$$\alpha \geqq k; \quad \beta + \alpha/2 \geqq k + 2; \quad \beta \geqq (k+2)/2 . \tag{2.2.1}$$

Assume L is k-very ample. Then $L \cdot f \geqq k$ gives $\alpha \geqq k$. Since E, $2\mathscr{E}$ are numerically equivalent to smooth effective elliptic curves, we know from (1.3.1) and (1.4.1) that

$L \cdot E \geqq k+2$, $2L \cdot \mathscr{E} \geqq k+2$. This leading to $\beta + \alpha/2 \geqq k+2$ and $\beta \geqq (k+2)/2$ respectively. Thus to conclude we have to show that $L \sim \alpha\mathscr{E} + \beta f$ is k-very ample if the conditions (2.2.1) hold true. Write

$$\mathscr{L} := L - K_S \sim (\alpha+2)\mathscr{E} + \beta f$$

and note that $\mathscr{L}$ is ample since $-K_S \sim 2E - f = 2\mathscr{E}$ is nef. Furthermore

$$\mathscr{L} \cdot \mathscr{L} = 2(\alpha+2)\beta \geqq (k+2)^2 \geqq 4k+5 .$$

Then $\mathscr{L}$ satisfies the assumptions of Theorem 0.3.2 and therefore, if $L = K_S + \mathscr{L}$ is not k-very ample, there exists an effective divisor D such that

$$\mathscr{L} \cdot D - k - 1 \leqq D \cdot D < \mathscr{L} \cdot D/2 < k+1 . \tag{2.2.2}$$

First, assume that $D = \lambda\mathscr{E}$ for some positive $\lambda \in \mathbb{Q}$. Then $D \cdot D = 0$, so we find $\lambda(k+2)/2 \leqq \lambda\beta = \mathscr{L} \cdot D \leqq k+1$ which gives $\lambda/2 < 1$. Since $E \cdot D = \lambda/2$ is an integer, we have a contradiction.

Let $D = \lambda f$, λ positive integer. Again $D \cdot D = 0$ so we find $\lambda(k+2) \leqq \lambda(\alpha+2) = \mathscr{L} \cdot D \leqq k+1$, which leads to the contradiction $\lambda = 0$.

Thus we can assume $D \sim \lambda\mathscr{E} + \mu f$ where $\lambda, \mu \in \mathbb{Q}$ and $\lambda\mu \neq 0$. Since $\lambda = D \cdot f \geqq 0$, $\mu = D \cdot \mathscr{E} \geqq 0$ we have $\lambda > 0$, $\mu > 0$. Note also that $\lambda = D \cdot f$ and $\mu + \lambda/2 = E \cdot D$ are integers. Now compute

$$\begin{aligned} 2k+1 \geqq \mathscr{L} \cdot D &= ((\alpha+2)\mathscr{E} + \beta f) \cdot (\lambda\mathscr{E} + \mu f) \\ &= \beta\lambda + (\alpha+2)\mu \geqq (k+2)(\mu + \lambda/2) \geqq k+2 . \end{aligned}$$

Therefore $D \cdot D > 0$ and $\mu + \lambda/2 = 1$. Hence $\lambda = 1$, $\mu = 1/2$, $D = \mathscr{E} + (1/2)f = E$, and so $D \cdot D = 1$. Then $\mathscr{L} \cdot D - k - 1 = \beta + \frac{\alpha}{2} - k$. Thus the relation, $\mathscr{L} \cdot D - k - 1 \leqq D \cdot D$, from (2.2.2), gives the contradiction $k+2 \leqq \beta + \alpha/2 \leqq k+1$. QED

(2.3) Corollary. *Let S be a $\mathbb{P}^1$ bundle, $p: S \to C$, over a smooth elliptic curve C of invariant $e = -1$. Let E, f denote a section of minimal self-intersection $E^2 = 1$ and a fiber of the ruling. Assume that $L \sim aE + bf$ is a k-very ample line bundle on S. Then $K_S + L$ is k-very ample unless either $a = k$, $a = k+1$ or $a = k+2$, $b = 0$.*

Proof. Note that $a = L \cdot f \geqq k$. If $a < k+2$, then $(K_S + L) \cdot f = (a-2) < k$, so $K_S + L$ is not k-very ample. Thus we can assume $a \geqq k+2$ and we apply (2.2). Since L is k-very ample we have $a + b \geqq k+2$ and $a + 2b \geqq k+2$. If $K_S + L \sim (a-2)E + (b+1)f$ is not k-very ample we have either

$$a - 2 + b + 1 = a + b - 1 < k+2 ,$$

or

$$a - 2 + 2(b+1) = a + 2b < k+2 .$$

The second case is excluded since $a + 2b \geqq k+2$. Hence we find $k+1 = a + b - 1$, i.e. $b = k + 2 - a$. Then from the inequality $a + 2b \geqq k+2$ we get $a \leqq k+2$, that is $a = k+2$ and $b = 0$. QED

(2.4) Remark. If S is a $\mathbb{P}^1$ bundle, $p: S \to C$, over a smooth elliptic curve C of invariant $e = 0, 1, 2$ we have the following characterization of k-very ampleness.

With the same notation as in (2.2), let $L \sim aE + bf$ be a line bundle on S. Then L is k-very ample if and only if $a \geqq k$ and $b \geqq ae + k + 2$. The proof runs parallel to that of Proposition 2.2, by taking $\mathscr{E} = E + ef, f$ as a basis for the second integral homology $H_2(S, \mathbb{Z})$ of S. □

If S is a $\mathbb{P}^1$ bundle, $p: S \to C$, over a smooth curve C, and L is a k-very ample line bundle on S, the k-very ampleness of $K_S + L$ is described by the following proposition.

(2.5) Proposition. *Let S be a $\mathbb{P}^1$ bundle, $p: S \to C$, over a smooth curve C, of genus $g(C)$. Let E be a section of minimal self-intersection, $E^2 = -e$, and let f be a fiber of p. Let L be a k-very ample line bundle on S. Assume that $k \geqq 2$ and that $L \cdot L \geqq 4k + 5$. Then for $K_S + L$ to be k-very ample it is necessary that $L \cdot f \geqq k + 2$. Under this additional condition the pairs, (S, L) with $K_S + L$ not k-very ample are characterized by:*

(2.5.1) *$C \cong \mathbb{P}^1$ and either*
  (2.5.1.1) *$S \cong \mathbb{F}_0$ and $L \cdot E = k, k + 1$ for any section E of p with $E \cdot E = 0$;*
  (2.5.1.2) *$S \cong \mathbb{F}_1$ and $L \cdot E = k$ for the section E of p with $E \cdot E = -1$;*

(2.5.2) *C is an elliptic curve, S is a $\mathbb{P}^1$ bundle of invariant $e = -1$, $L \sim (k + 2)E$.*

Proof. Let $L \sim aE + bf$. Clearly $a = L \cdot f \geqq k$. If $a < k + 2$, then $(K_S + L) \cdot f = a - 2 < k$, and thus $K_S + L$ is not k-very ample and we have the first necessary condition. Therefore we can assume that $a \geqq k + 2$.

If $C \cong \mathbb{P}^1$ then the result is an immediate consequence of (2.1). Therefore we can assume that the genus of C is positive.

If $K_S + L$ is not k-very ample, Theorem 0.3.2 applies to say that there exists an effective divisor D on S such that

$$L \cdot D - k - 1 \leqq D \cdot D < L \cdot D/2 < k + 1 . \tag{2.5.3}$$

We have $L \cdot D \geqq k$ since L is k-very ample. Let $L \cdot D = k, k + 1$. From (0.3.1) we know that D is reduced and irreducible, and furthermore D is a smooth $\mathbb{P}^1$ by (1.3.1), (1.4.1). Then, since $g(C) > 0$, D is a fiber of p and we contradict the assumption $a \geqq k + 2$.

