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Approximation on a disk II

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1 An approximation result

This paper is a continuation of [P]. The main result of [P] is that there are functions G defined in a neighborhood of the origin in the complex plane, which behave in a sense as $\bar{z}^2$, such that G together with z^2 separates the points of (small) disks D around the origin, and such that the function algebra $[z^2, G; D]$ on D is *not* the same as the algebra $C(D)$ of all continuous functions on D. In this paper we show that the other possibility also can occur: for a large class of functions G defined in a neighborhood of the origin we show $[z^2, G; D] = C(D)$ for sufficiently small disks D around 0. We will adopt notation from [P]. In the following it will be convenient to write the function G in the form

$$G(z) = \bar{z}^2(1 + g(z))^2 .$$

We like to mention that Pascal Thomas, independently from us and at the same time, worked out a special case of our main result, i.e. the case $g(z) = z$, [T].

Theorem. *Let g be defined in a neighborhood of the origin in the complex plane, of class C^1, with $g(0) = 0$, and such that $|g_z(0)| > |g_{\bar{z}}(0)|$. Then $[z^2, \bar{z}^2(1 + g(z))^2; D] = C(D)$ for sufficiently small disks D centered at the origin.*

Proof. Let $a = g_z(0)$ and $b = g_{\bar{z}}(0)$. By the change of coordinate $z = iw/a$ we may and will assume without loss of generality that $a = i$ and $|b| < 1$. Since the first order partial derivatives of g are continuous near 0, Taylor's formula can be applied to $\operatorname{Re} g$ and $\operatorname{Im} g$ to obtain that if ε is a number with $0 < \varepsilon < 1 - |b|$ the function

$$r(z) = g(z) - iz - b\bar{z}$$

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satisfies the inequality
$$|r(z)| \leqq \varepsilon|z|$$
for all z in a sufficiently small disk D around 0. Note also that the generators of the algebra separate the points of sufficiently small disks D.

We now follow the proof of Theorem 1 in [P].

Define $$X = \{(z^2, \bar{z}^2(1+g(z))^2) \colon z \in D\}\,.$$

Consider the map $\Pi \colon \mathbb{C}^2 \to \mathbb{C}^2$, defined by
$$\Pi(\zeta_1, \zeta_2) = (\zeta_1{}^2, \zeta_2{}^2)\,.$$

Then $\Pi^{-1}(X) = X_1 \cup X_2 \cup X_3 \cup X_4$ with
$$X_1 = \{(z, \bar{z}(1+g(z))) \colon z \in D\}$$
$$X_2 = \{(-z, -\bar{z}(1+g(z))) \colon z \in D\} = \{(z, \bar{z}(1+g(-z))) \colon z \in D\}$$
$$X_3 = \{(-z, \bar{z}(1+g(z))) \colon z \in D\}$$
$$X_4 = \{(z, -\bar{z}(1+g(z))) \colon z \in D\} = \{(-z, \bar{z}(1+g(-z))) \colon z \in D\}\,.$$

By Wermer's theorem it follows that the sets X_i are polynomially convex. Now Kallin's theorem is also valid if the two angular sectors are replaced by $S_+ = \{\operatorname{Im}\lambda > 0\} \cup \{0\}$ and $S_- = \{\operatorname{Im}\lambda < 0\} \cup \{0\}$ (see reference [9] of [P]). With $p(\zeta_1, \zeta_2) = \zeta_1 + \zeta_2$ we notice that for z in D:
$$p(z, \bar{z}(1+g(z))) = z + \bar{z} + \bar{z}g(z) = 2\operatorname{Re} z + i|z|^2 + b\bar{z}^2 + \bar{z}r(z)$$
where $|\bar{z}r(z)| \leqq \varepsilon|z|^2$.

It follows that $p(z, \bar{z}(1+g(z))) \in S_+$ so $p(X_1) \subset S_+$. In a similar way one shows that $p(X_2) \subset S_-$. Since $p^{-1}(0) \cap (X_1 \cup X_2)$ contains only the origin in $\mathbb{C}^2$ we can apply Kallin's theorem and conclude that $X_1 \cup X_2$ is polynomially convex.

Using the polynomial $p(\zeta_1, \zeta_2) = -\zeta_1 + \zeta_2$ one shows similarly that $X_3 \cup X_4$ is polynomially convex.

We apply Kallin's theorem for the third time, now with $p(\zeta_1, \zeta_2) = \zeta_1\zeta_2$. Since $p(X_1 \cup X_2)$ is contained in an angular sector near the positive real axis and $p(X_3 \cup X_4)$ in an angular sector near the negative real axis, it follows that $\Pi^{-1}(X) = X_1 \cup X_2 \cup X_3 \cup X_4$ is polynomially convex. By Sibony's theorem and the O'Farrell–Preskenis–Walsh result we conclude as in in the proof of Theorem 1 in [P] that $P(X) = C(X)$. This is equivalent to
$$[z^2, \bar{z}^2(1+g(z))^2; D] = C(D)\,.$$

2 Examples

Suppose g is of class C^1 and both $g_z(0)$ and $g_{\bar{z}}(0)$ are equal to 0. It can happen that the algebra $[z^2, \bar{z}^2(1+g(z))^2; D]$ is unequal to $C(D)$ and it is also possible that this algebra is equal to the algebra $C(D)$.

