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Nonlinear elliptic systems with dynamic boundary conditions

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Introduction and main results

Let us consider the following elliptic boundary value problem

(1) $$\lambda u-\partial_j(a_{jk}\,\partial_k u)=f(\cdot, u, \nabla u) \quad \text{in } \Omega\times(0,\infty),$$

(2) $$a_{jk}\,\nu^j\partial_k u=g(\cdot, u) \quad \text{on } \Gamma_1\times(0,\infty),$$

(3) $$u=0 \quad \text{on } \Gamma_2\times(0,\infty).$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Γ_1 and Γ_2 are both open and closed subsets of $\partial\Omega$. The outer normal on $\partial\Omega$ is denoted by $\nu=(\nu^1, \ldots, \nu^n)$.

We assume that the coefficients $a_{jk}(x)$ are $N\times N$-matrices, $N\geqq 1$, that they depend smoothly on $x\in\bar{\Omega}$ and that the uniform Legendre-Hadamard condition is satisfied, i.e.

$$a_{jk}^{rs}(x)\,\xi^j\xi^k\eta_r\,\eta_s>0 \qquad \text{for all } x\in\bar{\Omega},\ \xi\in\mathbb{R}^n\setminus\{0\},\ \eta\in\mathbb{R}^N\setminus\{0\}.$$

Note that problem (1)–(3) is a strongly coupled system of second order elliptic boundary value problems.

f and g are given functions. We assume that they are smooth with respect to all variables, i.e. $f\in C^\infty(\bar{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}, \mathbb{R}^N)$ and $g\in C^\infty(\Gamma_1\times\mathbb{R}^N, \mathbb{R}^N)$.

The elliptic problem (1)–(3) is completed by the following dynamic boundary condition

(4) $$\partial_t u+a_{jk}\,\nu^j\partial_k u=h(\cdot, u) \quad \text{on} \quad \Gamma_3\times(0,\infty),$$

and the initial condition

(5) $$u(\cdot, 0)=z_0 \quad \text{on} \quad \Gamma_3.$$

Here we assume that $\Gamma_3:=\partial\Omega\setminus(\Gamma_1\cup\Gamma_2)$ is not empty and that $h\in C^\infty(\Gamma_3\times\mathbb{R}^N, \mathbb{R}^N)$. Moreover $z_0\in B_{pp}^{1-1/p}(\Gamma_3)$, where $B_{pp}^s(\Gamma_3)$, $s\in\mathbb{R}$, $p\in(1,\infty)$, denote the Besov spaces over Γ_3.

Then we have the following *local* existence and uniqueness result for problem (1)–(5).

Theorem 1 *Suppose that $p>n$ and that there is an increasing function q: $\mathbb{R}^+ \to \mathbb{R}^+$ and a constant $\nu \in [1, 1+p/n)$ such that*

$$|\partial_2 f(x,\xi,\eta)| + (1+|\eta|)|\partial_3 f(x,\xi,\eta)| \leqq q(|\xi|)(1+|\eta|^\nu), \qquad (x,\xi,\eta)\in\bar{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}.$$

Then there is a $\tilde{\lambda}_0 \geqq 0$ such that for each $z_0 \in B_{pp}^{1-1/p}(\Gamma_3)$ and each $\lambda \geqq \tilde{\lambda}_0$ there exists a $\delta>0$ such that problem (1)–(5) possesses a unique weak solution $u\in C([0,\delta], W_p^1(\Omega))$.

Note that we only assumed growth restriction on f with respect to ∇u but not on f, g and h with respect to u.

The next theorem shows that we obtain *maximal* solutions, in the sense that there are no proper extensions, provided suitable growth restrictions on f with respect to $(u, \nabla u)$ and on g with respect to u are satisfied.

Theorem 2 *Suppose that $p>n$ and that there is a constant $M \geqq 0$ such that*

$$|\partial_2 f(x,\xi,\eta)| + |\partial_3 f(x,\xi,\eta)| \leqq M \qquad \textit{for } (x,\xi,\eta)\in\bar{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN},$$
$$|\partial_2 g(y,\xi)| \leqq M \qquad \textit{for } (y,\xi)\in\Gamma_1\times\mathbb{R}^N.$$

Then there is a $\lambda_0 \geqq 0$ such that for each $z_0\in B_{pp}^{1-1/p}(\Gamma_3)$ and each $\lambda \geqq \lambda_0$ there exists a unique maximal weak solution $u\in C([0,t^+), W_p^1(\Omega))$ of (1)–(5).