Therefore we can assume that $L \cdot D \geqq k + 2$. Note that $L \cdot D \leqq 2k$ by (0.3.3). Thus we can write

$$L \cdot D = k + \delta + 1 \quad \text{with } 1 \leqq \delta \leqq k - 1 . \tag{2.5.4}$$

Note also that (2.5.3), (2.5.4) give

$$D \cdot D \geqq \delta \geqq 1 . \tag{2.5.5}$$

We claim that D is reduced and irreducible. Indeed we have $L \cdot D \leqq 2k$. If $L \cdot D < 2k$, the claim follows from (0.3.1). If $L \cdot D = 2k$, either we are done, or $D = A + B$, where A, B are reduced, irreducible components such that $L \cdot A = L \cdot B = k$. Hence A, B are smooth $\mathbb{P}^1$'s by (1.3.1). Therefore, since $g(C) > 0$, A and B map to points under p. This means that A, B are fibers of p, and hence $D \cdot D = 0$, contradicting (2.5.5).

Step I. We can assume without loss of generality that $q(S) \geqq 2$ and that the invariant, e, of the $\mathbb{P}^1$ bundle, $p: S \to C$, is negative.

To see this assume that $e \geqq 0$. Then by (2.5.5), we see that $D \neq \lambda E$ and that $D \neq \lambda f$ for $\lambda \in \mathbb{Z}$. Therefore $D \sim xE + yf$ where $x, y \in \mathbb{Z}$ and $xy \neq 0$. Since $x = (xE + yf) \cdot f \geqq 0$ we have $x > 0$. Since $D \neq \lambda E$ and D is irreducible, we have that $y - xe = (xE + yf) \cdot E \geqq 0$, so that $y \geqq 0$, and hence $y > 0$. Note that $L \cdot E \geqq k + 2$ by (1.3.1), (1.4.1) since $g(C) > 0$. So we can find the contradiction

$$2k \geqq L \cdot D = xL \cdot E + yL \cdot f \geqq x(k + 2) + y(k + 2) \geqq 2(k + 2) .$$

Therefore we conclude that $e < 0$. Since $e \geqq -g(C)$ we have $g(C) = q(S) \geqq -e \geqq 1$. If $q(S) = 1$, Corollary 2.3 applies to say that $K_S + L$ is k-very ample unless (S, L) is as in (2.5.2). Thus we can assume $q(S) \geqq 2$.

Step II (Hartshorne's Formula [Ha]). Let $D \sim xE + yf$. One has

$$2g(D) - 2 = x(2q(S) - 2) + \frac{x - 1}{x} D \cdot D . \tag{2.5.6}$$

To see this note that $D \cdot D = x(2y - ex)$, $K_S \sim -2E + (2q(S) - 2 - e)f$ and

$$2g(D) - 2 = K_S \cdot D + D \cdot D = x(2q(S) - 2) - x^2 e + xe + 2yx - 2y .$$

It is a straightforward numerical check to show that $-x^2 e + xe + 2yx - 2y = \frac{x - 1}{x} D \cdot D$. Then (2.5.6) follows.

Step III. Let $D \sim xE + yf$. One has

$$D \cdot D = x . \tag{2.5.7}$$

Note that $D \cdot D = (2y - xe)x$, $L \cdot L = (2b - ae)a$, where x, a, $-e$ are positive integers. Hence in particular $L \cdot L/a$, $D \cdot D/x$ are positive integers. Compute

$$\begin{aligned} L \cdot D &= (aE + bf) \cdot (xE + yf) \\ &= x\left(b - \frac{ae}{2}\right) + a\left(y - \frac{xe}{2}\right) = \frac{x}{2a} L \cdot L + \frac{a}{2x} D \cdot D . \end{aligned} \tag{2.5.8}$$

Let $D \cdot D > 3x$. Then, since $D \cdot D$ is a multiple of x, $D \cdot D \geqq 4x$, and (2.5.8) leads to the contradiction

$$2k \geqq L \cdot D > 2a \geqq 2k + 4 .$$

Therefore $D \cdot D \leqq 3x$. If $D \cdot D = x$ we are done. Hence we must rule out the cases when $D \cdot D = 3x, 2x$.

Let $D \cdot D = 3x$. Since $D \cdot D \geqq \delta$ by (2.5.5), the equality (2.5.8) gives

$$L \cdot D = \frac{3x}{6a} L \cdot L + \frac{3a}{2} \geqq \frac{\delta}{6a} L \cdot L + \frac{3a}{2} . \tag{2.5.9}$$

Assume that $L \cdot L/a \geqq 4$. By combining (2.5.4) and (2.5.9) we find

$$k + \delta + 1 = L \cdot D \geqq \frac{2}{3} \delta + \frac{3}{2}(k + 2)$$

which leads to the contradiction $\frac{k - 1}{3} \geqq \frac{\delta}{3} + 1 \geqq \frac{k}{2} + 3$. Therefore $L \cdot L/a \leqq 3$ and the same argument as above rules out the remaining cases $L \cdot L/a = 1, 2, 3$. For example, if $L \cdot L/a = 1$, we have $a = L \cdot L \geqq 4k + 5$ and (2.5.9) yields

$$k + \delta + 1 = L \cdot D \geqq \frac{\delta}{6} + \frac{3(4k+5)}{2}$$

which leads to the contradiction $\frac{5(k-1)}{6} \geqq \frac{5}{6}\delta \geqq 5k + \frac{13}{2}$.

Let $D \cdot D = 2x$. Since $D \cdot D \geqq \delta$ by (2.5.5), the equality (2.5.8) gives

$$L \cdot D = \frac{2x}{4a} L \cdot L + a \geqq \frac{\delta}{4a} L \cdot L + a \,. \tag{2.5.10}$$

Assume that $L \cdot L/a \geqq 4$. By combining (2.5.4) and (2.5.10) we find

$$k + \delta + 1 = L \cdot D \geqq \delta + a \geqq \delta + k + 2$$

which gives the contradiction $k + 1 \geqq k + 2$. Therefore $L \cdot L/a \leqq 3$, and again, the same argument as above rules out the remaining cases $L \cdot L/a = 1, 2, 3$. For example, if $L \cdot L/a = 1$, then we have $a = L \cdot L \geqq 4k + 5$ and (2.5.10) yields

$$k + \delta + 1 = L \cdot D \geqq \frac{\delta}{4} + 4k + 5 \,.$$

which leads to the contradiction $\frac{3}{4}(k-1) \geqq \frac{3}{4}\delta \geqq 3k + 4$.

Step IV. Conclusion of the proof. Note that $g(D) \geqq 2$. Indeed by Step I we can assume that $g(C) = q(S) \geqq 2$. Then if $g(D) \leqq 1$, the image of D under p would be a point and hence D would be a fiber of p. This contradicts again (2.5.5). Therefore, Corollary 1.6 applies to give

$$g(D) \leqq L \cdot D - k - 2 = \delta - 1 \,. \tag{2.5.11}$$

By combining (2.5.6) and (2.5.7) we get

$$2g(D) - 1 = (D \cdot D)(2q(S) - 1) \,.$$

By using (2.5.11) and (2.5.5) this implies

$$2\delta - 3 \geqq \delta(2q(S) - 1)$$

which gives $q(S) = 1$. This contradicts our present assumption $q(S) \geqq 2$ and we are done. QED

We have the following low degree classification result.

