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# Local $q$-completeness of complements of smooth CR-submanifolds

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## 0 Introduction

It is an important goal of complex analysis to have quite simple criteria for $q$-convexity or $q$-completeness of a manifold: these imply cohomology finiteness or vanishing theorems of Grauert-Andreotti type [1] and, in connection with Morse theory [8], topological statements like Lefschetz type theorems (see [10]).

One early answer to the question, how $q$-convexity behaves under general processes, like removing subsets $M$ from complex manifolds $X$, was given by Barth [2] in 1970: he showed that $X \setminus M$ is $q$-convex if $M$ is a $q$-codimensional complex submanifold in projective space $X$. In 1973, Schneider [11] proved a similar result for complements of $q$-codimensional complex submanifolds with positive normal bundles in compact complex manifolds; this was generalized to local complete intersections by Fritzsche [6] in 1976.

The subject of the present paper is the question of what can be stated for the local $q$-completeness of $X \setminus M$ where $M$ is a CR-submanifold of a complex manifold $X$ – this simultaneously generalizes both the case of complex submanifolds $M \subset X$ and the case of real hypersurfaces $M \subset X$.

As a tool, we will use some results of Diederich and Fornaess [4, 5] and of Peternell [9] on $q$-convexity with corners, an important generalization of $q$-convexity which was introduced by Grauert [7] in 1981.

The article is organized as follows: The presentation of the basic definitions in Chap. 1 is followed by the proof of a necessary condition for local $q$-completeness of $\mathbf{C}^n \setminus M$ at $p \in M$, where $M \subset \mathbf{C}^n$ is a smooth CR-submanifold, in Chap. 2. This criterion then implies a lower estimate for $\nu(p; M) := \min\{q \leqq n \mid \mathbf{C}^n \setminus M$ is locally $q$-complete at $p\}$ in Chap. 3, where the lower bound $q'$ for $\nu(p; M)$ depends on the CR-codimension of $M$ and on the local index of convexity of $M$ at $p$ which is defined in 3.1.1. In Chap. 4, the quality of $q'$ is discussed; it becomes apparent that in the case of complex or of generic CR-submanifolds, the statement of Chap. 3 is optimal, in the sense that $q'$ coincides with an upper bound for $\nu(p; M)$. In the general case of non-generic CR-submanifolds, we only obtain $\nu(p; M) \leqq q' + 1$, and it remains an open question whether $\nu(p; M) \leqq q'$ is valid.

A geometric interpretation of the index of convexity in terms of the Levi form of $M$ at $p$ which is given in Chap. 5 makes it clear that the index of convexity is a CR-invariant, which implies that the local results given are independent of any choice of a CR-imbedding $M \hookrightarrow \mathbf{C}^n$. Transferring the local results to smooth CR-submanifolds $M$ in complex manifolds $X$, we give some globalizations of the local results in Chap. 6.

## 1 Preliminaries

In the sequel, let $M \subset \mathbf{C}^n$ be a smooth closed submanifold of real codimension $k \geqq 1$.

**1.1. Definition.** For $p \in M$, $m := \dim_{\mathbf{R}} M$, $k := \operatorname{codim}_{\mathbf{R}} M = 2n - m$ and $U = U(p) \subset \mathbf{C}^n$ an open neighborhood of $p$, we call a $k$-tuple $(\rho_1, \ldots, \rho_k)_U$ of functions $\rho_i : U \to \mathbf{R}$, $\rho_i \in C^\infty(U)$ $\forall i = 1, \ldots, k$, *defining functions* for $M$ on $U$ if:

(1) $M \cap U = \{z \in \mathbf{C}^n \mid \rho_i(z) = 0\ \forall i = 1, \ldots, k\}$ (2) $d\rho_1 \wedge \ldots \wedge d\rho_k \neq 0$ on $U$.

Given $(\rho_1, \ldots, \rho_k)_U$ defining functions for $M$ on the open neighborhood $U = U(p)$ of $p \in M$ in $\mathbf{C}^n$, we can describe the *holomorphic tangent space of M at p* as

$$T_p^{10} M \cong \ker \left( \frac{\partial \rho_i(p)}{\partial z_j} \right)_{1 \leqq i \leqq k,\, 1 \leqq j \leqq n}$$

where $\left( \frac{\partial \rho_i(p)}{\partial z_j} \right)_{i,j} : \mathbf{C}^n \to \mathbf{C}^n$ is the linear map

$$\left( \frac{\partial \rho_i(p)}{\partial z_j} \right)_{i,j} : \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix} \mapsto \left( \frac{\partial \rho_i(p)}{\partial z_j} \right)_{i,j} \cdot \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix}.$$

Obviously, $\dim_{\mathbf{C}} T_p^{10} M = n - \operatorname{rank} \left( \frac{\partial \rho_i(p)}{\partial z_j} \right)_{i,j}$, so $\dim_{\mathbf{C}} T_p^{10} M$ is minimal if $\left( \frac{\partial \rho_i(p)}{\partial z_j} \right)_{i,j}$ has maximal rank at $p$, i.e.

$$\operatorname{rank} \left( \frac{\partial \rho_i(p)}{\partial z_j} \right)_{i,j} = \begin{cases} n\ (m \leqq n) \\ 2n - m\ (m > n) \end{cases}.$$

In this case we define:

**1.2. Definition.** Let $M \subset \mathbf{C}^n$ be a closed smooth submanifold of real dimension $m$, $p \in M$. $M$ is said to be *generic* at $p$ if $\dim_{\mathbf{C}} T_p^{10} M = \max\{0, m - n\}$ holds. Being generic is of course an open property.

The particular class of smooth submanifolds we deal with is:

**1.3. Definition.** A smooth submanifold $M \subset \mathbf{C}^n$ is called *CR-submanifold of type* $(m, \ell)$ if:

(1) $\dim_{\mathbf{R}} M = m$ (2) $\forall p \in M$: $\dim_{\mathbf{C}} T_p^{10} M = \ell$.

Here, $\ell$ is called *CR-dimension of M*, the *CR-codimension of M* is given by $d := n - \ell$.

In this terminology, *generic* CR-submanifolds $M \subset \mathbf{C}^n$ of real dimension $m$ are of type $(m, 0)$ for $m \leqq n$ and of type $(m, m-n)$ for $m > n$.

Finally, in order to clarify the notation:

**1.4. Definition.** Let $M \subset \mathbf{C}^n$ be a smooth submanifold, $p \in M$ and $\Omega \subset \mathbf{C}^n$ an open set.

(1) A real-valued function $f \in C^\infty(\Omega)$ is called *q-convex* or *strictly q-convex* if for any $x \in \Omega$, the Levi form $\mathscr{L}_f(x)(\cdot,\cdot)$ of $f$ at $x$ has at least $n-q+1$ positive eigenvalues; $f$ is called *weakly q-convex* if the Levi form of $f$ has at least $n-q+1$ non-negative eigenvalues everywhere on $\Omega$.

(2) $f$ is called an *exhaustion function of* $\Omega$, if:

$\triangleright$ There is a $c_0 \in \mathbf{R} \cup \{+\infty\}$, such that $\Omega_c := \{z \in \Omega \mid f(z) < c\} \Subset \Omega \, \forall c < c_0$.

$\triangleright$ For each compact set $K \subset \Omega$, there is a constant $c < c_0$ with $K \subset \Omega_c$.

(3) We call $\mathbf{C}^n \setminus M$ *locally q-complete* at $p$ if there is an open neighborhood $U = U(p) \subset \mathbf{C}^n$ of $p$ such that $U \setminus M$ admits a $q$-convex exhaustion function.

## 2 A necessary condition for local $q$-completeness of $\mathbf{C}^n \setminus M$

During the course of this work, the following criterion will prove to be very useful:

**2.1. Theorem.** *Let $M \subset \mathbf{C}^n$ be a smooth submanifold such that $\mathbf{C}^n \setminus M$ is locally q-complete at $p \in M$. If $d_M$ denotes the euclidean distance to M, then there is an open neighborhood $V = V(p) \subset \mathbf{C}^n$ of p so that the function $-\log(d_M^2)$: $V \setminus M \xrightarrow{C^\infty} \mathbf{R}$ is at least weakly q-convex on $V \setminus M$.*

Our proof of this theorem is based on Lemma 2.3, a simplified version of a lemma by Peternell [9, Lemma 7] where the following terminology is used:

**2.2. Definition.** Let $U \subset \mathbf{C}^n$ be open, $f \colon U \to \mathbf{R}$ a real-valued function and $x \in U$.

(1) $f$ is called *q-convex with corners at x* if there is an open neighborhood $V = V(x) \subset U$ of $x$ and a finite number of functions $f_1, \ldots, f_r$ $q$-convex on $V$ such that $f_{|V} = \max(f_1, \ldots, f_r)$.

(2) $f$ is called *q-convex with corners on U* if $f$ is $q$-convex with corners at $x$ for every $x \in U$. The set of all functions $q$-convex with corners on $U$ will be denoted by $F_q(U)$.

(3) $U$ is called *q-convex with corners* if there is a compact set $K \subset U$ and an exhaustion function $g$ of $U$ such that $g \in F_q(U \setminus K)$.

**2.3. Lemma.** *Let X be a complex manifold, $\dim_{\mathbf{C}} X = n$, $S \subset X \times X$ an open subset and $\Delta := \{(z, w) \in X \times X \mid z = w\}$ the diagonal in $X \times X$; let $f \in F_{n+r}(S \setminus \Delta)$ be a function which is $(n+r)$-convex with corners outside $\Delta$. If $N_1 \Subset N_2 \Subset X$ are open sets such that $N_1 \times N_2 \Subset S$ and the function $s \colon N_1 \to \mathbf{R}$, $s(x) := \sup\{f(x, y) \mid y \in \overline{N_2} \setminus N_1\}$, satisfies $s(x) > f(x, y) \, \forall x \in N_1$, $y \in \partial N_2$, then the following statement holds:*

*If $N_1$ is q-convex with corners then for every open set $D \Subset N_1$ and for any $\varepsilon > 0$, there is a function $g \in F_{q+r}(D)$ satisfying $|g(x) - s(x)| < \varepsilon \, \forall x \in D$.*

Lemma 2.3 will be applied to a certain class of functions that are $n$-convex in $(\mathbf{C}^n\times\mathbf{C}^n)\setminus\Delta$; the construction of these functions makes use of several properties of the functions $h\colon \mathbf{C}^n\times\mathbf{C}^n\to\mathbf{R}_0^+$, $h(x,y):=\|x-y\|^2$, and $-\log(h)$ which are collected in the following lemma:

**2.4. Lemma.** *If the function $h\colon \mathbf{C}^{2n}\to\mathbf{R}_0^+$ is defined by $h(z):=\sum_{j=1}^{n}|z_j-z_{n+j}|^2$, then the following are satisfied:*

(1) *For any $z\in\mathbf{C}^{2n}$, $\zeta,\eta\in T_z^{10}\,\mathbf{C}^{2n}$,*

$$\mathscr{L}_h(z)(\zeta,\eta)=\sum_{j=1}^{2n}\zeta_j\cdot\bar\eta_j-\left(\sum_{j=1}^{n}\zeta_j\cdot\bar\eta_{j+n}+\sum_{j=n+1}^{2n}\zeta_j\cdot\bar\eta_{j-n}\right).$$

(2) *For any $z\in\mathbf{C}^{2n}$, $\mathscr{L}_h(z)_{|T_z^{10}\mathbf{C}^{2n}}\geqq 0$, and*

$$\mathscr{L}_h(z)_{|J_z}\equiv 0\qquad \textit{for } J_z:=\{\zeta\in T_z^{10}\,C^{2n}\,|\,\zeta_i=\zeta_{i+n}\,\forall i=1,\dots,n\}.$$

(3) *If $\Delta$ denotes the diagonal in $\mathbf{C}^n\times\mathbf{C}^n$ and if, for $z\in(\mathbf{C}^{2n}\setminus\Delta)$, $\Omega_z$ is the real hypersurface $\Omega_z:=\{w\in\mathbf{C}^{2n}\setminus\Delta\,|\,h(w)=h(z)\}$, then*

$$\mathscr{L}_{-\log(h)}(z)_{|J_z\oplus(T_z^{10}\Omega_z)^\perp}\equiv 0\,\forall z\in\mathbf{C}^{2n}\setminus\Delta.$$

Using the function $-\log(h)$, we now construct $n$-convex functions $f$ on $\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta$ with sub-level sets $\{f<c\}$ relatively closed in $(\mathbf{C}^n\times\mathbf{C}^n)\setminus\Delta$ as follows: For any real-valued function $\varphi\in C^\infty(\mathbf{R})$, $\varphi'>0$, $\varphi''>0$, the composed function $\varphi\circ(-\log(h))=:r\circ(-h)$ for $r\colon x\mapsto\varphi(-\log(-x))$, is weakly $n$-convex in $\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta$; more precisely we know, with the notations of Lemma 2.4:

(1) $\mathscr{L}_{r\circ(-h)}(z)_{|J_z\oplus(T_z^{10}\Omega_z)^\perp}\geqq 0$ and (2) $\mathscr{L}_{r\circ(-h)}(z)_{|(T_z^{10}\Omega_z)^\perp}>0$

for every $z\in\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta$. Now, for any $\varepsilon>0$, the plurisubharmonic functions

$$\psi_\varepsilon\colon \mathbf{C}^{2n}\to\mathbf{R}_0^+,\qquad \psi_\varepsilon\colon z\mapsto\varepsilon\cdot\sum_{j=1}^{n}|z_j|^2,$$

are strictly plurisubharmonic in the $J_z$-directions. Consequently, for any $\varepsilon>0$, the function

$$f_\varepsilon^{(r)}:=r\circ(-h)+\psi_\varepsilon,$$
$$f_\varepsilon^{(r)}(x,y)=r(-\|x-y\|^2)+\varepsilon\cdot\|x\|^2,\qquad (x,y)\in\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta,$$

is strictly $n$-convex in $(\mathbf{C}^n\times\mathbf{C}^n)\setminus\Delta$, the sub-level sets $\{f_\varepsilon^{(r)}<c\}$ being relatively closed in $(\mathbf{C}^n\times\mathbf{C}^n)\setminus\Delta$ for each $c\in\mathbf{R}$; in particular, $f_\varepsilon^{(r)}\in F_n(\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta)$.