If the trace of the solution u is bounded in $B_{pp}^{1-1/p}(\Gamma_3)$, then u is a global solution, i.e. $u\in C([0,\infty), W_p^1(\Omega))$.

Finally suppose that there is a positive constant c with

$$|h(y,\xi)| \leqq c(1+|\xi|), \qquad (y,\xi)\in\Gamma_3\times\mathbb{R}^N. \tag{6}$$

Then the solution u exists globally.

A proof of the assertions above is given in Sects. 4 and 6 of this paper.

Last of all, let us mention that we also specify conditions (see (6.3) and (6.4)) which imply global existence for any initial data $z_0\in B_{pp}^{1-1/p}(\Gamma_3)$ – but which allow a stronger growth rate for h than (6).

Equations of type (1)–(5) have been studied by several authors, cf. [11, 14, 18, 20, 24], see also [17]. In [20] Lions considered the following special case of (1)–(5): $N=1$, $a_{jk}=\delta_{jk}$ (Kronecker symbol), $f=0$, $\Gamma_1=\Gamma_2=\emptyset$ and $h(\xi)=-|\xi|^\rho\xi$, $\rho>0$. Using monotonicity and compactness methods he proved existence and uniqueness of global solutions of the following problem

$$\begin{aligned} \Delta u &= 0 \quad &&\text{in } \Omega\times(0,\infty),\\ \partial_t u+\partial_\nu u+|u|^\rho u &= 0 &&\text{on } \partial\Omega\times(0,\infty),\\ u(\cdot,0) &= z_0 &&\text{on } \partial\Omega. \end{aligned}$$

In this context, Lions introduced the following operator $\mathscr{B}\varphi := \partial_\nu \Phi$, acting on $B_{22}^{1/2}(\partial\Omega)$, where Φ is the unique solution of the following Dirichlet problem:

$$\Delta\Phi=0 \quad \text{in} \quad \Omega, \qquad \Phi=\varphi \quad \text{on} \quad \partial\Omega.$$

In a recent paper [18], Hintermann investigated the operator $\mathscr{B}$ (in a very general situation) by means of the theory of pseudo-differential operators and the Mikhlin-Hörmander multiplier theorem. The main result in [18] shows that $-\mathscr{B}$ generates an analytic semigroup on $B_{pp}^{1-1/p}(\partial\Omega)$, $1<p<\infty$, and that the *linear* problem

$$(7)\qquad \begin{aligned} \partial_j(a_{jk}\,\partial_k u)&=f && \text{in } \Omega\times(0,\infty),\\ \partial_t u+a_{jk}\,\nu^j\partial_k u&=h && \text{on } \partial\Omega\times(0,\infty),\\ u(\cdot,0)&=z_0 && \text{on } \partial\Omega, \end{aligned}$$

is well posed in $W_p^2(\Omega)$, i.e. for each $z_0\in B_{pp}^{2-1/p}(\partial\Omega)$ there is a unique solution $u(z_0,\cdot)$ and $[(z_0,t)\mapsto u(z_0,t)]\in C(B_{pp}^{2-1/p}(\partial\Omega)\times[0,\infty), W_p^2(\Omega))$.

It is the main purpose of this paper to develop a natural extension of the results in [18]. In particular, we show that the closure of $-\mathscr{B}$ in $B_{pp}^{s-1/p}(\partial\Omega)$ if $s<1$, respectively the maximal restriction of $-\mathscr{B}$ in $B_{pp}^{s-1/p}(\partial\Omega)$ if $s>1$, generates an analytic semigroup on $B_{pp}^{s-1/p}(\partial\Omega)$ and that problem (7) is well posed in $W_p^s(\Omega)$ for $s\in\mathbb{R}$. This weak formulation turns out to be very useful, essentially for two reasons: First of all, in this weak setting it is possible to treat the general case of (1)–(5), where not only h but also f and g are given *nonlinear* functions (with respect to u, of course).