(2.6) Proposition. *Let L be a k-very ample line bundle, $k \geqq 2$, on a smooth connected projective surface, S. Let $d := L \cdot L$, and assume that $d \leqq 2k + 3$. Then either*

(2.6.1) $k = 2$, $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, *or*
(2.6.2) $k = 3$, $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$.

Proof. Recall that the classification of pairs (S, L) for small values of $g(L)$ given in [BS2, §5], and stated for k-spanned line bundles, L, holds for k-very ample, L (see also (0.3)).

First, assume $g(L) \leqq 1$. Then we know from [BS2, (5.1)] that if $g(L) = 0$, then $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $k = 2$, and if $g(L) = 1$, then either $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, $k = 3$, or $(S, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2))$, $k = 2$. The last case is excluded by the assumption that $d \leqq 2k + 3 = 7$. The remaining cases give (2.6.1), (2.6.2) respectively.

Thus we can assume that $g(L) \geqq 2$. Therefore $d \geqq 2g(L) + k$ by Corollary 1.7.3. Then in particular $d \geqq K_S \cdot L + L \cdot L + k + 2$, or $K_S \cdot L \leqq -k - 2$. Hence

$$(L + K_S) \cdot L \leqq d - k - 2 \leqq k + 1 . \tag{2.6.3}$$

Assume that $K_S + L$ is not spanned. Therefore from [SV, (0.1)] we know that either $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a))$, $a = 1, 2$, or S is a $\mathbb{P}^1$ bundle over a smooth curve and $L \cdot f = 1$ for each fiber, f. If $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, we fall again in (2.6.1). The remaining cases are clearly excluded since L is k-very ample and $k \geqq 2$.

Therefore we can assume that $K_S + L$ is spanned. Let $\phi : S \to \mathbb{P}_{\mathbb{C}}$ be the adjoint morphism given by $\Gamma(K_S + L)$.

If $\dim \phi(S) = 0$, then $K_S \sim -L$ so that $g(L) = 1$, and we are done by the above.

Let $\dim \phi(S) = 1$. Then (S, L) is a conic bundle, $r : S \to Y$, over a smooth curve, Y, and $L \cdot f = 2$ for each fiber f or r. Note that r is a morphism with connected fibers given by the Remmert–Stein factorization of ϕ. Since $L \cdot f = 2$, all fibers of r are reduced and irreducible, and hence S is in fact a $\mathbb{P}^1$ bundle under r. We also have that $K_S + L \approx r^* H$ for some ample line bundle, H, on Y. If $\deg H \geqq 2$, then $(K_S + L) \cdot L \geqq 2k$, which contradicts (2.6.3). Therefore we conclude that $\deg H = 1$. Then $Y \cong \mathbb{P}^1$ and $S \cong \mathbb{F}_r$, $r \geqq 0$. Since $L \cdot f = 2$, we have that $k = 2$ and $L \sim 2E + bf$ for some integer b such that $b \geqq 2r + 2$ by (2.1.2). Therefore we get the contradiction

$$7 = 2k + 3 \geqq L \cdot L = -4r + 4b \geqq 4r + 8 .$$

Let $\dim \phi(S) = 2$. We claim that $K_S + L$ is ample. Indeed, if not, there exists a curve, γ, such that $(K_S + L) \cdot \gamma = 0$. By the Hodge index theorem we have $\gamma^2 < 0$ and, since $L \cdot \gamma \geqq k$, we also have $K_S \cdot \gamma \leqq -k < 0$. Therefore $\gamma^2 = K_S \cdot \gamma = -1$, which leads to the contradiction, $L \cdot \gamma = 1$.

Since $K_S + L$ is ample and spanned, there exists a smooth curve, C, in $|K_S + L|$. By combining (2.6.3), (1.3.1), and (1.4.1) we conclude that $C \cong \mathbb{P}^1$, i.e. $g(K_S + L) = 0$. It thus follows (see e.g. [LP, (2.3)]) that either $(S, K_S + L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, or $S \cong \mathbb{F}_r$, $r \geqq 0$, and $(K_S + L) \cdot f = 1$ for each fiber f of $\mathbb{F}_r$. In the first two cases we have $L \approx \mathcal{O}_{\mathbb{P}^2}(4)$, $k = 4$, and $L \approx \mathcal{O}_{\mathbb{P}^2}(5)$, $k = 5$, respectively. Both of these cases contradict the assumption that $d \leqq 2k + 3$. Then $S \cong \mathbb{F}_r$, $r \geqq 0$, and $L \sim 3E + bf$ for some integer b such that $b \geqq 3r + k$ by (2.1.2). Therefore we find

$$2k + 3 \geqq L \cdot L = -9r + 6b \geqq 9r + 6k ,$$

which leads to the contradiction $3 \geqq 9r + 4k$. QED

3 Existence theorems for k-reductions

Let S be a smooth surface and L a k-very ample line bundle on S. In this section we prove a main technical result of the paper, Theorem 3.1. This theorem is an existence theorem for the k-reduction.

(3.1) Theorem. *Let S be a smooth connected projective surface and L a k-very ample line bundle on S. Then $tK_S + L$, $t = 0, \ldots, k - 1$, is very ample and $kK_S + L$ in nef and big unless either:*

(3.1.1) $S = \mathbb{P}^2$ *and* $L \cong \mathcal{O}_{\mathbb{P}^2}(a)$ *for* $k \leqq a \leqq 3k$;
(3.1.2) $S = \mathbb{P}^1 \times \mathbb{P}^1$ *and* $L \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ *with* $k \leqq \min\{a, b\} \leqq 2k$;
(3.1.3) S *is a* $\mathbb{P}^1$ *bundle over a smooth curve and* $L \cdot f \leqq 2k$ *for every fiber* f;
(3.1.4) S *is a* k*-conic bundle, relative to* L*, over a smooth curve, or*
(3.1.5) S *is a Del Pezzo surface with* $L \approx -kK_S$, *with* $K_S^2 \geqq 2$ *for* $k \geqq 2$, *and with* $K_S^2 \geqq 3$ *for* $k = 1$.

Proof. Since, for $k = 1$, this result is contained in [So] and [V], we can assume that $k \geqq 2$. Let t be the minimum non-negative integer $t \leqq k - 2$ such that $K_S + (tK_S + L)$ is not very ample. Then in particular $\mathcal{L} := tK_S + L$ is very ample.

First, assume that $K_S + \mathcal{L}$ is not nef. Then well known results from adjunction theory (see [So] or [SV, (0.1)]) say that $(S, \mathcal{L})$ is either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $(Q, \mathcal{O}_Q(1))$, where Q is a smooth quadric in $\mathbb{P}^3$, or a $\mathbb{P}^1$ bundle over a smooth curve with $\mathcal{L} \cdot f = 1$ for every fiber f. Then we are in (3.1.1), (3.1.2) or (3.1.3) with $L \cdot f = 2t + 1 \leqq 2k - 3$.

Thus we can assume that $K_S + \mathcal{L}$ is nef. Let $\varphi: S \to Y$ be the morphism associated to $|m(K_S + \mathcal{L})|$ for $m \gg 0$. Again it is well known (see e.g. [SV, (0.2)]) that either

(i) $\dim \varphi(S) = 0$, $K_S \approx -\mathcal{L}$;
(ii) $\dim \varphi(S) = 1$, $(S, \mathcal{L})$ is a conic bundle, $\varphi: S \to C$, over a smooth curve C, or
(iii) $\dim \varphi(S) = 2$, $K_S + \mathcal{L}$ is nef and big.

In case (i) we have $L \approx -(1 + t)K_S$ and S is a Del Pezzo surface. If S is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ we fall again in cases (3.1.1), (3.1.2). If S is neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, there exists a line ℓ in S such that $\ell^2 = K_S \cdot \ell = -1$. Then we have

$$k \leqq L \cdot \ell = -(1 + t)K_S \cdot \ell = 1 + t\,,$$

which leads to the contradiction $t \geqq k - 1$.