Now we are ready for the

*Proof of Theorem 2.1* Let $U=U(p)\subset\mathbf{C}^n$ be an open $p$-neighborhood such that $U\setminus M$ is $q$-complete, and $U$ be chosen so small that $d_M\in C^\infty(U\setminus M)$. Let $V=V(p)\Subset U$ be small enough so that $d_M(x)=d_{M\cup\partial U}(x)\,\forall x\in V\setminus M$ (where $d_{M\cup\partial U}$ denotes the euclidean distance of points in $N_1:=U\setminus M$ to $M\cup\partial U$, $d_{M\cup\partial U}\in C^\infty(V\setminus M)$). Furthermore, let $N_2$ be an open neighborhood of $\bar U$ with

$N_1 \Subset N_2 \Subset \mathbf{C}^n$, $\varphi \in C^\infty(\mathbf{R})$ an arbitrary convex increasing function and $r$ the composed function $r\colon x \mapsto \varphi(-\log(-x))$. For any $\varepsilon > 0$, define $s_\varepsilon^{(r)}\colon N_1 \to \mathbf{R}_0^+$ by

$$\begin{aligned} s_\varepsilon^{(r)}(x) &:= \sup\{f_\varepsilon^{(r)}(x,y) \mid y \in \overline{N_2} \setminus N_1\} \\ &= \sup\{f_\varepsilon^{(r)}(x,y) \mid y \in \partial U \cup M\} \\ &= r \circ (-d^2_{M \cup \partial U})(x) + \varepsilon \cdot \|x\|^2. \end{aligned}$$

As $s_\varepsilon^{(r)}(x) > f_\varepsilon^{(r)}(x,y)\ \forall \varepsilon > 0$, $x \in N_1$, $y \in \partial N_2$, and as $N_1 = U \setminus M$ is $q$-complete one can, in view of Lemma 2.3, approximate each $s_\varepsilon^{(r)}$ on relatively-compact subsets $D \Subset N_1$ by functions $g \in F_q(D)$. The maximum principle then implies that the function $r \circ (-d_M^2) = \varphi(-\log d_M^2)$ is at least weakly $q$-convex in $V \setminus M$. As this holds for any real-valued, convex increasing function $\varphi \in C^\infty(\mathbf{R})$, the function $-\log d_M^2$ has to be weakly $q$-convex in $V \setminus M$, too. □

## 3 The best possible local $q$-completeness of $\mathbf{C}^n \setminus M$

Let again $M \subset \mathbf{C}^n$ be a smooth CR-submanifold and $p \in M$. From Theorem 2.1 we will deduce a lower estimate for $v(p; M) := \min\{q \leqq n \mid \mathbf{C}^n \setminus M \text{ is locally } q\text{-complete at } p\}$. The lower bound for $v(p; M)$ will be determined by the CR-codimension of $M$ and by an entity that will be called the "index of convexity"; some of its basic properties are introduced below.

### 3.1 *The index of convexity and its basic properties*

**3.1.1. Definition.** Let $M \subset \mathbf{C}^n$ be a smooth CR-submanifold of real dimension $\dim_{\mathbf{R}} M = m$.

(1) $E_j(p)$ holds at $p \in M$ if there are defining functions $\rho_1, \ldots, \rho_{2n-m}$ for $M$ near $p$ such that the Levi form of $\rho_1$ at $p$ has $j$ positive eigenvalues on $T_p^{10} M$.
(2) The *index of convexity* of $M$ at $p$ is defined by $b(p) := \max\{j \in \mathbf{N} \mid E_j(p) \text{ holds at } p\}$.
(3) The *local index of convexity of $M$ at $p$* will be given by

$$\hat{b}(p) := \varlimsup_{M \ni \tilde{p} \to p} b(\tilde{p}).$$

From the definition of the superior limit, respecting the fact that the function $b\colon M \to \mathbf{N}$, $b\colon p \mapsto b(p)$, has integer values, we immediately obtain that the local index of convexity has the following properties:

**3.1.2. Lemma.** *Let $M \subset \mathbf{C}^n$ be a smooth* CR-*submanifold, $p \in M$ and $\hat{b}(p)$ the local index of convexity of $M$ at $p$. Then:*

(1) *There is an open $p$-neighborhood $U \subset \mathbf{C}^n$ such that* $\hat{b}(p) = \max\{b(p') \mid p' \in U \cap M\}$.
(2) *Given an arbitrary open neighborhood $V = V(p)$ of $p$ in $\mathbf{C}^n$, there exists a point $p' \in V \cap M$ satisfying $b(p') = \hat{b}(p)$.*

The key for a better understanding of the index of convexity is the knowledge about the relation between different sets of defining functions for $M$ near $p$; a standard application of the inverse function theorem shows:

**3.1.3. Lemma.** *Let $M \subset \mathbf{C}^n$ a smooth submanifold of real codimension* $\operatorname{codim}_{\mathbf{R}} M = k$, *$p \in M$, $U$ and $\tilde{U}$ open neighborhoods of $p$ and $(\rho_1, \dots, \rho_k)_U$, $(\tilde{\rho}_1, \dots, \tilde{\rho}_k)_{\tilde{U}}$ two sets of defining functions for $M$ near $p$. Then, there is an open neighborhood $V = V(p) \subset U \cap \tilde{U}$ of $p$ and a matrix $H := (h_{ij})_{1 \leqq i,j \leqq k}$ of $C^\infty$-functions $h_{ij}\colon V \to \mathbf{R}$ such that:*

$$\begin{pmatrix} \tilde{\rho}_1 \\ \vdots \\ \tilde{\rho}_k \end{pmatrix} = H \cdot \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_k \end{pmatrix} \tag{1}$$

$$\det H(x) \neq 0 \; \forall x \in V. \tag{2}$$

An immediate consequence of Lemma 3.1.3 is:

**3.1.4. Lemma.** *Let $M \subset \mathbf{C}^n$ be a smooth submanifold of real codimension* $\operatorname{codim}_{\mathbf{R}} M = k$, *$p \in M$ and $b(p)$ the index of convexity of $M$ at $p$. If $(\rho_1, \dots, \rho_k)_U$ is an arbitrary set of defining functions for $M$ on an open neighborhood $U = U(p) \subset \mathbf{C}^n$ of $p$, then:*

(1) *There is $\alpha \in \mathbf{R}^k \setminus \{0\}$ and a real linear combination $\rho_\alpha := \sum_{i=1}^{k} \alpha_i \cdot \rho_i$ of the functions $\rho_i$, such that $\mathscr{L}_{\rho_\alpha}(p)_{|T_p^{10}M}$ has $b(p)$ positive eigenvalues.*

(2) *For no real linear combination $\rho_\beta := \sum_{i=1}^{k} \beta_i \cdot \rho_i$, $\beta \in \mathbf{R}^k$, does $\mathscr{L}_{\rho_\beta}(p)_{|T_p^{10}M}$ have more than $b(p)$ positive eigenvalues.*

Now we are ready to establish a lower estimate for $v(p; M)$; we will first do this in the case of generic CR-submanifolds $M$.

### *3.2 A negative result in the case of generic* CR-*submanifolds*

If $M \subset \mathbf{C}^n$ is a smooth CR-submanifold, $U = U(p) \subset \mathbf{C}^n$ an open neighborhood of $p \in M$ and if $\rho_i\colon U \xrightarrow{C^\infty} \mathbf{R}$, $1 \leqq i \leqq k$, are defining functions for $M$ on $U$, then the function

$$\rho\colon U \xrightarrow{C^\infty} \mathbf{R}_0^+, \qquad \rho\colon x \mapsto \rho(x) := \sum_{i=1}^{k} \rho_i^2(x),$$

is the square of a distance function to $M$; the Levi form of $\rho$ is influenced by the Levi forms of the defining functions as follows: for $x \in U$ and $\zeta, \eta \in T_x^{10}\mathbf{C}^n$, we have

$$\mathscr{L}_\rho(x)(\zeta, \eta) = 2 \sum_{i=1}^{k} \left( \sum_{j=1}^{n} \frac{\partial \rho_i(x)}{\partial z_j} \zeta_j \right) \cdot \left( \sum_{\mu=1}^{n} \frac{\partial \rho_i(x)}{\partial \bar{z}_\mu} \bar{\eta}_\mu \right) + 2 \sum_{i=1}^{k} \rho_i(x) \cdot \mathscr{L}_{\rho_i}(x)(\zeta, \eta).$$

If now $x \in U \setminus M$, $(\rho_1(x), \ldots, \rho_k(x)) =: \alpha \in \mathbf{R}^k \setminus \{0\}$, then the Levi form of the function $\rho_\alpha := \sum_{i=1}^{k} \alpha_i \cdot \rho_i$ satisfies

$$\mathscr{L}_\rho(x)(\zeta, \zeta) = 2 \cdot \mathscr{L}_{\rho_\alpha}(x)(\zeta, \zeta) \; \forall \zeta \in T_x^{10} M_x$$

where $M_x := \{z \in U \mid \rho_i(z) - \rho_i(x) = 0 \; \forall i = 1, \ldots, k\}$. So, at points $x \in U \setminus M$ suitably chosen, the Levi form of $\rho$ will have $b(p)$ positive eigenvalues on $T_x^{10} M_x$. On the other hand, if $\eta \in T_p^{10} \mathbf{C}^n$ is transverse to $T_p^{10} M$, then

$$\mathscr{L}_\rho(p)(\eta, \eta) = 2 \sum_{i=1}^{k} \left| \sum_{j=1}^{n} \frac{\partial \rho_i(p)}{\partial z_j} \eta_j \right|^2 > 0,$$

thus the Levi form of $\rho$ has $d$ positive eigenvalues in directions transverse to $T_p^{10} M$, where $d$ denotes the CR-codimension of $M$. This motivates the following theorem:

**3.2.1. Theorem.** *Let $M \subset \mathbf{C}^n$ be a generic* CR-*submanifold of real dimension* $\dim_{\mathbf{R}} M = m$, $k := 2n - m$ *its real codimension,* $d := \operatorname{corank}_{\mathbf{C}} T^{10} M$, $p \in M$, *and let* $b(p)$ *be the index of convexity of $M$ at $p$. Then, for any choice of defining functions* $(\rho_1, \ldots, \rho_k)_U$ *for $M$ near $p$, there exists a sequence* $(x_\nu)_\nu \subset U \setminus M$ *converging to $p$ such that the Levi form of* $\rho := \sum_{i=1}^{k} \rho_i^2$ *has* $d + b(p)$ *positive eigenvalues at each point* $x_\nu$.

*Proof.* For any system of defining functions $(\rho_1, \ldots, \rho_k)_U$ of $M$ near $p$, $\operatorname{rank}\left(\frac{\partial \rho_i(p)}{\partial z_j}\right)_{i,j} = \min\{n, k\}$ is maximal because $M$ is generic, so $\operatorname{rank}\left(\frac{\partial \rho_i(x)}{\partial z_j}\right)_{i,j}$ is maximal at each $x \in U$ if $U = U(p) \subset \mathbf{C}^n$ is chosen sufficiently small. Then, $M_x := \{z \in U \mid \rho_i(z) = \rho_i(x) \; \forall i = 1, \ldots, k\} \subset U$ is a generic CR-submanifold of type $(m, n-d)$ for every $x \in U$, coinciding with $M$ for $x \in M \cap U$, and

$$\mathscr{H} := \dot{\bigcup_{x \in U}} \{x\} \times T_x^{10} M_x, \qquad \operatorname{rank}_{\mathbf{C}} \mathscr{H} = n - d,$$

is a complex subbundle of $T^{10} U$. For $x \in U$ and $\zeta \in T_x^{10} \mathbf{C}^n$, the Levi form of $\rho := \sum_{i=1}^{k} \rho_i^2$ is:

$$(*) \qquad \mathscr{L}_\rho(x)(\zeta, \zeta) = 2 \sum_{i=1}^{k} \left| \sum_{j=1}^{n} \frac{\partial \rho_i(x)}{\partial z_j} \zeta_j \right|^2 + 2 \sum_{i=1}^{k} \rho_i(x) \, \mathscr{L}_{\rho_i}(x)(\zeta, \zeta).$$

We distinguish two cases:

*Case 1: $d = n$*

Then $M$ has CR-dimension 0, i.e. $M$ is totally real, which implies $T_p^{10} M = \{0\}$ and $b(p) = 0$. From $(*)$ it follows that the Levi form of $\rho$ is positive definite at $p$; obviously, the statement of Theorem 3.2.1 is then valid for any sequence $(x_\nu)_\nu \subset V \setminus M$.