In [14] only linear problems of type (1)–(5) are considered. In [18] Hintermann investigated problem (1)–(5) in the situation where $\Gamma_1=\Gamma_2=\emptyset$ and where f does not depend on $(u, \nabla u)$. Some nonlinear problems possessing special structural properties have been studied in [11, 20, 24]. Moreover, it should be mentioned that only in [18] the situation of strongly coupled systems is considered, whereas in [11, 14, 20, 24] the case $N=1$ is treated.

As a second advantage, the weak formulation turns out to be very useful in the discussion of the dynamic behaviour of problem (1)–(5). In particular, we show that the solution of (1)–(5) exists globally, provided an a priori bound for $\gamma_3 u$ in $B_{pp}^{1-1/p}(\Gamma_3)$ is known. Here, γ_3 denotes the trace operator with respect to Γ_3.

It should be mentioned that there are different approaches to problem (7) using Hilbert space methods. For example, in [24] the coerciveness in $W_2^1(\Omega)$ of the corresponding Dirichlet form is employed.

Recently, K. Gröger informed the author that it is possible to associate to problem (7) an equation of the following kind

$$(8)\qquad (\lambda+A+B)u=f+\gamma^* h,\qquad 0<t\leqq T,\qquad (\gamma u)(0)=z_0,$$

in the Hilbert space V_T', where $V_T:=L_2((0,T), W_2^1(\Omega))$.

Here, $\gamma\in\mathscr{L}(W_2^1(\Omega), B_{22}^{1/2}(\partial\Omega))$ denotes the trace operator, γ^* its dual operator and A the extension on V_T of the elliptic part in (7). B is an appropriate extension of the operator $C^1([0,T], W_2^1(\Omega))\to V_T'$, $u\mapsto\dfrac{d}{dt}(\gamma^*\gamma u)$. Then one can show that $\lambda+A$: $V_T\to V_T'$ is monotone and coercive and that B: $D(B)\subset V_T\to V_T'$ is maximal monotone. Thus the general theory of maximal monotone operators (cf. [15]) ensures the existence of a unique solution $u\in V_T=L_2((0,T), W_2^1(\Omega))$.

There are significant differences between these Hilbert space approaches and the one used in the main part of this paper.

First of all, the latter setting takes the dynamical properties of the problems under consideration – like global existence, blow up behaviour or stability of solutions – better into account than the more or less stationary appraoch of (8). For example, one should observe that the continuity of solutions with respect to time as well as the continuous dependence of solutions with respect to the intial data is an immediate consequence of our theory. Note also that weak solutions obtained by Theorem 1 are Hölder continuous with respect to the spatial variable, since $W_p^1(\Omega) \subset C^\mu(\bar{\Omega})$ for $\mu \in [0, 1-n/p]$. In contradistinction to this fact, it is well known that weak solutions, in the $W_2^1(\Omega)$-sense, of strongly coupled systems are not Hölder continuous, in general.

Secondly a "pure" Hilbert space approach is certainly the suitable setting to treat *linear* problems. However, in *nonlinear* problems it requires to assume growth restrictions, particularly for h in the problem considered here.

Finally, we mention that for certain inhomogeneities f and g the Legendre-Hadamard condition can be replaced by the weaker assumption that the elliptic part in (1)–(5) defines a normally elliptic boundary value problem in the sense of [7]. This is an important observation, since the extension to $W_2^1(\Omega)$ of a normally elliptic system is not coercive (in the sense of [15]), in general.

This paper is organized as follows. In Sects. 1 and 2 we construct an appropriate extension of the operator $\mathscr{B}$. In Sects. 3–5 we then prove existence and uniqueness results of an abstract version of (1)–(5). Some applications to local nonlinearities are derived in Sect. 6. Finally, in the Appendix of this paper, we prove a maximum principle for linear equations of type (1)–(5).