In case (ii), let f be a fiber of φ. Then $(K_S + \mathcal{L}) \cdot f = ((t + 1)K_S + L) \cdot f = 0$, so we find $2k - 2 \geqq 2(t + 1) = L \cdot f$. Therefore f is irreducible and S is a $\mathbb{P}^1$ bundle under φ. We are in case (3.1.3) with $L \cdot f = 2t + 2 \leqq 2k - 2$.

In case (iii), $K_S + \mathcal{L}$ is in fact ample. Indeed, if not, there exists an effective curve, say B, contracted by φ. Then $(K_S + \mathcal{L}) \cdot B = 0$ so by the Hodge index theorem one has $B^2 < 0$ and, since $\mathcal{L} \cdot B > 0$, $K_S \cdot B < 0$. Therefore $K_S \cdot B = B^2 = -1$, $\mathcal{L} \cdot B = 1$. Now $(tK_S + L) \cdot B = 1$ yields $k \leqq L \cdot B = t + 1$, the same contradiction as in case (i).

Let us study the very ampleness of $K_S + \mathcal{L}$. If $K_S + \mathcal{L}$ is not very ample, by the main theorem 2.1 of [SV] we know that either:

(a) S is a Del Pezzo surface with $K_S^2 = 2$, $\mathcal{L} \approx -2K_S$;
(b) S is a Del Pezzo surface with $K_S^2 = 1$, $\mathcal{L} \approx -3K_S$; or
(c) S is a $\mathbb{P}^1$ bundle over an elliptic curve of invariant $e = -1$ and $\mathcal{L} \approx 3E$, where E is a section of minimal self-intersection $E^2 = 1$.

Note that the further possible case of [SV, (2.1)] is excluded since $K_S + \mathcal{L}$ is ample.

In case (a), we have $tK_S + L \approx -2K_S$, i.e. $L \approx -(t + 2)K_S$. By (0.6) we know that L is k-very ample if and only if $t + 2 \geqq k$. Since $t \leqq k - 2$ we find $k = t + 2$. Then $L \approx -kK_S$ and we are in (3.1.5).

The remaining two cases are excluded. In case (b), we have $tK_S + L \approx -3K_S$, i.e. $L \approx -(t + 3)K_S$. By (0.6) we know that L is k-very ample if and only if $t + 3 \geqq k + 2$. Since $t \leqq k - 2$ we find a contradiction. In case (c), let f be a fiber of

the $\mathbb{P}^1$ bundle S. Then $K_S \sim -2E + f$ and $L \sim 3E - tK_S \sim (2t+3)E - tf$. Since $2t + 3 - t = t + 3 \leqq k + 1$, we know from (2.2) that L is not k-very ample, a contradiction.

Thus we can assume that $K_S + \mathscr{L} = (t+1)K_S + L$ is very ample for $t \leqq k-2$, i.e. $tK_S + L$ is very ample for $t = 0, \ldots, k-1$. In particular $\mathscr{L} := (k-1)K_S + L$ is very ample. If $kK_S + L = K_S + \mathscr{L}$ is not nef and big, the same general results from adjunction theory used before (see [SV]) apply to say that $(S, \mathscr{L})$ is either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$, a $\mathbb{P}^1$ bundle over a smooth curve with $\mathscr{L} \cdot f = 1$ for any fiber f, a Del Pezzo surface, i.e. $K_S \approx -\mathscr{L}$, or a conic bundle over a smooth curve. In the first four cases we fall again in (3.1.1), (3.1.2) or (3.1.3) with $L \cdot f = 2k - 1$. If $(S, \mathscr{L})$ is a Del Pezzo surface, we get $L \approx -kK_S$. Since L is k-very ample we see from (0.6) that $K_S^2 \geqq 2$ and we are in case (3.1.5). Let $(S, \mathscr{L})$ be a conic bundle, $\varphi : S \to C$, over a smooth curve C and let f be a fiber of φ. Then $(K_S + \mathscr{L}) \cdot f = (kK_S + L) \cdot f = 0$, so $L \cdot f = 2k$. If φ is a $\mathbb{P}^1$ bundle we are in case (3.1.3). Otherwise any reducible fiber, f, of φ consists of two irreducible components f_1, f_2 such that $L \cdot f_1 = L \cdot f_2 = k$ (recall (0.3.1)) and we are in case (3.1.4). QED

(*3.2*) *Remark.* It is easy to check the k-very ampleness of the adjoint bundle, $K_S + L$, for the surfaces (S, L) listed in Theorem 3.1. If S is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a $\mathbb{P}^1$ bundle with $L \cdot L \geqq 4k + 5$ this has been done in §2. If S is a k-conic bundle, relative to L, and $f = f_1 + f_2$ is a reducible fiber (see (0.5)), then $K_S \cdot f_i = -1$ and $(K_S + L) \cdot f_i = k - 1$, $i = 1, 2$. Thus $K_S + L$ is not k-very ample by (0.3.1). If S is a Del Pezzo surface and $L \approx -kK_S$ with $K_S^2 \geqq 2$, then $K_S + L \approx -(k-1)K_S$ is not k-very ample by (0.6). □

The following complements Theorem 3.1.

(3.3) Proposition. *Let S be a smooth connected surface, L a k-very ample line bundle on S, $k \geqq 2$. Assume that $tK_S + L$ is very ample, $t = 0, \ldots, k-1$ and $kK_S + L$ is ample. Then $tK_S + L$ is very ample for $t = 1, \ldots, k$ unless either:*

(3.3.1) *$t = k$, S is a Del Pezzo surface with $K_S^2 = 2$, $L \approx -(k+1)K_S$;*
(3.3.2) *$t = k$, S is a Del Pezzo surface with $K_S^2 = 1$, $L \approx -(k+2)K_S$.*

Proof. By assumption, we know that $tK_S + L$ is very ample for $t = 0, \ldots, k-1$. Let $\mathscr{L} := (k-1)K_S + L$. If $kK_S + L = K_S + \mathscr{L}$ is not very ample, from the main theorem (2.1) of [SV] we know that either

(i) S is a Del Pezzo surface with $K_S^2 = 2$ and $\mathscr{L} \approx -2K_S$;
(ii) S is a Del Pezzo surface with $K_S^2 = 1$ and $\mathscr{L} \approx -3K_S$; or
(iii) S is a $\mathbb{P}^1$ bundle over an elliptic curve of invariant $e = -1$ and $\mathscr{L} \approx 3E$, where E is a section of minimal self-intersection $E^2 = 1$.

Note that the further case (1.2) of [SV, (2.1)], does not occur here since $K_S + \mathscr{L}$ is ample.

In case (i) we have $(k-1)K_S + L \approx -2K_S$, i.e. $L \approx -(k+1)K_S$, which gives (3.3.1). In case (ii) we have $(k-1)K_S + L \approx -3K_S$, i.e. $L \approx -(k+2)K_S$, which gives (3.3.2). In case (iii), let f be a fiber of the $\mathbb{P}^1$ bundle S. We have $(k-1)K_S + L \approx 3E$, $K_S \sim -2E + f$ and hence $L \sim (2k+1)E - (k-1)f$. Since L is k-very ample we know from (2.2) that $2k + 1 - 2(k-1) = 3 \geqq k + 2$. This leads to the contradiction $k \leqq 1$. QED

(*3.4*) *Remark.* Notation as in (3.3). We use (0.6) to check the k-very ampleness of the adjoint bundle $K_S + L$. In case (3.3.1), $K_S + L \approx -kK_S$ is k-very ample. In case (3.3.2), $K_S + L \approx -(k+1)K_S$ is not k-very ample.