*Case 2: $d<n$*

Again, (*) implies that for every linear subspace $W_p \subset T_p^{10}\mathbf{C}^n$ transversely intersecting $T_p^{10}M$ at $p$, $\mathscr{L}_\rho(p)_{|W_p}$ is positive definite; in particular, this holds for the $d$-dimensional space

$$(T_p^{10}M)^\perp := \left\{ \eta = \sum_{j=1}^n \eta_j \frac{\partial}{\partial z_j}(p) \in T_p^{10}\mathbf{C}^n \,\middle|\, \sum_{j=1}^n \eta_j \cdot \bar{\zeta}_j = 0 \;\forall \zeta \in T_p^{10}M \right\},$$

so there is $c_1 > 0$ and an open neighborhood $V_1 = V_1(p) \Subset U$ of $p$ such that

$$(1) \qquad \boxed{\forall x \in V_1:\; \mathscr{L}_\rho(x)(\eta, \eta) \geqq c_1 \cdot \|\eta\|^2 \;\forall \eta \in (T_x^{10}M_x)^\perp.}$$

In view of Lemma 3.1.4, one can choose an $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k \setminus \{0\}$, a function $\rho_\alpha := \sum_{i=1}^k \alpha_i \cdot \rho_i$ and a $b(p)$-dimensional subspace $L_p \subset T_p^{10}M$ so that $\mathscr{L}_{\rho_\alpha}(p)_{|L_p}$ is positive definite.

Now let $\sigma_1, \ldots, \sigma_{b(p)}$ be sections of $\mathscr{H}$ such that $\{\sigma_1(p), \ldots, \sigma_{b(p)}(p)\}$ is a basis for $L_p$; then there is an open $p$-neighborhood $V_2$ and a constant $c_2 > 0$ such that $L_x := \operatorname{span}_\mathbf{C}\{\sigma_1(x), \ldots, \sigma_{b(p)}(x)\}$ is a $b(p)$-dimensional linear subspace of $T_x^{10}M_x$ for each $x \in V_2$ and that

$$(2) \qquad \boxed{\mathscr{L}_{\rho_\alpha}(x)(\zeta, \zeta) \geqq c_2 \cdot \|\zeta\|^2 \;\forall x \in V_2,\; \forall \zeta \in L_x}$$

is satisfied.

For $\nu \in \mathbf{N}$, let $B_\nu := B\left(p, \frac{1}{\nu}\right) \subset \mathbf{C}^n$ be the ball with radius $\frac{1}{\nu}$ and center $p$, and $\tilde{B}_\nu := B_\nu \cap V_1 \cap V_2 \cap U$. Choose an increasing unbounded sequence $(R_\nu)_{\nu \in \mathbf{N}} \subset \mathbf{R}^+$, such that $A_\nu := \tilde{B}_\nu \cap \left\{ x \in U \,\middle|\, \rho_i(x) = \frac{\alpha_i}{R_\nu} \;\forall i = 1, \ldots, k \right\}$ is non-empty for all $\nu \in \mathbf{N}$, and assign one element $x_\nu \in A_\nu$ to each $\nu \in \mathbf{N}$. Then $(x_\nu)_{\nu \in \mathbf{N}} \subset U \setminus M$ converges to $p$ and, for each $\nu \in \mathbf{N}$, $P_{x_\nu} := (T_{x_\nu}^{10}M_{x_\nu})^\perp \oplus L_{x_\nu} \subset T_{x_\nu}^{10}\mathbf{C}^n$ is a linear subspace of complex dimension $d + b(p)$ where we can estimate the Levi form of $\rho$ as follows:

If $\xi := \zeta + \eta \in P_{x_\nu} \setminus \{0\}$, $\eta \in (T_{x_\nu}^{10}M_{x_\nu})^\perp$, $\zeta \in L_{x_\nu}$, then in view of (1) and (2),

$$\mathscr{L}_\rho(x_\nu)(\xi, \xi) \geqq c_1 \cdot \|\eta\|^2 - \frac{4}{R_\nu} \cdot c_3 \cdot \|\eta\| \cdot \|\zeta\| + \frac{2}{R_\nu} \cdot c_2 \cdot \|\zeta\|^2$$

holds for $c_3 := \sup\{|\mathscr{L}_{\rho_\alpha}(x)(u, v)| \,|\, x \in \overline{V_1 \cap V_2}, u, v \in T_x^{10}\mathbf{C}^n, \|u\| = \|v\| = 1\}$.

If now $\zeta = 0$, we know that $\eta \neq 0$, and we obtain: $\mathscr{L}_\rho(x_\nu)(\xi, \xi) \geqq c_1 \cdot \|\eta\|^2 > 0$.

In case $\zeta \neq 0$, we can go on estimating

$$\begin{aligned} \mathscr{L}_\rho(x_\nu)(\xi, \xi) &\geqq c_1 \cdot \left( \|\eta\| - \frac{2c_3}{R_\nu c_1} \|\zeta\| \right)^2 + \|\zeta\|^2 \cdot \left( \frac{2c_2}{R_\nu} - \frac{4c_3^2}{R_\nu^2 c_1} \right) \\ &\geqq \|\zeta\|^2 \cdot \frac{2}{R_\nu} \cdot \left( c_2 - \frac{4c_3^2}{c_1 R_\nu} \right) \end{aligned}$$

which is positive for all $\nu \geqq \nu_0$, $\nu_0 \in \mathbf{N}$ being chosen so that $c_2 > \frac{1}{R_{\nu_0}} \cdot \frac{4c_3^2}{c_1}$ holds.

Hence, $(x_\nu)_{\nu \geqq \nu_0} \subset U \setminus M$ is a sequence with limit $p$ such that $\mathscr{L}_\rho(x_\nu)_{|P_{x_\nu}}$ is positive definite at each $x_\nu$, $\nu > \nu_0$, and the proof of Theorem 3.2.1 is complete. □

Theorem 2.1 and Theorem 3.2.1 yield the following lower estimate for $\nu(p; M)$:

**3.2.2. Theorem.** *Let $M \subset \mathbf{C}^n$ be a generic* CR-*submanifold, $d := \operatorname{corank}_{\mathbf{C}} T^{10} M$, $p \in M$ and $\hat{b}(p)$ the local index of convexity of $M$ at $p$. Then, $\nu(p; M) \geqq d + \hat{b}(p)$, i.e. $\mathbf{C}^n \setminus M$ is not locally $q$-complete at $p$ for $q = d + \hat{b}(p) - 1$.*

*Proof.* By contradiction:

Suppose $\mathbf{C}^n \setminus M$ is locally $q$-complete at $p$ for $q := d + \hat{b}(p) - 1$. Then, by Theorem 2.1, there is an open $p$-neighborhood $V = V(p) \subset \mathbf{C}^n$ so that the function $-\log d_M^2$ is at least weakly $q$-convex in $V \setminus M$. Now choose $\tilde{U} = \tilde{U}(p) \Subset V$ and defining functions $(\rho_1, \ldots, \rho_k)_{\tilde{U}}$ of $M$ on $\tilde{U}$ so that $d^2_{M|\tilde{U}} = \sum_{i=1}^{k} \rho_i^2 =: \rho$ holds, which is clearly possible if $\tilde{U}$ is chosen so small that the real normal bundle of $M$ has an orthonormal frame on $\tilde{U}$. According to Lemma 3.1.2(2), there is $p_0 \in \tilde{U} \cap M$ with $\hat{b}(p) = b(p_0)$, and from Theorem 3.2.1 we get a sequence $(x_\nu)_{\nu \in \mathbf{N}} \subset \tilde{U} \cap M$ converging to $p_0$ such that $\mathscr{L}_\rho(x_\nu) = \mathscr{L}_{d_M^2}(x_\nu)$ has $d + b(p_0) = d + \hat{b}(p)$ positive eigenvalues at each $x_\nu$. So, for $x := x_1 \in \tilde{U} \setminus M \subset V \setminus M$, there is a complex linear subspace $L_x \subset T_x^{10} \mathbf{C}^n$, $\dim_{\mathbf{C}} L_x = d + \hat{b}(p)$, where $\mathscr{L}_{d_M^2}(x)_{|L_x}$ is positive definite. If now $\Omega_x$ denotes the real hypersurface $\Omega_x := \{z \in \tilde{U} \mid d_M^2(z) = d_M^2(x)\}$ and if $\tilde{L}_x := L_x \cap T_x^{10} \Omega_x$, then $\dim_{\mathbf{C}} \tilde{L}_x \geqq d + \hat{b}(p) - 1 = q$, and we see that

$$\mathscr{L}_{-\log d_M^2}(x)_{|\tilde{L}_x}(\zeta, \zeta) = -\frac{1}{d_M^2(x)} \cdot \mathscr{L}_{d_M^2}(x)_{|\tilde{L}_x}(\zeta, \zeta) \; \forall \zeta \in T_x^{10} \Omega_x.$$

As this is strictly negative definite on $\tilde{L}_x$, $\mathscr{L}_{-\log(d_M^2)}(x)$ can have no more than $n - \dim_{\mathbf{C}} \tilde{L}_x = n - q$ non-negative eigenvalues – in view of Theorem 2.1, this contradicts the fact that $-\log d_M^2$ is weakly $q$-convex in $x$. □

**3.2.3. Corollary.** *If $M \subset \mathbf{C}^n$ is a totally real smooth submanifold, then $\mathbf{C}^n \setminus M$ is not locally $(n-1)$-complete.*

### *3.3 A negative result for non-generic* CR-*submanifolds*

If $M$ is non-generic, the dimensions of the holomorphic tangent spaces of the manifolds $M_x$ introduced in the previous section may possibly be smaller than $\operatorname{rank}_{\mathbf{C}} T^{10} M$, and one can no longer identify the Levi form of $\rho$ with the Levi form of a suitable function $\rho_\alpha$ in directions tangential to $M_x$. However, the extra terms that occur can be controlled, and one obtains an analogue of Theorem 3.2.1:

**3.3.1. Theorem.** *Let $M \subset \mathbf{C}^n$ be a non-generic* CR-*submanifold of real dimension $\dim_{\mathbf{R}} M = m$, $k := 2n - m$ its real codimension, $d := \operatorname{corank}_{\mathbf{C}} T^{10} M$, $p \in M$ and let $b(p)$ be the index of convexity of $M$ at $p$. Then, for any choice of defining functions $(\rho_1, \ldots, \rho_k)_U$ for $M$ near $p$, there exists a sequence $(x_\nu)_\nu \subset U \setminus M$ converging to* *p such that the Levi form of* $\rho := \sum_{i=1}^{k} \rho_i^2$ *has* $d + b(p)$ *positive eigenvalues at each point* $x_\nu$.

*Proof.* Let $(\rho_1, \ldots, \rho_k)_U$ be defining functions of $M$ on the open $p$-neighborhood $U \subset \mathbf{C}^n$, and $\rho := \sum_{i=1}^{k} \rho_i^2$. As $M$ is non-generic, $d < \min\{k, n\}$, and we can assume that

$$\operatorname{rank}\left(\frac{\partial \rho_i(x)}{\partial z_j}\right)_{1 \leq j \leq n;\, 1 \leq i \leq d} = d \ \forall x \in \tilde{U}$$

holds on a sufficiently small neighborhood $\tilde{U} \Subset U$ of $p$. Now define, for $x \in \tilde{U}$,

$$\hat{M}_x := \{z \in \tilde{U} \mid \rho_1(z) = \rho_1(x), \ldots, \rho_d(z) = \rho_d(x)\} \subset \tilde{U},$$

and put $\hat{M} := \{z \in \tilde{U} \mid \rho_1(z) = \ldots = \rho_d(z) = 0\}$. Then $\hat{M}_x = \hat{M} \ \forall x \in \tilde{U} \cap M$, and for all $x \in \tilde{U}$, $\hat{M}_x$ is generic of type $(2n-d, n-d)$, containing $M_x = \{z \in \tilde{U} \mid \rho_i(z) = \rho_i(x) \ \forall i = 1, \ldots, k\}$ and satisfying $T_{\tilde{p}}^{10} M = T_{\tilde{p}}^{10} \hat{M} \ \forall \tilde{p} \in \tilde{U} \cap M$. Arguing with the function $\hat{\rho} := \sum_{i=1}^{d} \rho_i^2 \in C^\infty(\tilde{U})$ as we did with $\rho$ in the proof of Theorem 3.2.1, we find a $p$-neighborhood $\tilde{V}_1 = \tilde{V}_1(p) \Subset \tilde{U}$ and a constant $c_1 > 0$ such that

$$(1') \quad \boxed{\forall x \in V_1 : \ \mathscr{L}_{\hat{\rho}}(x)(\eta, \eta) + \sum_{i=d+1}^{k} \rho_i(x) \cdot \mathscr{L}_{\rho_i}(x)(\eta, \eta) \geqq c_1 \cdot \|\eta\|^2 \ \forall \eta \in (T_x^{10} \hat{M}_x)^\perp}$$

holds.