Notations. Let E and F be Banach spaces over $\mathbb{R}$. Then $\mathscr{L}(E, F)$ denotes the Banach space of all bounded linear operators from E to F. For the set of all isomorphisms in $\mathscr{L}(E, F)$ we write $\mathrm{Isom}(E, F)$. Moreover, $\mathscr{L}_s(E, F)$ denotes the vector space $\mathscr{L}(E, F)$, equipped with the strong topology. Furthermore, $\mathscr{L}(E, F; \mathbb{R})$ stands for the Banach space of all continuous bilinear forms from $E \times F$ to $\mathbb{R}$. The norm in $\mathscr{L}(E, F; \mathbb{R})$ is given by $\|a\| := \sup\{|a(x, y)|; \|x\|_E \leqq 1, \|y\|_F \leqq 1\}$ for $a \in \mathscr{L}(E, F; \mathbb{R})$. Finally, we write $E \hookrightarrow F$, if E is continuously embedded in F. If, in addition, E is dense in F, we write $E \overset{d}{\hookrightarrow} F$.

1 A scale of analytic semigroups

The purpose of this section is to construct an appropriate extension of a pseudo-differential operator introduced by Hintermann in [18]. Using the main result in [18] and some interpolation techniques, we obtain a scale of generators of analytic semigroups on some Besov spaces.

In the following, Ω denotes a bounded domain in $\mathbb{R}^n$, $n \geqq 1$, of class C^∞. Suppose that Γ_i, $1 \leqq i \leqq 3$, are both open and closed in $\partial\Omega$, the boundary of Ω, with $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. Moreover, we assume that $\Gamma_3 \neq \emptyset$.

For $1 < q < \infty$ and $s \in \mathbb{R}$ we denote by $H_q^s(\Omega) := H_q^s(\Omega, \mathbb{R}^N)$, $N \geqq 1$, the Bessel potential spaces over Ω with norm $|\cdot|_{s,q}$ and by $B_q^s(\Gamma_i) := B_{qq}^{s-1/q}(\Gamma_i, \mathbb{R}^N)$ the Besov spaces over Γ_i with norm $\|\cdot\|_{i,s,q}$, $1 \leqq i \leqq 3$, cf. [8] or [27]. Furthermore $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_i$ stands for the duality pairing in $L_q(\Omega) := (L_q(\Omega, \mathbb{R}^N), |\cdot|_q)$ and $L_q(\Gamma_i) := (L_q(\Gamma_i, \mathbb{R}^N), \|\cdot\|_{i,q})$, $1 \leqq i \leqq 3$, respectively. We fix $T > 0$ and we suppose that for some $\rho \in (0, 1)$:

$$(1.1) \qquad a_{jk} = a_{kj},\ a_j,\ a_0 \in C^\rho([0, T], C^\infty(\bar{\Omega}, \mathscr{L}(\mathbb{R}^N))), \qquad 1 \leqq j, k \leqq n.$$

Then we define the following second order differential operators

$$\begin{aligned}(1.2)\qquad \mathscr{A}(t)u &:= -\partial_j(a_{jk}(t,\cdot)\,\partial_k u)+a_j(t,\cdot)\,\partial_j u+a_o(t,\cdot)u,\\ \mathscr{A}^{\#}(t)\varphi &:= -\partial_j(a_{jk}^T(t,\cdot)\,\partial_k\varphi)-\partial_j(a_j^T(t,\cdot)\varphi)+a_0^T(t,\cdot)\varphi,\end{aligned}$$

for $u\in H_p^2(\Omega)$ and $\varphi\in H_{p'}^2(\Omega)$, where $p\in(1,\infty)$, $\frac{1}{p'}:=1-\frac{1}{p}$ and where a^T stands for the transposed of $a\in\mathscr{L}(\mathbb{R}^N)$.

Denoting by $\gamma_i\in\mathscr{L}(H_p^s(\Omega), B_p^s(\Gamma_i))$, $s>\frac{1}{p}$ and $\gamma_i^{\#}\in\mathscr{L}(H_{p'}^t(\Omega), B_{p'}^t(\Gamma_i))$, $t>\frac{1}{p'}$, the trace operators with respect to Γ_i, $1\leqq i\leqq 3$, we define the following boundary operators

$$\begin{aligned}(1.3)\qquad \mathscr{B}_i(t)u &:= a_{jk}(t,\cdot)\,\nu^j\gamma_i\,\partial_k u+b_i(t,\cdot)\,\gamma_i u,\\ \mathscr{B}_i^{\#}(t)\varphi &:= a_{jk}^T(t,\cdot)\,\nu^j\gamma_i^{\#}\,\partial_k\varphi+(a_j^T(t,\cdot)\,\nu^j+b_i^T(t,\cdot))\,\gamma_i^{\#}\varphi,\end{aligned}$$

for $u\in H_p^2(\Omega)$, $\varphi\in H_{p'}^2(\Omega)$, $t\in[0,T]$ and $1\leqq i\leqq 3$. Here $\nu=(\nu^1,\ldots,\nu^n)$ is the outer normal on $\partial\Omega$ and

$$(1.4)\qquad b_i\in C^\rho([0,T], C^\infty(\Gamma_i,\mathscr{L}(\mathbb{R}^N))),\qquad 1\leqq i\leqq 3.$$