Note that these cases really occur. Indeed, in case (3.3.1) (respectively (3.3.2)) $L \approx -(k+1)K_S$ (respectively $L \approx -(k+2)K_S$) is k-very ample again by (0.6).

4 k-very ampleness of the adjoint bundle

Let S be a smooth surface and L a k-very ample line bundle on S. If $kK_S + L$ is nef and big (respectively merely nef), let (S', L') be the k-reduction of (S, L) (respectively a k-reduction of (S, L)). If $L \cdot L \geqq 4k + 5$, the results of the previous sections apply to study the k-very ampleness of either $K_S + L$ when k-reductions do not exist (see Proposition 4.1) or of $K_{S'} + L'$, when k-reductions exist (see Proposition 4.2, Remark 4.2.4 and Theorem 4.4). Note that under the assumptions of Theorem 4.4, S is its own k-reduction. The results of this section specialize to the main result of [AP] in the case $k = 2$.

Here, if S is a $\mathbb{P}^1$ bundle, E, f denote as usual a section of minimal self-intersection and a fiber of the ruling.

(4.1) Proposition. *Let S be a smooth surface and let L be a k-very ample line bundle on S, $k \geqq 2$. Assume that $L \cdot L \geqq 4k + 5$. Furthermore assume that $kK_S + L$ is not nef and big. Then $K_S + L$ is k-very ample unless:*

(4.1.1) $S = \mathbb{P}^2$, $L \cong \mathcal{O}_{\mathbb{P}^2}(a)$, $k \leqq a \leqq k + 2$;
(4.1.2) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ *and either* $k \leqq a \leqq k + 1$, *or* $k \leqq b \leqq k + 1$;
(4.1.3) $S = \mathbb{F}_1$, $L \sim aE + bf$ *with either* $a = k$, $k + 1$, *or* $k + 2 \leqq a \leqq 2k$ *and* $b = a + k$;
(4.1.4) *S is a $\mathbb{P}^1$ bundle, $p: S \to C$, over a smooth curve C of invariant e, $L \sim aE + bf$ with either $a = k$, $k + 1$ or C is an elliptic curve, $E^2 = -e = 1$ and $L \sim (k + 2)E$;*
(4.1.5) *S is a k-conic bundle relative to L, which is not a $\mathbb{P}^1$ bundle;*
(4.1.6) *S is a Del Pezzo surface with $L \approx -kK_S$, $K_S^2 \geqq 2$.*

Proof. Following Theorem 3.1 we must consider the cases (3.1.1) to (3.1.5) of that theorem. Note that (2.1) gives the cases (4.1.1), (4.1.2) and (4.1.3), while Proposition 2.5 gives the case (4.1.4). Note also that the condition $a \leqq 2k$ in (4.1.3) follows from the assumption that $kK_S + L$ is not nef and big. Furthermore Theorem 3.1 and Remark 3.2 lead to the cases (4.1.5) and (4.1.6) QED

(4.2) Proposition. *Let L be a k-very ample line bundle on a smooth, connected, projective surface, S. Assume that $L \cdot L \geqq 4k + 5$ and $k \geqq 2$. If $kK_S + L$ is nef, but not ample, let (S', L'), $\pi: S \to S'$, be an associated k-reduction. Then $K_{S'} + L'$ is k-very ample unless $S \cong S'$ and S is a $\mathbb{P}^1$ bundle (see (2.5) in this case).*

Proof. If $K_{S'} + L'$ is not k-very ample, in view of Theorem 0.3.2, there exists an effective divisor D on S' such that

$$L' \cdot D - k - 1 \leqq D \cdot D < L' \cdot D/2 < k + 1 \,. \tag{4.2.1}$$

Let B denote the set of points blown up under π and let $\mathscr{E} = \pi^{-1}(B)$ be the exceptional locus of π. Let γ denote the number of points making up B. Let $\#$ be

the number of points in $D \cap B$ and let $D^{\wedge}$ be the proper transform of D under π. Recall that

$$L' \cdot D = \pi^* L' \cdot D^{\wedge} = (L + k\mathscr{E}) \cdot D^{\wedge} \geqq k + k\# \geqq k \,. \tag{4.2.2}$$

We have that $\# = 0$ or 1 from (4.2.1) and (4.2.2). If $\# = 1$, then $\gamma \geqq 1$ and $L' \cdot L' = L \cdot L + \gamma k^2 \geqq k^2 + 4k + 5$. From (4.2.2) we conclude that $L' \cdot D \geqq 2k$, and by Lemma 0.3.3, we conclude that $L' \cdot D = 2k$. From (4.2.1) we conclude that $D \cdot D = k - 1$. From the Hodge index theorem we conclude that $L' \cdot L' \leqq 4k^2/(k-1)$. Since $L' \cdot L' \geqq k^2 + 4k + 5$ we must have $4k^2 \geqq (k-1)(k^2 + 4k + 5)$, which is absurd for $k \geqq 2$. Thus we can assume that $D \cap B = \emptyset$, so that $D^{\wedge} \cong D$, and $L \cdot D^{\wedge} = L' \cdot D$.

Let either $L' \cdot D = k$, or $L' \cdot D = k + 1$. Then by (0.3.1), D is reduced and irreducible, and by (1.3.1), (1.4.1) we know that in both the cases $D \cong \mathbb{P}^1$. If $L' \cdot D = k$ we have $D \cdot D \geqq -1$. Note that $D \cdot D \neq -1$, since otherwise, we would have a contradiction of Lemma 0.4.4. Thus $D \cdot D \geqq 0$. Likewise if $L' \cdot D = k + 1$ we have $D \cdot D \geqq 0$. Hence in either case, $L' \cdot D = k$, or $L' \cdot D = k + 1$, we have $K_{S'} \cdot D = -2 - D \cdot D \leqq -2$. This gives $(kK_{S'} + L') \cdot D \leqq -2k + k + 1 = -k + 1 < 0$. Note that this is impossible since $kK_{S'} + L'$ is nef (see (0.4)).

Thus we can assume that $L' \cdot D \geqq k + 2$. From this we conclude that $L' \cdot L' \leqq (k+2)^2$. To see this note that, since $L' \cdot D < 2k + 2$ by (4.2.1), we can write $L' \cdot D = k + 1 + \delta$ where $1 \leqq \delta < k + 1$. Hence by (4.2.1), $D \cdot D \geqq \delta$. By (0.3.4) we have

$$L' \cdot L' \leqq (L' \cdot D)^2/(D \cdot D) \leqq (k + 1 + \delta)^2/\delta \leqq (k+2)^2 \,. \tag{4.2.3}$$

Recalling (0.4.3) we have

$$L \cdot L + k^2\gamma = L' \cdot L' \leqq (k+2)^2 \,.$$

Therefore if $\gamma > 0$, we find $L \cdot L \leqq 4k + 4$ which contradicts the hypothesis, $L \cdot L \geqq 4k + 5$.