As in the proof of Theorem 3.2.1, we choose $\alpha \in \mathbf{R}^k \setminus \{0\}$ such that the Levi form of $\rho_\alpha := \sum_{i=1}^{k} \alpha_i \cdot \rho_i$ is positive definite on a $b(p)$-dimensional complex linear subspace $L_p \subset T_p^{10} M = T_p^{10} \hat{M}$, and smooth sections $\sigma_1, \ldots, \sigma_{b(p)}$ of the bundle

$$\mathscr{H} := \dot{\bigcup_{x \in \tilde{U}}} \{x\} \times T_x^{10} \hat{M}_x, \qquad \operatorname{rank}_{\mathbf{C}} \mathscr{H} = n - d,$$

such that $L_p = \operatorname{span}_{\mathbf{C}}\{\sigma_1(p), \ldots, \sigma_{b(p)}(p)\}$. Again, there is an open neighborhood $V_2 = V_2(p) \Subset \tilde{U}$ and a constant $c_2 > 0$ so that, for each $x \in V_2$, $L_x := \operatorname{span}_{\mathbf{C}}\{\sigma_1(x), \ldots, \sigma_{b(p)}(x)\} \subset T_x^{10} \hat{M}_x$ is $b(p)$-dimensional and

$$(2') \quad \boxed{\forall x \in V_2 : \ \mathscr{L}_{\rho_\alpha}(x)(\zeta, \zeta) \geqq c_2 \cdot \|\zeta\|^2 \ \forall \zeta \in L_x}$$

is satisfied.

Now, for $x \in V_1 \cap V_2$ and $\xi := \zeta + \eta \in L_x \oplus^{\perp} (T_x^{10} \hat{M}_x)^{\perp}$,

$$\begin{aligned}\mathscr{L}_\rho(x)(\xi,\xi) &= \mathscr{L}_{\hat{\rho}}(x)(\eta,\eta) + 2\sum_{i=d+1}^{k} \rho_i(x)\cdot \mathscr{L}_{\rho_i}(x)(\eta,\eta) + 4\,\mathrm{Re}\sum_{i=1}^{k} \rho_i(x)\cdot \mathscr{L}_{\rho_i}(x)(\zeta,\eta)\\ &\quad + 2\sum_{i=1}^{k} \rho_i(x)\cdot \mathscr{L}_{\rho_i}(x)(\zeta,\zeta) + 2\sum_{i=d+1}^{k} \left|\sum_{\nu=1}^{n} \frac{\partial \rho_i(x)}{\partial z_\nu}\cdot(\zeta_\nu + \eta_\nu)\right|^2\\ &\geqq c_1\cdot \|\eta\|^2 + 4\,\mathrm{Re}\sum_{i=1}^{k} \rho_i(x)\cdot \mathscr{L}_{\rho_i}(x)(\zeta,\eta) + 2\sum_{i=1}^{k} \rho_i(x)\cdot \mathscr{L}_{\rho_i}(x)(\zeta,\zeta)\end{aligned}$$

in view of (1′). The right-hand side of the above estimate is exactly the same sum of terms which was estimated in the proof of Theorem 3.2.1 after the consideration of (1); repeating these arguments, we conclude the proof of Theorem 3.3.1. □

Obviously, applying Theorem 2.1, one gets a result for non-generic CR-submanifolds which is completely analogous to Theorem 3.2.2:

**3.3.2. Theorem.** *Let $M \subset \mathbf{C}^n$ be a non-generic* CR*-submanifold, $d := \mathrm{corank}_{\mathbf{C}}\, T^{10} M$, $p \in M$ and $\hat{b}(p)$ the local index of convexity of $M$ at $p$. Then, $\nu(p; M) \geqq d + \hat{b}(p)$, i.e. $\mathbf{C}^n \setminus M$ is not locally $q$-complete at $p$ for $q = d + \hat{b}(p) - 1$.*

## 4 Positive results on the local $q$-completeness of $\mathbf{C}^n \setminus M$

We will first generalize the local version of Barth's [2] results (which was mentioned in the introduction of this paper) to complex foliated CR-submanifolds, using methods of Peternell [9].

### *4.1 Complex submanifolds and complex-foliated* CR*-submanifolds $M \subset \mathbf{C}^n$*

If

$$d^F\colon \mathbf{C}^n \times \mathbf{C}^n \to [0,1], \qquad d^F(x,y) := 1 - \frac{\left|1 + \sum_{i=1}^{n} x_i \cdot \bar{y}_i\right|^2}{\left(1 + \sum_{i=1}^{n} |x_i|^2\right)\cdot\left(1 + \sum_{i=1}^{n} |y_i|^2\right)},$$

then $h^F := -\log(d^F) \in C^\infty((\mathbf{C}^n \times \mathbf{C}^n)\setminus \Delta)$ (where $\Delta$ again denotes the diagonal in $\mathbf{C}^n \times \mathbf{C}^n$), and $h^F$ is $n$-convex near but outside of $\Delta$ (see [9]). The Fubini distance of points in $\mathbf{C}^n$ to a smooth submanifold $M \subset \mathbf{C}^n$ is given by $d_M^F$: $x \mapsto \inf\{d^F(x,y) \mid y \in M\}$, $x \in \mathbf{C}^n$, and there is a neighborhood $U$ of $M$ so that $d_M^F \in C^\infty(U \setminus M)$ and $d d_M^F(x) \neq 0\ \forall x \in U \setminus M$ (see [2]). From these facts we will derive:

**4.1.1. Theorem.** *Let $M \subset \mathbf{C}^n$ be a smooth submanifold, $p \in M$, $h^F$, $d_M^F$ as before. Let $K \subset \mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$ be a compact set so that $h^F$ is $n$-convex outside $K$, and $V = V(p) \subset \mathbf{C}^n$ a sufficiently small open neighborhood of $p$ so that $d_M^F \in C^\infty(V \setminus M)$*

*and* $(V\setminus M)\times(V\cap M)\subset(\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta)\setminus K$. *Let* $d^F_{M\cap V}$ *denote the Fubini distance of points in* $V$ *to* $M\cap V$, *and let* $x\in V\setminus M$, $y\in V\cap M$ *satisfy* $d^F_{M\cap V}(x)=d^F(x,y)$.

*If there exists an s-dimensional complex manifold* $A_y\subset M$ *passing through* $y\in A_y$, *then the function* $-\log(d^F_{M\cap V})$ *is* $(n-s)$*-convex at* $x$.

*Proof.* Given $x\in V\setminus M$ and $y\in V\cap M$, choose open neighborhoods $U=U(x)\Subset V\setminus M$ and $W=W(y)\subset\mathbf{C}^n$ such that $A_y\cap W\subset M\cap V$ and $U\times W\Subset(\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta)\setminus K$. Then, the function $h^F$: $U\times W\to\mathbf{R}$, $h^F:=-\log d^F$, is $n$-convex in $U\times W$, and so is $h^F_{|U\times A_y}$, its restriction to the complex manifold $U\times A_y$. As $\sup h^F_{|U\times A_y}<\infty$, the function

$$g:\ U\to\mathbf{R},\qquad g(u):=\sup\{-\log(d^F(u,v))\,|\,v\in A_y\}$$

has an $(n-s)$-convex support function at $x$ (see [9, Lemma 4]). But $g$ itself is differentiable at $x$, so $g$ is $(n-s)$-convex at $x$ and on a small neighborhood $\tilde{U}=\tilde{U}(x)\Subset U$ of $x$. At the same time, $g$ is a support function for $-\log(d^F_{M\cap V})$ at $x$; therefore, $-\log(d^F_{M\cap V})$ *must be* $(n-s)$-convex at $x$, and the proof of Theorem 4.1.1 is complete. □

An immediate consequence of Theorem 4.1.1 is:

**4.1.2. Corollary.** *Let* $M\subset\mathbf{C}^n$ *be a* CR*-submanifold and* $p\in M$. *If, near* $p$, $M$ *is foliated by s-dimensional complex manifolds, then* $\mathbf{C}^n\setminus M$ *is locally q-complete at p for* $q=n-s$.

*Remark.* In view of Theorem 3.2.2 and Theorem 3.3.2, the result in Corollary 4.1.2 is optimal in the case of a complex submanifold $M\subset\mathbf{C}^n$ as well as in the case of a maximally complex foliated CR-submanifold $M\subset\mathbf{C}^n$. The following examples show that the situation is different if $M$ is not maximally complex foliated:

*4.1.3. Examples.* For $n\geqq 4$, the real hypersurfaces

$$M_1:=\{z\in\mathbf{C}^n\,|\,|z_{n-2}|^2+|z_{n-1}|^2+|z_n|^2-1=:\rho_1(z)=0\},$$
$$M_2:=\{z\in\mathbf{C}^n\,|\,|z_{n-2}|^2-|z_{n-1}|^2+|z_n|^2-1=:\varphi_1(z)=0\}$$

are foliated by $(n-3)$-dimensional complex manifolds near $p:=(0,\ldots,0,1)\in M_1\cap M_2$, so by Corollary 4.1.2, $\mathbf{C}^n\setminus M_1$ and $\mathbf{C}^n\setminus M_2$ are locally 3-complete at $p$.

The local index of convexity of $M_1$ at $p$ is $\hat{b}_{M_1}(p)=2$, therefore by Theorem 3.2.2, $\nu(p;M_1)=3$. For $\nu(p;M_2)$ we only know $2\leqq\nu(p;M_2)\leqq 3$ as $\hat{b}_{M_2}(p)=1$. The following results will enable us to see that $\nu(p;M_2)=2$ is correct.

## *4.2 A positive result for generic* CR*-submanifolds* $M\subset\mathbf{C}^n$

In the case of a generic CR-submanifold $M\subset\mathbf{C}^n$, the lower bound for $\nu(p;M)$ given by Theorem 3.2.2 is an upper bound for $\nu(p;M)$ at the same time; we show:

**4.2.1. Theorem.** *Let* $M\subset\mathbf{C}^n$ *be a generic* CR*-submanifold,* $d:=\operatorname{corank}_{\mathbf{C}}T^{10}M$, $p\in M$, *and* $\hat{b}(p)$ *the local index of convexity of* $M$ *at* $p$. *Then,* $\mathbf{C}^n\setminus M$ *is locally q-complete at* $p$ *for* $q:=d+\hat{b}(p)$ *which means that* $\nu(p;M)\leqq d+\hat{b}(p)$. *Actually,* $\nu(p;M)=d+\hat{b}(p)$ *in view of Theorem* 3.2.2.

*Proof.* For $d=n$, the statement is trivial. For $d<n$, we have $d=\operatorname{codim}_{\mathbf{R}} M$ because $M$ is generic; the proof is done in two steps described by the following theorems:

**4.2.2. Theorem.** *Let $M \subset \mathbf{C}^n$ be a generic* CR-*submanifold of real codimension $k$, $p \in M$, $(\rho_1, \ldots, \rho_k)_{\tilde{U}}$ defining functions for $M$ on the open neighborhood $\tilde{U}=\tilde{U}(p)$ of $p$ in $\mathbf{C}^n$, and $\rho := \sum_{i=1}^{k} \rho_i^2$. Then, there is a ball $B=B(p) \Subset \tilde{U}$ centered at $p$ and a constant $c_2>0$ such that for all $x \in B \setminus M$:*

(1) *The level set $\Omega_x := \{z \in U \mid \rho(z)=\rho(x)\}$ of $\rho$ is a real hypersurface*

(2) $\forall \eta \in (T_x^{10}\, \Omega_x)^{\perp}$: $\mathscr{L}_{-\log\rho}(x)(\eta, \eta) \geqq \frac{c_2}{\rho(x)} \cdot \|\eta\|^2$.

**4.2.3. Theorem.** *Let $M \subset \mathbf{C}^n$ be a generic* CR-*submanifold of type $(m, \ell)$, $\ell=m-n>0$, $k := \operatorname{codim}_{\mathbf{R}} M$, $d := \operatorname{corank}_{\mathbf{C}} T^{10} M$, $p \in M$ and $\hat{b}(p)$ the local index of convexity of $M$ at $p$. If then $(\rho_1, \ldots, \rho_k)_U$ are defining functions of $M$ over $U=U(p) \subset \mathbf{C}^n$ and if $\rho := \sum_{i=1}^{k} \rho_i^2$, it follows:*

(1) *There is a $p$-neighborhood $\tilde{U} \Subset U$ so that $M_x := \{z \in \tilde{U} \mid \rho_i(z)=\rho_i(x)\ \forall i =1, \ldots, k\} \subset \tilde{U}$ is a generic* CR-*submanifold of type $(m, \ell)$ for each $x \in \tilde{U}$.*

(2) *There is a ball $B=B(p) \Subset \tilde{U}$ centered at $p$ and a constant $c_1>0$ such that, at each point $x \in B \setminus M$, there is a complex linear subspace $W_x \subset T_x^{10} M_x$, $\dim_{\mathbf{C}} W_x \geqq \ell - \hat{b}(p)$, such that*

$$\mathscr{L}_{-\log\rho}(x)_{|W_x}(\zeta, \zeta) \geqq -c_1 \cdot \|\zeta\|^2.$$

Theorem 4.2.2 and Theorem 4.2.3 imply Theorem 4.2.1 as follows:

Given defining functions $(\rho_1, \ldots, \rho_k)_{\tilde{U}}$ for $M$ over the open $p$-neighborhood $\tilde{U}=\tilde{U}(p)$, choose a ball $B=B(p) \Subset \tilde{U}$ which is so small that the statements of Theorem 4.2.2 and of Theorem 4.2.3 hold simultaneously. Then,

$$\mathscr{L}_{-\log\rho}(x)(\xi, \xi) = \mathscr{L}_{-\log\rho}(x)(\zeta, \zeta) + \mathscr{L}_{-\log\rho}(x)(\eta, \eta) + 2 \operatorname{Re} \mathscr{L}_{-\log\rho}(x)(\zeta, \eta)$$

for $x \in B \setminus M$ and for $\xi := \zeta + \eta \in W_x \oplus^{\perp} (T_x^{10}\, \Omega_x)^{\perp}$. Estimating the first two terms on the right-hand side by Theorem 4.2.2 and by Theorem 4.2.3 and observing that

$$|\mathscr{L}_{-\log\rho}(x)(\zeta, \eta)| \leqq \frac{c_3}{\sqrt{\rho(x)}} \cdot \|\zeta\| \cdot \|\eta\|$$

for a suitable constant $c_3>0$, we obtain

$$\begin{aligned} \mathscr{L}_{-\log\rho}(x)(\xi, \xi) &\geqq -c_1 \cdot \|\zeta\|^2 + c_2 \cdot \frac{1}{\rho(x)} \cdot \|\eta\|^2 - 2c_3 \cdot \frac{1}{\sqrt{\rho(x)}} \cdot \|\zeta\| \cdot \|\eta\| \\ &\geqq -\left(c_1 + \frac{c_3^2}{c_2}\right) \cdot \|\zeta\|^2 \geqq : -c\, \|\xi\|^2. \end{aligned}$$

If now $\varphi \in C^{\infty}(B)$ is a strictly plurisubharmonic exhaustion function of the ball $B$, then the function

$$\psi\colon B \setminus M \to \mathbf{R}, \qquad \psi(x) := -\log\rho(x) + 2c\, \|x\|^2 + \varphi(x),$$

is a smooth $(d+\hat{b}(p))$-convex exhaustion function of $B\setminus M$.