(1) In [P] it is shown that $[z^2, \bar{z}^2(1+\bar{z}^3)^{-2/3}; D] \neq C(D)$ for (sufficiently small) disks D.

(2) Let f be a real-valued function of class C^1, defined in a neighborhood of 0, such that f is even, and such that $f(0) = 0, f(z) > 0$ if $z \neq 0$.

The functions z^2 and $\bar{z}^2(1 + izf(z))^2$ separate the points of (small) disks D around 0, and as in the proof of the theorem above we find $[z^2, \bar{z}^2(1 + izf(z))^2; D] = C(D)$.

(3) Also $[z^2, \bar{z}^2(1 + iz^3)^2; D] = C(D)$ if D is a disk centered at the origin. Using the same pull-back Π as in the proof of the theorem and with

$$X = \{(z^2, \bar{z}^2(1 + iz^3)^2 : z \in D\}$$

one now finds

$$X_1 = \{(z, \bar{z}(1 + iz^3)) : z \in D\}$$

$$X_2 = \{(-z, -\bar{z}(1 + iz^3)) : z \in D\} = \{(z, \bar{z}(1 - iz^3)) : z \in D\}$$

$$X_3 = \{(-z, \bar{z}(1 + iz^3)) : z \in D\}$$

$$X_4 = \{(z, -\bar{z}(1 + iz^3)) : z \in D\} = \{(-z, \bar{z}(1 - iz^3)) : z \in D\}.$$

Use $p(\zeta_1, \zeta_2) = \zeta_1{}^3 + \zeta_2{}^3$ to show that $X_1 \cup X_2$ is polynomially convex and $p(\zeta_1, \zeta_2) = -\zeta_1{}^3 + \zeta_2{}^3$ to show that $X_3 \cup X_4$ is polynomially convex. It follows as in the proof of the theorem that $[z^2, \bar{z}^2(1 + iz^3)^2; D] = C(D)$.

3 Remarks

(1) Is it true (if z^2 and G separate the points of D) that $[z^2, G; D] \neq C(D)$ for every antiholomorphic function G? In the light of the theorem and the examples above one might even conjecture that $[z^2, \bar{z}^2(1 + g(z))^2; D] \neq C(D)$ for every g with $|g_z(0)| < |g_{\bar{z}}(0)|$.

(2) It is not clear whether the theorem can be generalized to the situation where F and G behave like z^m and $\bar{z}^m$ with $m > 2$. So there is nothing known about $[F, G; D]$ for this case (except for even values of m: in this situation we know that there exist examples with $[F, G; D] \neq C(D)$).

(3) Consider once again the situation that F and G are of the form $F(z) = z^m(1 + f(z)), G(z) = \bar{z}^n(1 + g(z))$ where f and g are functions defined in a neigborhood of the origin, with $f(0) = 0, g(0) = 0$. The functions f and g were supposed to be of class C^1 but if one is willing to drop this differentiability condition, just assuming continuity of f and g, then one can find a counterexample for the case $m = n$ in the following way.

Choose sequences (a_k), (r_k), (R_k) of positive numbers converging to 0 and such that $0 < r_k < R_k$ and $a_{k+1} + R_{k+1} < a_k - R_k$ for each k.

Let $D_k = \{|z - a_k| \leqq r_k\}$ and $E_k = \{|z - a_k| \leqq R_k\}, k = 1, 2, 3, \ldots$ Let $F(z) = z^m$ and define a modification G of the function $\bar{z}^m + \bar{z}^{m+1}$ on the complex plane in the following manner:

$$G(z) = \bar{z}^m + \bar{z}^{m+1} \text{ outside } E_1 \cup E_2 \cup \ldots, \text{ in particular } g(0) = 0$$

$$G(z) = a_k{}^m + a_k{}^{m+1} \text{ on } D_k.$$

For an appropriate choice of the sequences (r_k) and (R_k) and the values of G on the sets $E_k - D_k$ the function g is continuous and moreover the functions F and

G separate the points. For any disk D centered at 0 the elements of $[F, G; D]$ are analytic on the interior of all sets D_k which belong to D. So for any such disk $D: [F, G; D] \neq C(D)$.

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References

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[T] Thomas, P.: Private communication

Note added in proof

The second author recently proved a generalization of the theorem for the situation where F and G behave like z^m and $\bar{z}^m$ with $m > 2$.