If $\gamma = 0$, then $S \cong S'$. Since $kK_S + L$ is not ample, by Lemma 0.4.5 we conclude that S is either a $\mathbb{P}^1$ bundle, or $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$. It is easy to see, by using (2.1), that $K_S + L$ is k-very ample in the case of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ with $kK_S + L$ nef. For the $\mathbb{P}^1$ bundle case see (2.5). QED

(4.2.4) Remark. Note that in cases (4.1.5) and (4.1.6) of Proposition 4.1, the line bundle, $kK_S + L$, is nef. Then a k-reduction, (S', L'), of (S, L) exists and by Proposition 4.2 we have that $K_{S'} + L'$ is k-very ample. Note also that, in case (4.1.5), S' is a $\mathbb{P}^1$ bundle obtained by blowing down one irreducible component for each reducible fiber of the k-conic bundle, S (see also (0.5)). In case (4.1.6), S' is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

(4.2.5) Lemma. *Let L be a k-very ample line bundle on a smooth, connected, projective surface, S. Assume that $L \cdot L \geqq 4k + 5$. If $kK_S + L$ is ample, and if $K_S + L$ is not k-very ample, then the curve, D, which exists by Theorem* 0.3.2 *and satisfies*

$$L \cdot D - k - 1 \leqq D \cdot D < L \cdot D/2 < k + 1$$

must also satisfy $L \cdot D \geqq k + 2$.

Proof. The lemma which will be useful in the sequel follows from the argument in (4.2). □

The following result will be used in the proof of Theorem 4.4.

(4.3) Lemma. *Let S be a smooth surface and L a k-very ample line bundle on S, $k \geqq 2$. Let $L \cdot L \geqq 4k + 5$ and assume that $K_S + L$ is not k-very ample. Then there exists an effective divisor, D, satisfying the numerical condition of* (0.3.2). *If $kK_S + L$ is ample, then D is reduced and irreducible.*

Proof. From (0.3.2) we have $L \cdot D \leqq 2k + 1$. The case $L \cdot D = 2k + 1$ does not occur by Lemma 0.3.3. If $L \cdot D < 2k$, then D is reduced and irreducible by (0.3.1). Thus we have to consider the case when $L \cdot D = 2k$, $D = A + B$, where A, B are irreducible, reduced components such that $L \cdot A = L \cdot B = k$. Hence in particular A, B are smooth $\mathbb{P}^1$'s by (1.3.1). From (0.3.2) we have $D \cdot D = k - 1$. By the Hodge index theorem we get

$$(4k + 5)(k - 1) \leqq (L \cdot L)(D \cdot D) \leqq 4k^2 , \tag{4.3.1}$$

which gives $k \leqq 5$.

First note that D is reduced. If not, $D = 2A$ and $A \cdot A = (k - 1)/4$. Then $k = 5$, $L \cdot A = 5$, and $A \cdot A = 1$. Therefore the Hodge index theorem yields $L \sim 5A$. Hence A is ample and $(S, A) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, i.e. $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5))$. This contradicts the ampleness of $5K_S + L$.

Thus we can assume that A, B are distinct. Since $kK_S + L$ is ample we have $(kK_S + L) \cdot A = k(K_S \cdot A + 1) > 0$ which gives $K_S \cdot A \geqq 0$. Similarly $K_S \cdot B \geqq 0$. Then $K_S \cdot D \geqq 0$, so that

$$2A \cdot B - 4 = K_S \cdot D + D \cdot D \geqq k - 1 . \tag{4.3.2}$$

This yields

$$A \cdot B \geqq (k + 3)/2 \tag{4.3.3}$$

as well as

$$g(D) \geqq (k + 1)/2 . \tag{4.3.4}$$

Let L_D be the restriction of L to D. We can assume that $h^0(L_D) \geqq k + 2$, since otherwise $D \cong \mathbb{P}^1$ by (1.2). We will prove the result in the case $k = 2$ and in the hardest case, $k = 5$. The proof in the cases $k = 3, 4$ runs parallel to that of the $k = 2$ case.

Let $k = 2$. Take a smooth point $x \in A$. By Lemma 1.1, $\mathcal{L} := L_D \otimes [x]^{-1}$ is very ample and embeds D as a degree 3 curve in $\mathbb{P}^n$ for $n \geqq 2$. If $n = 2$, then $g(D) = 1$, contradicting (4.3.4). Hence $\mathcal{L}$ embeds D as a non-degenerate curve in $\mathbb{P}^n$, $n \geqq 3$, in such a way that A goes to a line and B goes to a conic. Recalling (4.3.3) we see that any hyperplane $\mathcal{P} = \mathbb{P}^{n-1}$ in $\mathbb{P}^n$ containing A satisfies the condition $\mathcal{P} \cdot B \geqq A \cdot B \geqq 3$. Then $\mathcal{P}$ contains B. Therefore D would be contained in $\mathcal{P}$, which is a contradiction.

Let $k = 5$. Take 4 distinct, smooth points x_1, x_2, x_3, x_4 in A and let $\mathcal{L} := L_D \otimes [x_1]^{-1} \otimes [x_2]^{-1} \otimes [x_3]^{-1} \otimes [x_4]^{-1}$. By Lemma 1.1, $\mathcal{L}$ is very ample and embeds D as a non-degenerate curve of degree 6 in $\mathbb{P}^n$, $n \geqq 2$ in such a way that A goes to a line and B goes to a rational quintic curve. Since there is no smooth, rational, quintic curve in $\mathbb{P}^2$, we conclude that $n \geqq 3$. By (4.3.3) we have that $A \cdot B \geqq 4$. We can choose a $\mathcal{P} = \mathbb{P}^{n-1}$ in $\mathbb{P}^n$ containing A with $x \in \mathcal{P} \cap B$ and with $x \notin A \cap B$. Hence $\mathcal{P} \cdot B \geqq A \cdot B + 1$. If $A \cdot B \geqq 5$, then $\mathcal{P}$ contains B. Therefore D would be contained in $\mathcal{P}$ and this is not possible. Thus we can assume that $A \cdot B = 4$. Since $k = 5$, we have equalities in (4.3.1) and hence $L \sim \lambda D$ for some $\lambda \in \mathbb{Q}$. Since $L \cdot D = 10$, $D \cdot D = 4$ we find that $\lambda = 5/2$, i.e. $2L \sim 5D$. Then there

exists an ample divisor, E, on S such that $L \sim 5E$, $D \sim 2E$, $E \cdot E = 1$. From (4.3.2) we get $K_S \cdot D = 0$. Therefore $K_S \cdot E = 0$, so that $E \cdot E = 1$ gives a parity contradiction. QED

The following is the main result of this paper.

(4.4) Theorem. *Let S be a smooth surface and let L be a k-very ample line bundle on S, $k \geqq 2$. Assume that $kK_S + L$ is ample. Assume that $L \cdot L \geqq 4k + 5$. Then $K_S + L$ is k-very ample, unless S is a Del Pezzo surface with $K_S^2 = 1$ and $L \approx -(k+2)K_S$.*

Proof. By combining Proposition 3.3 and Remark 3.4 we can further assume that $tK_S + L$ is very ample for $1 \leqq t \leqq k$. Indeed, if not, S is a Del Pezzo surface with $K_S^2 = 1$ and $L \approx -(k+2)K_S$, so we are in the exceptional case of the theorem. If $K_S + L$ is not k-very ample, in view of Theorem 0.3.2, there exists an effective divisor D on S such that

$$L \cdot D - k - 1 \leqq D \cdot D < L \cdot D/2 < k + 1 \,. \tag{4.4.1}$$

By Lemma 4.2.5, $L \cdot D \geqq k + 2$. By Lemma 4.3, we know that D is irreducible and reduced. Furthermore $D \cdot D \geqq 1$ by (4.4.1).

(*4.4.2*) *Claim.* If S is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, a $\mathbb{P}^1$ bundle, a Del Pezzo surface with $L \approx -tK_S$, t a positive integer, or a surface in $\mathbb{P}^3$, then $K_S + L$ is k-very ample.

Proof. Let $S \cong \mathbb{P}^2$. Then $L \approx \mathcal{O}_{\mathbb{P}^2}(a)$, $a \geqq k$. Since $kK_S + L \approx \mathcal{O}_{\mathbb{P}^2}(a - 3k)$ is very ample we have $a \geqq 3k + 1$. Therefore $K_S + L \approx \mathcal{O}_{\mathbb{P}^2}(a - 3)$ is k-very ample by (2.1.1) since $a - 3 \geqq 3k - 2 \geqq k$.