So, it suffices to prove Theorem 4.2.2 and Theorem 4.2.3.

*Proof of Theorem 4.2.2* Given defining functions $(\rho_1, \dots, \rho_k)_{\tilde{U}}$ of $M$ over $\tilde{U} = \tilde{U}(p) \subset \mathbf{C}^n$, the vectors $\partial\rho_1(p), \dots, \partial\rho_k(p)$ are linearly independent in the holomorphic cotangent space $(T_p^{10}\,\mathbf{C}^n)^*$ of $\mathbf{C}^n$ at $p$ as $M$ is generic. Let the coordinates of $\mathbf{C}^n$ be chosen such that at $p$,

$$\langle \partial\rho_i(p), \partial\rho_j(p)\rangle = \delta_{ij}, \qquad 1 \leqq i, j \leqq k,$$

where $\langle\ ,\ \rangle$ denotes the unitary product which is induced on $(T_p^{10}\,\mathbf{C}^n)^*$ by the euclidean metric. If now $V = V(p) \subset \tilde{U}$ is so small that $\partial\rho_1(x), \dots, \partial\rho_k(x)$ are linearly independent at each $x \in V$, then $\partial\rho(x) \neq 0\ \forall x \in V\setminus M$, so $\Omega_x$ is a real hypersurface. A basis for $(T_x^{10}\,\Omega_x)^\perp := \{\eta \in T_x^{10}\,\mathbf{C}^n \mid \langle \eta, \zeta\rangle = 0\ \forall \zeta \in T_x^{10}\,\Omega_x\}$ is the vector $\eta(x) := \sum_{\nu=1}^{n} \frac{\partial\rho(x)}{\partial\bar{z}_\nu}\frac{\partial}{\partial z_\nu}(x)$, $x \in V\setminus M$, and the square of the euclidean length of $\eta(x)$ is just

$$\text{(1)} \qquad \|\eta(x)\|^2 = \sum_{\nu=1}^{n} \left|\frac{\partial\rho(x)}{\partial z_\nu}\right|^2 = 4 \sum_{i,\,j=1}^{k} \rho_i(x)\,\rho_j(x)\langle \partial\rho_i(x), \partial\rho_j(x)\rangle.$$

By continuity, for every $\varepsilon > 0$ there is an open neighborhood $U_\varepsilon = U_\varepsilon(p) \subset V$ of $p$ such that

$$\text{(2)} \qquad \forall x \in U_\varepsilon:\ |\langle \partial\rho_i(x), \partial\rho_j(x)\rangle - \delta_{ij}| < \varepsilon$$

holds; for the moment, let us fix an $\varepsilon_0 \ll 1$ and an accompanying $U_{\varepsilon_0}$. If $x \in U_{\varepsilon_0}\setminus M$ and $\eta := \eta(x) \in (T_x^{10}\,\Omega_x)^\perp$, we obtain:

$$\text{(3)} \qquad \mathcal{L}_{-\log\rho}(x)(\eta, \eta) \geqq \frac{\|\eta\|^2}{\rho(x)}\left(\frac{\|\eta\|^2}{\rho(x)} - \frac{8}{\|\eta\|^2}\sum_{i=1}^{k}\left|\sum_{j=1}^{k} \rho_j(x)\langle \partial\rho_i(x), \partial\rho_j(x)\rangle\right|^2\right) - \frac{2}{\rho(x)}\|\eta\|^2\sqrt{\rho(x)}\cdot c$$

where $c := \max\left\{\sum_{i=1}^{k} |\mathcal{L}_{\rho_i}(y)(\xi, \xi)| \mid y \in \overline{U_{\varepsilon_0}},\ \xi \in T_y^{10}\,\mathbf{C}^n,\ \|\xi\| = 1\right\}$. If we were now given a constant $\gamma > 0$ satisfying

$$\text{(4)} \qquad \left(\frac{\|\eta(y)\|^2}{\rho(y)} - \frac{8}{\|\eta(y)\|^2}\sum_{i=1}^{k}\left|\sum_{j=1}^{k} \rho_j(y)\langle \partial\rho_i(y), \partial\rho_j(y)\rangle\right|^2\right) \geqq \gamma > 0\ \forall y \in U_{\varepsilon_0}\setminus M,$$

then

$$\mathcal{L}_{-\log\rho}(x)(\eta, \eta) \geqq \frac{\|\eta\|^2}{\rho(x)^2}\cdot(\gamma - 2\cdot\sqrt{\rho(x)}\cdot c)$$

would follow from (3). If we then choose $c_2>0$ and $B=B(p)\Subset U_{\varepsilon_0}$ so small that $\forall x\in B: (\gamma-2\cdot\sqrt{\rho(x)\cdot c})\geqq c_2>0$, we get

$$\mathscr{L}_{-\log\rho}(x)(\eta,\eta)\geqq\frac{c_2}{\rho(x)}\cdot\|\eta\|^2\ \forall x\in B\setminus M,\ \forall\eta\in(T_x^{10}\,\Omega_x)^{\perp}$$

and the proof of Theorem 4.2.2 is complete.

Indeed, (4) holds if the above $\varepsilon_0$ is small enough:

For any $y\in U_{\varepsilon_0}$, $\|\eta(y)\|^2=4\sum_{i,j=1}^{k}\rho_i(y)\cdot\rho_j(y)\cdot\langle\partial\rho_i(y),\partial\rho_j(y)\rangle$ according to (1), which would equal $4\rho(y)$ if the vectors $\partial\rho_i(y)$, $1\leqq i\leqq k$, were pairwise orthonormal at $y$. But the vectors $\partial\rho_i(p)$, $1\leqq i\leqq k$, are pairwise orthonormal; so if $\varepsilon_0$ is appropriately small, we can always achieve $\|\eta(y)\|^2\geqq\frac{7}{2}\rho(y)\ \forall y\in U_{\varepsilon_0}$. By similar arguments we can see that

$$\sum_{i=1}^{k}\left|\sum_{j=1}^{k}\rho_j(y)\langle\partial\rho_i(y),\partial\rho_j(y)\rangle\right|^2\leqq\tfrac{3}{2}\rho(y)\ \forall y\in U_{\varepsilon_0}$$

if $\varepsilon_0$ is small enough, so we get

$$\left(\frac{\|\eta(y)\|^2}{\rho(y)}-\frac{8}{\|\eta(y)\|^2}\cdot\sum_{i=1}^{k}\left|\sum_{j=1}^{k}\rho_j(y)\langle\partial\rho_i(y),\partial\rho_j(y)\rangle\right|^2\right)\geqq\left(\frac{7}{2}-\frac{16}{7\rho(y)}\cdot\frac{3}{2}\rho(y)\right)=:\gamma>0$$

for all $y\in U_{\varepsilon_0}\setminus M$. The proof is complete. □

*Proof of Theorem 4.2.3* (1) follows immediately from the fact that the matrix $\left(\frac{\partial\rho_i}{\partial z_j}\right)_{i,j}$ has maximal rank $k$ at $p$. If $(\rho_1,\dots,\rho_k)_{\tilde U}$ are defining functions for $M$ on $\tilde U=\tilde U(p)\subset\mathbf{C}^n$, $\partial\rho_1\wedge\dots\wedge\partial\rho_{k|\tilde U}\neq0$, and $\rho:=\sum_{j=1}^{k}\rho_j^2$, we can locally describe the euclidean distance $d_M$ to $M$ in terms of the functions $\rho_i$, so there is a small neighborhood $V=V(p)\Subset\tilde U$ of $p$ and a constant $\tilde c>0$ so that on $V$, we can control $d_M$ by the estimate

$$(1)\qquad d_M^2\leqq\tilde c\cdot\rho.$$

Now choose a ball $B=B(p)\Subset V$ so small that $\hat b(p)=\max\{b(\tilde p)\,|\,\tilde p\in M\cap\bar B\}$ holds (see Lemma 3.1.2(1)) and so that the vector bundle $\mathscr{H}:=\bigcup_{x\in\tilde U}\{x\}\times T_x^{10}M_x$ has an orthonormal frame $\{\zeta_1,\dots,\zeta_\ell\}$ over $B$. Then, for any $x\in B\setminus M$ and for every $\zeta\in T_x^{10}M_x$,

$$(2)\qquad \mathscr{L}_{-\log\rho}(x)_{|T_x^{10}M_x}(\zeta,\zeta)=\frac{-2}{\sqrt{\rho(x)}}\cdot\mathscr{L}_{\rho_\alpha}(x)_{|T_x^{10}M_x}(\zeta,\zeta)$$

holds for $\rho_\alpha := \sum_{i=1}^{k} \alpha_i \rho_i$, $\alpha_i := \frac{\rho_i(x)}{\sqrt{\rho(x)}}$, $1 \leqq i \leqq k$, $\alpha = (\alpha_1, \ldots, a_k) \in \mathbf{R}^k$, $\|\alpha\| = 1$. In order to estimate the Levi form of $\rho_\alpha$, fix a point $\tilde{p} \in M$ with $d_M(x) = \|x - \tilde{p}\|$. By definition of $\hat{b}(p)$ and by choice of $B$, $\mathscr{L}_{\rho_\alpha}(\tilde{p})_{|T_{\tilde{p}}^{10}M}$ has at least $\ell - \hat{b}(p)$ non-positive eigenvalues which means that there are linear subspaces $W_{\tilde{p}}^0$, $W_{\tilde{p}}^- \subset T_{\tilde{p}}^{10}M$ with the following properties:

(a) $\mathscr{L}_{\rho_\alpha}(\tilde{p})_{|W_{\tilde{p}}^-}$ is negative definite

(b) $\mathscr{L}_{\rho_\alpha}(\tilde{p})(\zeta, \eta) = 0\ \forall \zeta \in W_{\tilde{p}}^0, \eta \in T_{\tilde{p}}^{10}M$

(c) $W_{\tilde{p}}^0 \perp W_{\tilde{p}}^-$

(d) $\dim_{\mathbf{C}}(W_{\tilde{p}}^0 \oplus W_{\tilde{p}}^-) \geqq \ell - \hat{b}(p)$.

Now define

$$V^0 := \left\{\tau \in \mathbf{C}^\ell \,\middle|\, \sum_{i=1}^{\ell} \tau_i \cdot \zeta_i(\tilde{p}) \in W_{\tilde{p}}^0\right\}; \qquad V^- := \left\{\sigma \in \mathbf{C}^\ell \,\middle|\, \sum_{j=1}^{\ell} \sigma_j \cdot \zeta_j(\tilde{p}) \in W_{\tilde{p}}^-\right\},$$

and put $W_x^0 := \left\{\zeta = \sum_{i=1}^{\ell} \tau_i \cdot \zeta_i(x) \,\middle|\, \tau \in V^0\right\}$, $W_x^- := \left\{\eta = \sum_{j=1}^{\ell} \sigma_j \cdot \zeta_j(x) \,\middle|\, \sigma \in V^-\right\}$. Then, $\dim_{\mathbf{C}} W_x^0 = \dim_{\mathbf{C}} W_{\tilde{p}}^0$, $\dim_{\mathbf{C}} W_x^- = \dim_{\mathbf{C}} W_{\tilde{p}}^-$, $W_x^0 \perp W_x^-$, and $W_x := W_x^0 \oplus W_x^- \subset T_x^{10} M_x$ is a subspace of dimension at least $\ell - \hat{b}(p)$. Take any $\xi(x) \in W_x^0$, $\eta(x) \in W_x$, $\|\xi(x)\| = \|\eta(x)\| = 1$. Then there are $\tau = (\tau_1, \ldots, \tau_\ell) \in V^0$, $\sigma = (\sigma_1, \ldots, \sigma_\ell) \in \mathbf{C}^\ell$, $\|\tau\| = \|\sigma\| = 1$, such that $\xi(x) = \sum_{i=1}^{\ell} \tau_i \cdot \zeta_i(x)$, $\eta(x) = \sum_{j=1}^{\ell} \sigma_j \cdot \zeta_j(x)$ hold. Putting $\xi(\tilde{p}) := \sum_{i=1}^{\ell} \tau_i \cdot \zeta_i(\tilde{p}) \in W_{\tilde{p}}^0$, according to (b), one obtains for $\eta(\tilde{p}) := \sum_{j=1}^{\ell} \sigma_j \cdot \zeta_j(\tilde{p})$ that $\mathscr{L}_{\rho_\alpha}(\tilde{p})(\xi(\tilde{p}), \eta(\tilde{p})) = 0$. Now define

$$f_{\alpha,\tau,\sigma}\colon B \to \mathbf{C}, \qquad f_{\alpha,\tau,\sigma}(z) := \mathscr{L}_{\rho_\alpha}(z)\left(\sum_{i=1}^{\ell} \tau_i \zeta_i(z), \sum_{j=1}^{\ell} \sigma_j \zeta_j(z)\right).$$

Looking at the power series expansion of the smooth function $\operatorname{Re} f_{\alpha,\tau,\sigma}$ about $\tilde{p}$, one obtains

$$(3) \quad |\operatorname{Re} \mathscr{L}_{\rho_\alpha}(x)(\xi(x), \eta(x))| = |\operatorname{Re} f_{\alpha,\tau,\sigma}(x)| \leqq \|x - \tilde{p}\| \cdot (\|\partial f_{\alpha,\tau,\sigma}(w)\| + \|\bar{\partial} f_{\alpha,\tau,\sigma}(w)\|)$$

for a suitable $w$ on the straight line between $\tilde{p}$ and $x$. The norm of $\partial f_{\alpha,\tau,\sigma}$, $\bar{\partial} f_{\alpha,\tau,\sigma}$ can be estimated uniformly on $B$, independent of the parameters $\alpha$, $\tau$, $\sigma$. In view of (1) and (3), we conclude:

$$(4) \quad |\operatorname{Re} \mathscr{L}_{\rho_\alpha}(x)(\xi(x), \eta(x))| \leqq \frac{c_1}{2} \sqrt{\rho(x)}\ \forall \xi(x) \in W_x^0, \eta(x) \in W_x, \|\xi(x)\| = \|\eta(x)\| = 1$$

where the constant $c_1 > 0$ does not depend on $\alpha$.

If, however, $\eta(x) \in W_x^-$, $\|\eta(x)\| = 1$, there is a unit vector $\sigma = (\sigma_1, \ldots, \sigma_\ell) \in V^-$ with $\eta(x) = \sum_{j=1}^{\ell} \sigma_j \cdot \zeta_j(x)$. By (a), we know that $\mathscr{L}_{\rho_\alpha}(\tilde{p})(\eta(\tilde{p}), \eta(\tilde{p})) < 0$ for $\eta(\tilde{p}) := \sum_{j=1}^{\ell} \sigma_j \cdot \zeta_j(\tilde{p}) \in W_{\tilde{p}}^-$.

*Case 1:* $\mathscr{L}_{\rho_\alpha}(x)(\eta(x), \eta(x)) \leqq 0$.

Then, in view of (2),

$$\mathscr{L}_{-\log\rho}(x)(\eta(x),\eta(x)) = -\frac{2}{\sqrt{\rho(x)}}\mathscr{L}_{\rho_\alpha}(x)(\eta(x),\eta(x)) \geqq 0 \geqq -c_1.$$

*Case 2:* $\mathscr{L}_{\rho_\alpha}(x)(\eta(x),\eta(x))>0$.

As $\mathscr{L}_{\rho_\alpha}(\tilde{p})(\eta(\tilde{p}),\eta(\tilde{p}))<0$, there must be a $w$ on the straight line between $\tilde{p}$ and $x$ with $\mathscr{L}_{\rho_\alpha}(w)(\eta(w),\eta(w))=0$. Just as in (4), we find that

$$|\mathscr{L}_{\rho_\alpha}(x)(\eta(x),\eta(x))| \leqq \frac{c_1}{2}\cdot\sqrt{\rho(x)}$$

which, in view of (2), again yields

$$\mathscr{L}_{-\log\rho}(x)(\eta(x),\eta(x)) = -\frac{2}{\sqrt{\rho(x)}}\mathscr{L}_{\rho_\alpha}(x)(\eta(x),\eta(x)) \geqq -c_1.$$

So, we always have

$$\mathscr{L}_{-\log\rho}(x)(\eta(x),\eta(x)) \geqq -c_1 \ \forall\, \eta(x)\in W_x^-, \qquad \|\eta(x)\|=1. \tag{5}$$

Combining the results above, we obtain the desired estimate for the Levi form of the function $-\log\rho$ on $W_x$: if $\zeta=\xi+\eta\in W_x=W_x^0\oplus^\perp W_x^-$, then

$$\mathscr{L}_{-\log\rho}(x)(\zeta,\zeta) \geqq -c_1(\|\xi\|^2+\|\eta\|^2) = -c_1\|\zeta\|^2$$

as $\xi$ and $\eta$ are orthogonal.

This completes the proof of Theorem 4.2.3 (and of Theorem 4.2.1 at the same time). □

### *4.3 A positive result in the general case of non-generic* CR*-submanifolds* $M\subset\mathbf{C}^n$

Each non-generic CR-submanifold $M\subset\mathbf{C}^n$ can locally be described as a transverse intersection of generic CR-submanifolds, the holomorphic tangent spaces of which coincide with the holomorphic tangent spaces of $M$. This is stated in the lemmata below:

**4.3.1. Lemma.** *Let $M\subset\mathbf{C}^n$ be a* CR*-submanifold of type $(m,\ell)$, non-generic, $p\in M$, $k:=\operatorname{codim}_{\mathbf{R}}M$ and $d:=\operatorname{corank}_{\mathbf{C}}T^{10}M$. Then there is an open $p$-neighborhood $U=U(p)\subset\mathbf{C}^n$ and defining functions $(\rho_1,\dots,\rho_k)_U$ of $M$ over $U$ such that for every subset $I\subset\{1,\dots,k\}$, $|I|=d$, the family $\{\partial\rho_i(p)\,|\,i\in I\}$ is complex linearly independent.*

*Proof.* If $(\tilde{\rho}_1,\dots,\tilde{\rho}_k)_U$ are defining functions for $M$ on an open neighborhood $U=U(p)$ of $p$ in $\mathbf{C}^n$ with

$$\operatorname{rank}\left(\frac{\partial\tilde{\rho}_i(p)}{\partial z_j}\right)_{1\leqq i\leqq d;\,1\leqq j\leqq n}=d,$$

a transformation of the form

$$\begin{pmatrix}\rho_1\\ \vdots\\ \rho_k\end{pmatrix} := \begin{pmatrix}E_d & O\\ A & \varepsilon E_{k-d}\end{pmatrix}\cdot\begin{pmatrix}\tilde\rho_1\\ \vdots\\ \tilde\rho_k\end{pmatrix}$$

where $E_d$ or $E_{k-d}$ denote the identity matrices with $d$ or $(k-d)$ rows and columns, will yield defining functions $(\rho_1, \dots, \rho_k)_U$ of $M$ over $U$ with the desired properties; for this purpose, $A$ has to be a real $(k-d)\times d$-matrix with rank $A=k-d$ and $a_{ij}\neq 0\ \forall i=1, \dots, k-d, \forall j=1, \dots, d$, and $\varepsilon>0$ has to be sufficiently small – then, in view of the Steinitz exchange principle, any subfamily of length $d$ of $\{\partial\rho_1(p), \dots, \partial\rho_k(p)\}$ is complex linearly independent. □

The following is now obvious:

**4.3.2. Lemma.** *Let $M\subset\mathbf{C}^n$ be a* CR*-submanifold of type $(m,\ell)$, non-generic, $k:=\operatorname{codim}_{\mathbf{R}} M$, $d:=\operatorname{corank}_{\mathbf{C}} T^{10}M$, and $p\in M$. Then there is an open neighborhood $U=U(p)$ of $p$ in $\mathbf{C}^n$ and defining functions $(\rho_1, \dots, \rho_k)_U$ of $M$ over $U$ such that:*

*(1) For any $I\subset\{1, \dots, k\}$, $|I|=d$, and for each $x\in U$, $M_x^{(I)}:=\{z\in U\mid \rho_i(z)=\rho_i(x)\ \forall i\in I\}$ is a generic* CR*-submanifold of type $(2n-d,\ell)$.*

*(2) If for arbitrary $I\subset\{1, \dots, k\}$, $|I|=d$, one defines $M^{(I)}:=\{z\in U\mid \rho_i(z)=0\ \forall i\in I\}$, then $M^{(I)}=M_{\tilde p}^{(I)}$ and $T_{\tilde p}^{10}M=T_{\tilde p}^{10}M^{(I)}$ for every $\tilde p\in U\cap M$.*

Now we are ready to prove a counterpart of Theorem 4.2.3 in the category of non-generic CR-submanifolds $M\subset\mathbf{C}^n$:

**4.3.3. Theorem.** *Let $M\subset\mathbf{C}^n$ be a* CR*-submanifold of type $(m,\ell)$, non-generic, $k:=\operatorname{codim}_{\mathbf{R}} M$, $d:=\operatorname{corank}_{\mathbf{C}} T^{10}M$, $p\in M$, and $\hat b(p)$ the local index of convexity of $M$ at $p$. If an open $p$-neighborhood $U=U(p)\subset\mathbf{C}^n$ and defining functions $(\rho_1, \dots, \rho_k)_U$ for $M$ on $U$ are chosen with the properties described in Lemma* 4.3.2 *and if $\rho:=\sum_{i=1}^{k}\rho_i^2\in C^\infty(U)$, then with the notations of Lemma* 4.3.2, *the following holds:*

*There is a ball $B=B(p)\Subset U$ centered at $p$ and a constant $c_1>0$ such that at each $x\in B\setminus M$ and for every $I\subset\{1, \dots, k\}$, $|I|=d$, there exists a complex linear subspace $W_x^{(I)}\subset T_x^{10}M_x^{(I)}$ of dimension $\dim_{\mathbf{C}} W_x^{(I)}\geqq\ell-\hat b(p)$ such that*

$$\mathscr{L}_{-\log\rho}(x)_{|W_x^{(I)}}(\xi,\xi)\geqq -c_1\cdot\|\xi\|^2.$$

*Proof.* Let $I\subset\{1, \dots, k\}$, $|I|=d$, be fixed and put $I':=\{1, \dots, k\}\setminus I$. As in the proof of Theorem 4.2.3, choose a ball $B^{(I)}=B^{(I)}(p)$ centered at $p$ so small that

(1) there is a constant $\tilde c>0$ with $d_M^2(x)\leqq\tilde c\cdot\rho(x)\ \forall x\in\bar B^{(I)}$,

(2) the vector bundle $\mathscr{H}:=\bigcup_{x\in U}\{x\}\times T_x^{10}M_x^{(I)}$ has an orthonormal frame $\{(\zeta_1, \dots, \zeta_\ell\}$ on $B^{(I)}$,

(3) $\hat b(p)=\max\{b(\tilde p)\mid\tilde p\in M\cap\bar B^{(I)}\}$

hold, where $d_M$ again denotes the euclidean distance to $M$. Now, if $x\in B^{(I)}\setminus M$, $\xi\in T_x^{10}M_x^{(I)}$, and if $\rho_\alpha:=\sum_{i=1}^{k}\alpha_i\cdot\rho_i$, $\alpha_i:=\dfrac{\rho_i(x)}{\sqrt{\rho(x)}}\ \forall 1\leqq i\leqq k$, $\alpha=(\alpha_1, \dots, \alpha_k)\in\mathbf{R}^k$, $\|\alpha\|=1$, we obtain

$$\mathscr{L}_{-\log\rho}(x)(\xi,\xi)\geqq -\frac{2}{\rho(x)}\cdot\sum_{i\in I'}\left|\sum_{\mu=1}^{n}\frac{\partial\rho_i(x)}{\partial z_\mu}\cdot\xi_\mu\right|^2-\frac{2}{\sqrt{\rho(x)}}\cdot\mathscr{L}_{\rho_\alpha}(x)(\xi,\xi).$$

As has been shown in the proof of Theorem 4.2.3, there is a complex linear subspace $W_x^{(I)} \subset T_x^{10} M_x^{(I)}$ of dimension at least $\ell - \hat{b}(p)$ and a constant $c_1^{(I)} > 0$ (independent of $x$) such that

$$-\frac{2}{\sqrt{\rho(x)}} \cdot \mathscr{L}_{\rho_\alpha}(x)_{|W_x^{(I)}}(\xi, \xi) \geqq -c_1^{(I)} \cdot \|\xi\|^2$$

holds; the other term satisfies a similar inequality on the whole of $T_x^{10} M_x^{(I)}$, namely

$$-\frac{2}{\rho(x)} \cdot \sum_{i \in I'} \left| \sum_{\mu=1}^{n} \frac{\partial \rho_i(x)}{\partial z_\mu} \cdot \xi_\mu \right|^2 \geqq -c_2^{(I)} \cdot \|\xi\|^2 \qquad \forall \xi \in T_x^{10} M_x^{(I)}$$

where the constant $c_2^{(I)} > 0$ does not depend on $x$ and is determined by the maxima of the second order partial derivatives of the function

$$g\colon\; x \mapsto \sum_{i \in I'} \sum_{\nu=1}^{\ell} \left| \sum_{\mu=1}^{n} \frac{\partial \rho_i(x)}{\partial z_\mu} \cdot (\zeta_\nu(x))_\mu \right|^2$$

on the compact set $\bar{B}^{(I)}$, taking into account the fact that $g$ vanishes of second order on $B^{(I)} \cap M$. So putting $c^{(I)} := c_1^{(I)} + c_2^{(I)} > 0$, we obtain

$$\mathscr{L}_{-\log \rho}(x)_{|W_x^{(I)}}(\xi, \xi) \geqq -c^{(I)} \cdot \|\xi\|^2,$$

and the statement of Theorem 4.3.3 is true for $c_1 := \max\{c^{(I)}\}$ and $B := \bigcap B^{(I)}$. □

It is quite easy to show that there is a constant $c_3 > 0$ and a ball $B = B(p)$ centered at $p$ such that the estimate

$$2 \operatorname{Re} \mathscr{L}_{-\log \rho}(x)(\zeta, \eta) \geqq -\frac{2c_3}{\sqrt{\rho(x)}} \cdot \|\zeta\| \cdot \|\eta\| \qquad \forall \zeta \in T_x^{10} M_x^{(I)}, \forall \eta \in T_x^{10} \mathbf{C}^n$$

holds at each $x \in B \setminus M$ and for every $I \subset \{1, \ldots, k\}$, $|I| = d$. If we then had an analogue of Theorem 4.2.2 in the category of non-generic CR-submanifolds $M \subset \mathbf{C}^n$, using Theorem 4.3.3 we could conclude, just as we did in the proof of Theorem 4.2.1, that the complement of a non-generic CR-submanifold $M \subset \mathbf{C}^n$ of CR-codimension $d$ were locally $(d + \hat{b}(p))$-complete at $p \in M$. However, the complex gradients $\partial \rho_1(p), \ldots, \partial \rho_k(p)$ being complex linearly independent was essential for our proof of Theorem 4.2.2; as this is no longer fulfilled in the case of non-generic CR-submanifolds we will not have a substitute for Theorem 4.2.2 here.