Let $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $L \approx \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$, $a \geqq k$, $b \geqq k$. Since $kK_S + L \approx \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a - 2k, b - 2k)$ is very ample we have that $a \geqq 2k + 1$, $b \geqq 2k + 1$. Therefore since $a - 2 \geqq 2k - 1 \geqq k, b - 2 \geqq 2k - 1 \geqq k, K_S + L \approx \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a - 2, b - 2)$ is k-very ample by (2.1.1).

Let S be a Del Pezzo surface, $L \approx -tK_S$. Assume that $K_S \cdot K_S \geqq 2$. Since $kK_S + L \approx (k - t)K_S$ is very ample, we have by (0.6.4), $t - k \geqq 1$. Therefore $K_S + L \approx (1 - t)K_S$ is k-very ample, again by (0.6.4), since $t - 1 \geqq k$. If $K_S \cdot K_S = 1$, the same argument as above, by using (0.6.3), gives the result.

Let S be a $\mathbb{P}^1$ bundle over a smooth curve, C, with $S \ncong \mathbb{P}^1 \times \mathbb{P}^1$. Since $kK_S + L$ is very ample we have $L \cdot f \geqq 1 + 2k$ and hence $L \cdot f \geqq k + 3$. Then Proposition 2.5 applies to say that $K_S + L$ is k-very ample unless $S \cong \mathbb{F}_1$, $L \sim aE + (a + k)f$. But in this case $kK_S + L \sim (a - 2k)E + (a - 2k)f$ is not very ample, by (2.1.2).

Let S be a degree m surface in $\mathbb{P}^3$. Then $L \approx \mathcal{O}_S(a)$, for some positive integer a. By [BS3, (0.1.1)], we know that $K_S + L \approx \mathcal{O}_S(m - 4 + a)$ is k-very ample unless $m - 4 + a \leqq k - 1$. Since $kK_S + L$ is very ample we have $k(m - 4) + a \geqq 1$. By using the previous inequality, this gives $(k - 1)(m - 3) \geqq 1$. Hence in particular, $m \geqq 4$ and therefore K_S is spanned. It is easy to see directly from the definition of k-very ampleness, that the tensor product of a spanned line bundle and a k-very ample line bundle is k-very ample. □

Let $L \cdot D = k + 2$. Then by (1.4.2) we know that either D is isomorphic to an elliptic curve of degree 3 in $\mathbb{P}^2$ or D is a smooth $\mathbb{P}^1$. In the first case the genus formula gives $K_S \cdot D = -D \cdot D \leqq -1$ and we find $(kK_S + L) \cdot D \leqq 2$. Since D is a degree 3 elliptic curve, this contradicts the fact that $kK_S + L$ is very ample. In the second case $K_S \cdot D = -2 - D \cdot D \leqq -3$ and we have the contradiction $(kK_S + L) \cdot D \leqq 2 - 2k < 0$.

Therefore we can assume that $L \cdot D \geqq k+3$. From (0.3.3) we know that $L \cdot D \leqq 2k$. Thus we can write

(4.4.3) $$L \cdot D = k+1+\delta \quad \text{with } 2 \leqq \delta \leqq k-1 .$$

If $k \geqq 6$, by using (0.3.3) with $a = 2$, one has $L \cdot D \leqq 2k-1$. Therefore by (4.4.3) we have that

(4.4.4) $$\delta \leqq k-2 \quad \text{if } k \geqq 6 .$$

Note that if $K_S \cdot D \geqq 0$, then since $\delta \geqq 2$ by (4.4.3) and $D \cdot D \geqq \delta$, we have $g(D) \geqq 2$. By Corollary 1.7.3, we have that $L \cdot D \geqq 2g(D)+k$. This implies that $\delta \geqq 2g(D)-1$. Since $D \cdot D \geqq \delta$, we conclude that $K_S \cdot D < 0$. Because of this the following lemma will complete the proof of Theorem 4.4.

(4.4.5) Lemma. *Using the notation as above, if $K_S \cdot D < 0$ then $K_S + L$ is k-very ample.*

Proof. We know by assumption that $K_S \cdot D < 0$. Let $\mathscr{L} := kK_S + L$. Since $\mathscr{L}$ is (very) ample, we have

$$\mathscr{L} \cdot D = kK_S \cdot D + k + \delta + 1 \geqq 1$$

which implies that $K_S \cdot D \geqq -1$ since $\delta \leqq k-1$. Recall that $D \cdot D \geqq \delta$ by (4.4.1). Furthermore $\delta \geqq 2g(D)-1$. This is clear if $g(D) \leqq 1$. If $g(D) \geqq 2$ this follows as above by using Corollary 1.7.3. Thus we find

$$2g(D)-2 = (K_S + D) \cdot D \geqq -1 + 2g(D) - 1 = 2g(D) - 2 .$$

This shows that

(4.4.6) $$K_S \cdot D = -1; \quad D \cdot D = \delta = 2g(D) - 1 .$$

Hence in particular we conclude that δ must be odd, and S is birationally ruled since $K_S \cdot D < 0$ and $D \cdot D > 0$.

(4.4.7) Claim. $h^0(\mathscr{L}) \leqq (\delta+3)/2$.

Proof. Consider the exact sequence

$$0 \to \mathscr{L} - D \to \mathscr{L} \to \mathscr{L}_D \to 0 .$$

We claim that $h^0(\mathscr{L} - D) = 0$. Assume otherwise. Then there exists an effective divisor $\mathscr{R} \in |\mathscr{L} - D|$. Note that, by (4.4.3) and (4.4.6), $\mathscr{L} \cdot D = \delta + 1$. Hence one has

(4.4.7.1) $$\mathscr{L} \cdot \mathscr{R} = \mathscr{L} \cdot \mathscr{L} - \mathscr{L} \cdot D = \mathscr{L} \cdot \mathscr{L} - (\delta+1) .$$

Since $\mathscr{L}$ is very ample, either $\mathscr{L} \cdot \mathscr{R} \geqq 1$ or $\mathscr{R} \approx \mathscr{O}_S$. If $\mathscr{R} \approx \mathscr{O}_S$, we have $\mathscr{L} \approx D$, which leads to the contradiction $\delta + 1 = \mathscr{L} \cdot D = D \cdot D = \delta$. Therefore $\mathscr{L} \cdot \mathscr{R} \geqq 1$, so that (4.4.7.1) gives $\mathscr{L} \cdot \mathscr{L} \geqq \delta + 2$. The Hodge index relation

$$(\mathscr{L} \cdot \mathscr{L})\delta = (\mathscr{L} \cdot \mathscr{L})(D \cdot D) \leqq (\mathscr{L} \cdot D)^2 = (\delta+1)^2$$

yields $\mathscr{L} \cdot \mathscr{L} \leqq \delta + 2$. Therefore we have $\mathscr{L} \cdot \mathscr{L} = \delta + 2$, that is, by (4.4.7.1), $\mathscr{L} \cdot \mathscr{R} = 1$. Then $\mathscr{R}$ is a line relative to $\mathscr{L}$. Moreover $\mathscr{R} \cdot \mathscr{R} = (\mathscr{L} - D)^2 = \delta + 2 - 2(\delta+1) + \delta = 0$. It thus follows that S is a $\mathbb{P}^1$ bundle. In this case we are done by Claim 4.4.2.