So, in the general case of a non-generic CR-submanifold $M \subset \mathbf{C}^n$, Theorem 4.3.3 only implies:

**4.3.4. Theorem.** *Let $M \subset \mathbf{C}^n$ be a non-generic* CR-*submanifold of type $(m, \ell)$, $k := \operatorname{codim}_{\mathbf{R}} M$, $d := \operatorname{corank}_{\mathbf{C}} T^{10} M$, $p \in M$ and $\hat{b}(p)$ the local index of convexity of $M$ at $p$. Then, $\mathbf{C}^n \setminus M$ is locally $q$-complete at $p$ for $q := d + \hat{b}(p) + 1$ which means that $\nu(p; M) \leqq d + \hat{b}(p) + 1$. More precisely, in view of Theorem* 3.3.2, *$d + \hat{b}(p) \leqq \nu(p; M) \leqq d + \hat{b}(p) + 1$.*

It remains an open question whether $\nu(p; M) = d + \hat{b}(p)$ or $\nu(p; M) = d + \hat{b}(p) + 1$ is true in the general non-generic case.

## 5 The index of convexity as a CR-invariant

From the preceding sections we know that the local $q$-completeness of the complement of a CR-manifold $M$ which is imbedded in $\mathbf{C}^n$ is determined by the CR-codimension of $M$ and by the local indices of convexity of $M$. The results presented here do not depend on the imbedding $M \hookrightarrow \mathbf{C}^n$; in order to realize this, we first give a geometric interpretation of the index of convexity.

### *5.1 A geometric interpretation of the index of convexity*

If $M \subset \mathbf{C}^n$ is a CR-submanifold, $p \in M$, $T_p M$ the real tangent space of $M$ at $p$, and if $J\colon T_p\mathbf{C}^n \to T_p\mathbf{C}^n$ denotes the complex structure map, *the CR-tangent space of $M$ at $p$* is given by $HT_pM := T_pM \cap JT_pM$.

We define the intrinsic Levi form of $M$ at $p$ following [3]:

**5.1.1. Definition.** The *(intrinsic) Levi form* of $M$ at $p$ is the map

$$\mathscr{L}_M(p)\colon\ HT_pM \simeq T_p^{10}M \to T_pM/HT_pM, \qquad \mathscr{L}_M(p)(\zeta) := \pi\left(\frac{i}{2}[L, \bar{L}](p)\right),$$

where $\pi\colon T_pM \to T_pM/HT_pM$ is the quotient map and $L$ is a smooth $T^{10}M$-valued vector field satisfying $L(p) = \zeta$ (this definition does not depend on the choice of such an $L \in \Gamma(M, T^{10}M)$, of course).

Now let $V_p \subset T_p\mathbf{C}^n$ be any real hyperplane containing $T_pM$ and let $HV_p := V_p \cap JV_p$ denote the maximal complex linear subspace of $V_p$, then, for $m := \dim_{\mathbf{R}} M$, it is clear that $\dim_{\mathbf{R}}(HV_p \cap T_pM) \geqq m-1$. So, the following makes sense:

**5.1.2. Definition.** Let $M \subset \mathbf{C}^n$ be a CR-submanifold, $p \in M$ and $V_p \subset T_p\mathbf{C}^n$ a real hyperplane containing $T_pM$ such that $\dim_{\mathbf{R}}(HV_p \cap T_pM) = m-1$ and $T_pM$ is split up into the half-spaces $(T_pM)^+$, $(T_pM)^-$ by $HV_p \cap T_pM$.

*$M$ has the property $G_j(p)$ with respect to $V_p$* if there is a complex linear subspace $I_p \subset T_p^{10}M$ of dimension $j$ and $S \in \{(T_pM)^+, (T_pM)^-\}$ such that $\frac{i}{2}[L, \bar{L}](p) \in S$ holds for all $L \in \Gamma(M, T^{10}M)$ that satisfy $L(p) \in I_p \setminus \{0\}$.

In the above situation, put $\tilde{c}(V_p) := \max\{j \in \mathbf{N} \mid M$ has the property $G_j(p)$ with respect to $V_p\}$; for hyperplanes $V_p \subset T_p\mathbf{C}^n$ satisfying $T_pM \subset HV_p$ define $\tilde{c}(V_p) := 0$. If $c(p) := \max\{\tilde{c}(V_p) \mid V_p \subset T_p\mathbf{C}^n$ is a real hyperplane so that $T_pM \subset V_p\}$, then we obtain the following geometric notion of the index of convexity:

**5.1.3. Lemma.** *Let $M \subset \mathbf{C}^n$ be a* CR-*submanifold of type $(m, \ell)$, $p \in M$, $b(p)$ the index of convexity of $M$ at $p$, and let $\tilde{c}$ be defined as above. Then $b(p) = c(p)$.*

*Proof.* First we show $b(p) \leqq c(p)$.

If $b(p) > 0$, choose defining functions $(\rho_1, \ldots, \rho_k)_U$ of $M$ on an open $p$-neighborhood $U \subset \mathbf{C}^n$ and a complex linear subspace $I_p \subset T_p^{10}M$ of dimension $\dim_{\mathbf{C}} I_p = b(p)$ such that $\mathscr{L}_{\rho_1}(p)_{|I_p}$ is positive definite. Then, $\hat{M} := \{z \in U \mid \rho_1(z) = 0\} \supset (M \cap U)$ is a real hypersurface, and $T_p\hat{M}$ is a real hyperplane containing $T_pM$. As $\mathscr{L}_{\rho_1}(p)_{|I_p \setminus \{0\}} > 0$, we conclude that $\mathscr{L}_{\hat{M}}(p)_{|T_p^{10}M} \not\equiv 0 \in T_p\hat{M}/HT_p\hat{M}$ and therefore $T_pM \not\subset HT_p\hat{M}$; so $\dim_{\mathbf{R}}(HT_p\hat{M} \cap T_pM) = m-1$, and there is

$\eta \in T_p M \setminus HT_p\hat{M}$ such that $T_p M = (T_p M \cap HT_p\hat{M}) \oplus \mathrm{span}_{\mathbf{R}}\{\eta\}$ holds. If now $L$ is a $T^{10}M$-valued vector field with $L(p) := \zeta \in I_p \setminus \{0\}$, then there are unique $v \in T_p M \cap HT_p\hat{M}$, $v' \in \mathrm{span}_{\mathbf{R}}\{\eta\}$ satisfying

$$\frac{i}{2}[L, \bar{L}](p) = v + v'$$

where $v' := a \cdot \mathscr{L}_{\rho_1}(p)(\zeta, \zeta) \cdot \eta$ and the constant $a \in \mathbf{R} \setminus \{0\}$ depends on the choice of $\eta$.

If $(T_p M)^+$ denotes the halfspace of $T_p M$ with respect to $(T_p M \cap HT_p\hat{M})$ that contains $v'$, then

$$\frac{i}{2}[W, \bar{W}](p) \in (T_p M)^+ \ \forall W \in \Gamma(M, T^{10}M) \text{ satisfying } W(p) \in I_p \setminus \{0\},$$

as the Levi form of $\rho_1$ at $p$ does not change sign on $I_p$. So, $\tilde{c}(T_p\hat{M}) \geqq \dim_{\mathbf{C}} I_p = b(p)$ which implies $c(p) \geqq b(p)$, and it remains to show that $c(p) \leqq b(p)$.

If $c(p) > 0$, choose a real hyperplane $V_p \subset T_p\mathbf{C}^n$ with $V_p \supset T_p M$ and $c(p) = \tilde{c}(V_p)$. The real conormal space of $M$ at $p$, $N_p M$, obviously satisfies $N_p M \supset \{w \in (T_p\mathbf{C}^n)^* \mid w(v) = 0 \ \forall v \in V_p\} =: (V_p^\perp)^*$. Given defining functions $\rho_1, \ldots, \rho_k$ for $M$ near $p$, the family $\{d\rho_1(p), \ldots, d\rho_k(p)\}$ is an $\mathbf{R}$-basis of $N_p M$; as $\dim_{\mathbf{R}}(V_p^\perp)^* = 1$, there is $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k \setminus \{0\}$ such that $\sum_{i=1}^k \alpha_i \cdot d\rho_i(p)$ spans $(V_p^\perp)^*$. So, $(V_p^\perp)^* = N_p\hat{M}$ and $V_p = T_p\hat{M}$ for $\rho_\alpha := \sum_{i=1}^k \alpha_i \rho_i$ and $\hat{M} := \{z \in U \mid \rho_\alpha(z) = 0\}$.

Let now $I_p \subset T_p^{10}M$ be a $c(p) = \tilde{c}(V_p)$-dimensional complex linear subspace and let $(T_p M)^+$ be the half-space of $T_p M$ with respect to $T_p M \cap HT_p\hat{M}$ such that

$$\frac{i}{2}[W, \bar{W}](p) \in (T_p M)^+ \ \forall W \in \Gamma(M, T^{10}M) \text{ satisfying } W(p) \in I_p \setminus \{0\};$$

then, $\mathscr{L}_{\rho_\alpha}(p)_{|I_p}$ is either positive definite or negative definite. Anyhow, we find a linear combination $\rho_\beta$, $\beta \in \{\pm\alpha\}$, of the defining functions $\rho_1, \ldots, \rho_k$ of $M$ such that $\mathscr{L}_{\rho_\beta}(p)_{|T_p^{10}M}$ has at least $c(p)$ positive eigenvalues, clearly $c(p) \leqq b(p)$ in view of Lemma 3.1.4(2).

This completes the proof of Lemma 5.1.4. □

### *5.2 The behaviour of the index of convexity under* CR*-transformations*

From the geometric notion of the index of convexity, one can easily conclude that it is a CR-invariant. First we observe that the Lie bracket fulfills the following compatibility relation:

**5.2.1. Lemma.** *Let $M_1, M_2 \subset \mathbf{C}^n$ be smooth submanifolds, $\varphi\colon M_1 \xrightarrow{C^\infty} M_2$ a diffeomorphism and $\varphi_*^{\mathbf{C}}\colon \Gamma(M_1, T^{\mathbf{C}}M_1) \to \Gamma(M_2, T^{\mathbf{C}}M_2)$ the smooth map between* *the sections of the complexified tangent bundles of* $M_1$, $M_2$ *which is induced by* $\varphi$. *Then, for every* $Z, W \in \Gamma(M_1, T^{\mathbf{C}} M_1)$:

$$[\varphi_*^{\mathbf{C}} Z, \varphi_*^{\mathbf{C}} W] = \varphi_*^{\mathbf{C}}([Z, W]).$$

Now suppose that $M_1$, $M_2$ carry additional structure, namely that there are CR-structures $T^{10} M_1 \subset T^{\mathbf{C}} M_1$ and $T^{10} M_2 \subset T^{\mathbf{C}} M_2$. If $\varphi$: $M_1 \to M_2$ is a CR-diffeomorphism, then the induced **C**-vector bundle isomorphism $\varphi_*^{\mathbf{C}}$: $T^{\mathbf{C}} M_1 \to T^{\mathbf{C}} M_2$ restricts to a **C**-vector bundle isomorphism

$$\varphi_{*|T^{10}M_1}^{\mathbf{C}}: T^{10} M_1 \to T^{10} M_2$$

between the holomorphic tangent bundles of $M_1$ and $M_2$. This is used to show:

**5.2.2. Theorem.** *If* $M_1$, $M_2 \subset \mathbf{C}^n$ *are* CR-*submanifolds of type* $(m, \ell)$, $U_1 = U_1(p_1)$, $U_2 = U_2(p_2) \subset \mathbf{C}^n$ *open neighborhoods of the points* $p_1 \in M_1$, $p_2 \in M_2$ *and if* $\varphi$: $U_1 \cap M_1 \xrightarrow{\simeq} U_2 \cap M_2$ *is a* CR-*diffeomorphism with* $\varphi(p_1) = p_2$, *then* $b(p_1) = b(p_2)$.

*Proof.* We only have to show $b(p_1) \leqq b(p_2)$, then the statement of the theorem follows by analogous arguments for $\varphi^{-1}$: $U_2 \cap M_2 \xrightarrow{\simeq} U_1 \cap M_1$.

If $b(p_1) > 0$, from Lemma 5.1.3 we are given a real hyperplane $V_{p_1} \subset T_{p_1} \mathbf{C}^n$, $T_{p_1} M_1 \subset V_{p_1}$, such that $b(p_1) = \tilde{c}(V_{p_1})$. By definition of $\tilde{c}(V_{p_1})$, there is a complex linear subspace $J_{p_1} \subset T_{p_1}^{10} M_1$ of dimension $b(p_1)$ such that for all $T^{10} M_1$-valued vectorfields $L$ satisfying $L(p_1) \in J_{p_1} \setminus \{0\}$, the Lie brackets $\frac{i}{2}[L, \bar{L}]$ are always situated in the same halfspace of $T_{p_1} M_1$ with respect to $HV_{p_1} \cap T_{p_1} M_1$; let this halfspace be denoted by $(T_{p_1} M_1)^+$.

As the CR-diffeomorphism $\varphi$: $U_1 \cap M_1 \to U_2 \cap M_2$ induces a **C**-vector bundle isomorphism

$$\varphi_*^{\mathbf{C}}: T^{10} M_{1|U_1} \to T^{10} M_{2|U_2}$$

it follows that $J_{p_2} := \varphi_*^{\mathbf{C}}(J_{p_1}) \subset T_{p_2}^{10} M_2$ is a complex $b(p_1)$-dimensional linear subspace. If now $Y \in \Gamma(U_2 \cap M_2, T^{10} M_2)$ is a vectorfield satisfying $Y(p_2) \in J_{p_2} \setminus \{0\}$, then the vectorfield $\varphi_*^{\mathbf{C}^{-1}}(Y) =: L \in \Gamma(U_1 \cap M_1, T^{10} M_1)$ satisfies $L(p_1) \in J_{p_1} \setminus \{0\}$, and

$$\frac{i}{2}[Y, \bar{Y}](p_2) = \varphi_*^{\mathbf{C}}\left(\frac{i}{2}[L, \bar{L}](p_1)\right)$$

is situated in the halfspace

$$(T_{p_2} M_2)^+ := \varphi_*^{\mathbf{C}}(T_{p_1} M_1)^+ = \varphi_*(T_{p_1} M_1)^+$$

of $T_{p_2} M_2$ with respect to the codimension 1 – real linear subspace

$$\varphi_*^{\mathbf{C}}(HV_{p_1} \cap T_{p_1} M_1) = \varphi_*(HV_{p_1} \cap T_{p_1} M_1) \subset T_{p_2} M_2.$$

Putting $V_{p_2}:=\varphi_*^{\mathbf{C}}(HV_{p_1})\oplus \operatorname{span}_{\mathbf{R}}\{\sigma\}$ for any $0\neq\sigma\in T_{p_2}M_2\setminus\varphi_*(HV_{p_1}\cap T_{p_1}M_1)$, one obviously obtains a real hyperplane $V_{p_2}\subset T_{p_2}\mathbf{C}^n$ containing $T_{p_2}M_2$ which satisfies $\tilde{c}(V_{p_2})\geqq \dim_{\mathbf{C}} J_{p_2}=b(p_1)$. It follows that

$$b(p_2)\underset{(5.1.3)}{=}c(p_2)\underset{\text{Def.}}{\geqq}\tilde{c}(V_{p_2})\geqq b(p_1),$$

and the proof of the theorem is complete. □

## 6 Some globalizations of the local results

We conclude the article with some remarks on global consequences of the local theorems proved in Chap. 4.

By means of Theorem 2.1 it is easy to see that, for smooth submanifolds $M\subset\mathbf{C}^n$, there is no difference between local and global $q$-completeness of the complement $\mathbf{C}^n\setminus M$:

**6.1. Theorem.** *If the complement $\mathbf{C}^n\setminus M$ of a smooth submanifold $M\subset\mathbf{C}^n$ is locally $q$-complete, then $\mathbf{C}^n\setminus M$ is globally $q$-complete.*

The equivalence between local and global $q$-completeness of $C^n\setminus M$ follows from two facts:

[1] $\mathbf{C}^n$ has strictly plurisubharmonic exhaustion functions
[2] There is an open neighborhood $U=U(M)\subset\mathbf{C}^n$ of $M$ and a smooth function $d_M^2$ on $U$ satisfying $d_M^2\geqq 0$, $M=\{d_M^2=0\}$, such that the following holds: If $\mathbf{C}^n\setminus M$ is locally $q$-complete at $p\in M$, then there is an open $p$-neighborhood $V=V(p)\subset U$ so that the function $-\log d^2_{M|V\setminus M}$ is at least weakly $q$-convex in $V\setminus M$ (see Theorem 2.1).

If $X$ is a complex manifold and $M\subset X$ is a smooth submanifold, then local and global $q$-completeness of $X\setminus M$ are equivalent if [1] and [2] remain valid after substituting $X$ for $\mathbf{C}^n$. Now it is precisely the Stein manifolds that are characterized by the existence of strictly plurisubharmonic smooth exhaustion functions, so in order to see that an analogue of Theorem 6.1 is valid for submanifolds $M$ of Stein manifolds $X$, we only have to prove a generalization of Theorem 2.1:

**6.2. Theorem.** *If $X$ is a Stein manifold and $M\subset X$ is a smooth submanifold, then there is an open neighborhood $U=U(M)\subset X$ and a function $d_M^{X\,2}: U\to\mathbf{R}_0^+$ satisfying $M=\{z\in U\mid d_M^{X\,2}(z)=0\}$ and $-\log(d_M^{X\,2})\in C^\infty(U\setminus M)$ so that the following holds:*

*If $X\setminus M$ is locally $q$-complete at $p\in M$, then there exists an open $p$-neighborhood $V=V(p)\subset U$ such that $-\log(d_M^{X\,2})_{|V\setminus M}$ is at least weakly $q$-convex in $V\setminus M$.*

*Proof.* It suffices to find an open neighborhood $S\supset\Delta_X$ of the diagonal $\Delta_X\subset X\times X$ and a function $\psi=-\log\tilde{\psi}$, $\tilde{\psi}\geqq 0$, $\tilde{\psi}_{|\Delta_X}\equiv 0$ which is strictly $n$-convex in $S\setminus\Delta_X$ such that

$$d_M^{X\,2}:\ z\mapsto \inf\{\tilde{\psi}(z,y)\mid y\in M\}$$

is smooth on an open neighborhood $U=U(M)\subset X$ of $M$; then, using Lemma 2.3 as in the proof of Theorem 2.1, it follows that the function $-\log(d_M^{X\,2})=\sup\{\psi(\cdot,y)\mid y\in M\}$ has the required properties.

First, there is a neighborhood $\tilde{S}$ of $\Delta_X$ and a function $h$: $\tilde{S} \to \mathbf{R}$ with the following properties (see [10, Theorem 3.1]):

(1) $h \in C^\infty(\tilde{S})$, $h \geqq 0$, $\Delta_X = \{h=0\}$

(2) For every $(x, x) \in \Delta_X$ one can find an open neighborhood $V_x \subset \tilde{S}$ and a function $\theta_x \in C^\infty(V_x)$ such that $-\log h_{|V_x \setminus \Delta_X} + \theta_{x|V_x \setminus \Delta_X}$ is strictly $n$-convex in $V_x \setminus \Delta_X$.

The union $\bigcup_{(x,x) \in \Delta_X} V_x \supset \Delta_X$ of all those $V_x$ is an open covering of $\Delta_X$, and we can choose a locally finite refinement $\{V_j\}_{j \in J}$ and functions $\theta_j \in C^\infty(V_j)$ such that $-\log h_{|V_j \setminus \Delta_X} + \theta_{j|V_j \setminus \Delta_X}$ is strictly $n$-convex in $V_j \setminus \Delta_X$ for each $j \in J$.

Now define $S := \bigcup_{j \in J} V_j \supset \Delta_X$ and, given a strictly plurisubharmonic smooth exhaustion function $\varphi$ of $X \times X$, choose a convex increasing function $r$ so that $r \circ \varphi - \theta_j$ is strictly plurisubharmonic in $V_j \setminus \Delta_X$ for each $j \in J$. Then the function

$$\psi := -\log\left(\frac{h}{\exp(r \circ \varphi)}\right) =: -\log(\tilde{\psi})$$

is strictly $n$-convex on $S \setminus \Delta_X$ and satisfies $\tilde{\psi} \geqq 0$ and $\tilde{\psi}_{|\Delta_X} = 0$.

The smoothness of the function $d_M^{X\,2}$: $z \mapsto \inf\{\tilde{\psi}(z, y) \mid y \in M\}$ will follow from the implicit function theorem, applied to

$$\left(\frac{\partial \tilde{\psi}}{\partial y_1}(z_0, y_0), \ldots, \frac{\partial \tilde{\psi}}{\partial y_m}(z_0, y_0)\right) = 0,$$

where $y_1, \ldots, y_m$ are real local coordinates of $M$ at $y_0 \in M$ and where $z_0 \in X$ satisfies $(z_0, y_0) \in S$ and $d_M^{X\,2}(z_0) = \tilde{\psi}(z_0, y_0)$. □

**6.3. Corollary.** *If the complement $X \setminus M$ of a smooth submanifold $M \subset X$ in a Stein manifold $X$ is locally $q$-complete, then $X \setminus M$ is globally $q$-complete.*

In the general case of smooth submanifolds $M$ in complex manifolds $X$ which are not Stein one can no more expect that local $q$-completeness of $X \setminus M$ implies global $q$-completeness of $X \setminus M$; however, if $X$ is a compact complex manifold such that for smooth submanifolds $M \subset X$, property [2] keeps valid, after substituting $X$ for $\mathbf{C}^n$ and strict $q$-convexity for the weak $q$-convexity of the function $-\log d_M^2$ in $V \setminus M$, then local $q$-completeness of $X \setminus M$ implies global $q$-convexity of $X \setminus M$. Such a result holds, for example, in projective space:

**6.4. Theorem.** *If the complement $\mathbf{P}_n \setminus M$ of the smooth manifold $M$ in the $n$-dimensional complex projective space $\mathbf{P}_n$ is locally $q$-complete, then $\mathbf{P}_n \setminus M$ is $q$-convex.*

*Proof.* According to [9] the Fubini metric $d^F$ is $n$-convex near the diagonal $\Delta_{\mathbf{P}_n}$ in $\mathbf{P}_n \times \mathbf{P}_n \setminus \Delta_{\mathbf{P}_n}$, and the Fubini distance $d_M^F$: $z \mapsto \inf\{d^F(z, y) \mid y \in M\}$ to $M$ is smooth near $M$ (see [2]). As in the proof of Theorem 6.2 we obtain, applying Lemma 2.3, a criterion which says that for each $p \in M$, there is an open $p$-neighborhood $U = U(p) \subset \mathbf{P}_n$ so that $-\log(d_M^{F2})_{|U(p) \setminus M}$ is $q$-convex. Choosing $U := \bigcup_{p \in M} U(p)$ an open neighborhood of $M$ in $\mathbf{P}_n$ and putting $K := \mathbf{P}_n \setminus U$, the function $-\log(d_M^{F2})$ is an exhaustion function for $\mathbf{P}_n \setminus M$ which is strictly $q$-convex outside the compact set $K \subset \mathbf{P}_n \setminus M$, and the statement of Theorem 6.4 follows. □

By means of the smoothing techniques of Diederich and Fornaess in [4] and [5], one can derive global $\tilde{q}$-completeness of $\mathbf{P}_n \setminus M$ from local $q$-completeness of $\mathbf{P}_n \setminus M$ where, in general, $\tilde{q} > q$ will be valid:

**6.5. Theorem.** *If the complement $\mathbf{P}_n \setminus M$ of the smooth submanifold $M$ in $\mathbf{P}_n$ is locally $q$-complete, then $\mathbf{P}_n \setminus M$ is globally $\tilde{q}$-complete for $\tilde{q} := n - \left[\frac{n}{q}\right] + 1$ where $\left[\frac{n}{q}\right]$ denotes the integer value of $\frac{n}{q}$.*

*Proof.* If $\mathbf{P}_n \setminus M$ is locally $q$-complete, then $\mathbf{P}_n \setminus M$ is globally $q$-convex by Theorem 6.4 and as well globally $q$-complete with corners (see [9, Theorem 4]). Therefore, Theorem 6.5 is an immediate consequence of the smoothing theorems of Diederich and Fornaess. □

Combining the Theorems 6.1, ..., 6.5 with the results of Chap. 4 one obtains global theorems on the analytic convexity of complements of smooth submanifolds $M$ in Stein manifolds or in $\mathbf{P}_n$. We only have to observe that the index of convexity is a biholomorphic invariant which is trivial in view of Chap. 5, and we have to substitute the local index of convexity $\hat{b}(p)$ of $M$ at $p$ by $b := \max\{b(p) \mid p \in M\}$ and $\mathbf{C}^n$ by a complex manifold $X$ of dimension $n$ in the local theorems of Chap. 4.

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