Thus we conclude that $h^0(\mathscr{L} - D) = 0$, so that $h^0(\mathscr{L}) \leqq h^0(\mathscr{L}_D)$. From (4.4.6) we know that $\delta + 1 = \deg \mathscr{L}_D > 2g(D) - 2$, and hence that $h^1(\mathscr{L}_D) = 0$. Therefore

by the Riemann–Roch theorem we have $h^0(\mathscr{L}_D) = \deg \mathscr{L}_D - g(D) + 1 = (\delta + 3)/2$ and we are done. □

We can assume that

$$\delta \geqq 7; \quad k \geqq 9 . \tag{4.4.8}$$

To see this, note that by Claim 4.4.2 we can assume that $h^0(\mathscr{L}) \geqq 5$. Therefore Claim 4.4.7 implies that $\delta \geqq 7$. Hence $k \geqq 9$ by (4.4.3) and (4.4.4).

(*4.4.9*) *Claim.* We have $K_S \cdot K_S = 0$.

Proof. Let $\mathscr{M} := (k+1)K_S + L = K_S + \mathscr{L}$. We can assume that $\mathscr{M}$ is spanned. Otherwise, since $\mathscr{L}$ is very ample, general results from adjunction theory (see e.g. [SV, (0.1)], apply to say that S is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a $\mathbb{P}^1$ bundle. In all these cases we are done by Claim 4.4.2. Writing $\lambda := \mathscr{L} \cdot \mathscr{L}$ and $\mu := \mathscr{M} \cdot \mathscr{M}$, we have

$$\lambda = k(kK_S^2 + 2K_S \cdot L) + L^2; \tag{4.4.9.1}$$

$$\mu = (k+1)((k+1)K_S^2 + 2K_S \cdot L) + L^2 .$$

Recall that since $k \geqq 9$ by (4.4.8), it follows from (4.4.4) that $\delta \leqq k - 2$. Thus

$$3 \leqq \lambda \leqq \delta + 2 \leqq k; \quad 0 \leqq \mu \leqq \delta \leqq k - 2 . \tag{4.4.9.2}$$

Indeed, $\mu \geqq 0$ since $\mathscr{M}$ is spanned and $\lambda = \mathscr{L} \cdot \mathscr{L} \leqq \delta + 2$ follows as in the proof of Claim 4.4.7. Note that, by (4.4.3) and (4.4.6), $\mathscr{M} \cdot D = K_S \cdot D + \mathscr{L} \cdot D = \delta$ and $D \cdot D = \delta$. Then the Hodge index relation gives $\mu = \mathscr{M} \cdot \mathscr{M} \leqq (\mathscr{M} \cdot D)^2/\delta = \delta$. Furthermore we can assume that $\lambda = \mathscr{L} \cdot \mathscr{L} \geqq 3$. Otherwise, since $\mathscr{L}$ is very ample, either $S \cong \mathbb{P}^2$ or $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and in these cases we are done by Claim (4.4.2). Let

$$\alpha := -(kK_S \cdot K_S + 2K_S \cdot L); \quad \alpha^* := -((k+1)K_S \cdot K_S + 2K_S \cdot L) .$$

Then (4.4.9.1) becomes $L \cdot L = \alpha k + \lambda = \alpha^*(k+1) + \mu$. Therefore we have

$$\alpha^* = (\alpha - \alpha^*)k + \lambda - \mu , \tag{4.4.9.3}$$

where $\alpha - \alpha^* = K_S \cdot K_S$. We have to show that $\alpha = \alpha^*$. Assume that $\alpha^* > \alpha$. Hence, by combining (4.4.9.2) and (4.4.9.3), and recalling that $k \geqq 9$, we find

$$\alpha^* \leqq -k + \lambda - \mu \leqq \lambda - k \leqq k - k = 0 .$$

Thus we get

$$4k + 5 \leqq L \cdot L = \alpha^*(k+1) + \mu \leqq 0 + k - 2 = k - 2 ,$$

which is a contradiction. Assume now that $\alpha > \alpha^*$. Then (4.4.9.3) yields $\alpha^* \geqq k + \lambda - \mu$. By using (4.4.9.2) we find

$$\begin{aligned} L \cdot L = \alpha^*(k+1) + \mu &\geqq (k + \lambda - \mu)(k+1) + \mu \\ &= k(k+1) + \lambda(k+1) - \mu k = k(k+1) + \lambda + k(\lambda - \mu) \\ &\geqq k(k+1) + 3 + k(3 - \delta) . \end{aligned}$$

Thus we have that

$$L \cdot L + \delta k \geqq k^2 + 4k + 3 . \tag{4.4.9.4}$$

Recalling (4.4.3) and (4.4.6), the Hodge index relation yields

$$(L\cdot L)\,\delta \leqq (k+\delta+1)^2 = k^2+2k+1+2\delta k+\delta^2+2\delta\,.$$

Since (4.4.9.4) gives $k^2+2k+1 \leqq L\cdot L+\delta k-2k-2$, the above relation yields

$$(L\cdot L)\,\delta \leqq (L\cdot L+\delta k-2k-2)+2\delta k+\delta^2+2\delta\,.$$

Rewriting this we have

$$(L\cdot L)\,(\delta-1) \leqq 3(\delta-1)k+k+1+(\delta^2-1)+2(\delta-1)\,.$$

Since $\delta \geqq 7$ we can divide by $\delta-1$ to obtain

$$L\cdot L \leqq 3k+(k+1)/(\delta-1)+\delta+3\,.$$

Since $7 \leqq \delta \leqq k-2$ we obtain

$$L\cdot L \leqq 4k+1+(k+1)/(\delta-1) \leqq 4k+k/6+7/6.$$

Using $\delta \leqq k-2$ in (4.4.9.4) gives $L\cdot L \geqq 6k+3$ which leads to the contradiction $6k+3 \leqq 4k+k/6+7/6$. This proves the claim. □

Since S is birationally ruled one has that either $S \cong \mathbb{P}^2$, or $K_S\cdot K_S \leqq 8(1-q(S))$ with equality only if S is a $\mathbb{P}^1$ bundle. Thus from Claim 4.4.9 we conclude that either $q(S)=0$ and S is rational, or $q(S)=1$ and S is a $\mathbb{P}^1$ bundle. In this last case we are done by Claim 4.4.2. Therefore we can assume that S is rational. Hence in particular $h^0(2K_S)=0$. Therefore the Riemann–Roch theorem gives

$$h^0(-K_S)=\chi(-K_S)+h^1(-K_S) \geqq \chi(-K_S)=1\,.$$

Thus, since $-K_S$ is effective and L is k-very ample, we have $-L\cdot K_S \geqq k$. Recall that we can assume $\mathscr{L}\cdot\mathscr{L} \geqq 3$. Therefore from $3 \leqq \mathscr{L}\cdot\mathscr{L}=(kK_S+L)^2 = L\cdot L+2kK_S\cdot L$ we get $L\cdot L \geqq 3+2k^2$. Recalling that $k \geqq 9$ since we are assuming $\delta \geqq 7$, this contradicts

$$L\cdot L \leqq (1+k+\delta)^2/\delta \leqq (k+8)^2/7\,,$$

where the last inequality follows from (0.3.4) and $\delta \geqq 7$.

The arguments above show that the assumption $K_S\cdot D<0$ leads to a contradiction. This completes the proof of Lemma 4.4.5 and hence also the proof of the theorem. QED

(*4.4.10*) *Remark.* Let (S,L) be the exceptional pair in Theorem 4.4. From [SV, (1.3)] we know that $L+kK_S \approx -2K_S$ is ample and spanned and gives a double cover $p: S \to Q$, where $Q \subset \mathbb{P}^3$ is a quadric cone. The branch locus of p consists of the vertex of Q and a smooth curve which is the transversal intersection of Q with a cubic surface.

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