Finally, we assume that $\mathscr{A}(t)$ satisfies the uniform Legendre-Hadamard condition, i.e.

$$\begin{aligned}(1.5)\qquad & a_{jk}^{rs}(t,x)\,\xi^j\xi^k\eta_r\,\eta_s>0\qquad \text{for } (t,x)\in[0,T]\times\bar\Omega\\ & \text{and}\quad \xi\in\mathbb{R}^n\setminus\{0\},\eta\in\mathbb{R}^N\setminus\{0\}.\end{aligned}$$

and that, if $\Gamma_1\neq\emptyset$, the boundary operator $(\mathscr{B}_1(t),\gamma_2,\gamma_3)$ satisfies the normal complementing condition with respect to $\mathscr{A}(t)$ in the sense of [7].

For the remainder of this section we fix $t\in[0,T]$ and suppress it in our notation. The following a priori estimate is of fundamental importance. A proof can be found in [16, 3]. See also [1].

There exist constant $c>0$, $\lambda_*>0$, $\vartheta\in(0,\pi/2)$ such that

$$(1.6)\qquad (\lambda+\mathscr{A},\mathscr{B}_1,\gamma_2,\gamma_3)\in\operatorname{Isom}(H_p^2(\Omega), L_p(\Omega)\times B_p^1(\Gamma_1)\times B_p^2(\Gamma_2)\times B_p^2(\Gamma_3))$$

and

$$\begin{aligned}(1.7)\qquad |\lambda|\,|u|_p+|u|_{2,p} &\leqq c\{|(\lambda+\mathscr{A})u|_p+(1+|\lambda|)^{(1-1/p)/2}\,\|\mathscr{B}_1 u\|_{1,1,p}\\ &\quad+(1+|\lambda|)^{(2-1/p)/2}(\|\gamma_2 u\|_{2,2,p}+\|\gamma_3 u\|_{3,2,p})\}\end{aligned}$$

for all $\lambda\in S(\vartheta,\lambda_*):=\{z\in\mathbb{C};\,|\arg(z)|\leqq\vartheta+\pi/2 \text{ and } |z|\geqq\lambda_*\}$ and $u\in H_p^2(\Omega)$.

It should be mentioned that assumption (1.5) can be considerably weakened. For example, it is sufficient to assume that $(\mathscr{A}(t),\mathscr{B}_1(t),\gamma_2,\gamma_3)$, $t\in[0,T]$, defines a normally elliptic boundary value problem in the sense of [7].

Due to (1.6) we define for $\lambda\geqq\lambda_*$ the following Green operators

$$\begin{aligned}(1.8)\qquad & \mathscr{R}:=\mathscr{R}_\lambda:=(\lambda+\mathscr{A},\mathscr{B}_1,\gamma_2,\gamma_3)^{-1}\,|\,L_p(\Omega)\times B_p^1(\Gamma_1)\times\{0\}\times B_p^2(\Gamma_3),\\ & \mathscr{S}:=\mathscr{S}_\lambda:=\mathscr{R}_\lambda\,|\,L_p(\Omega)\times B_p^1(\Gamma_1)\times\{0\},\qquad \mathscr{T}:=\mathscr{T}_\lambda:=\mathscr{R}_\lambda\,|\,\{0\}\times B_p^2(\Gamma_3).\end{aligned}$$

Let $(f, g, h) \in L_p(\Omega) \times B_p^1(\Gamma_1) \times B_p^2(\Gamma_3)$ be given. Then $u := \mathscr{R}(f, g, h)$ is the unique solution of the following elliptic boundary value problem:

$$(\lambda + \mathscr{A})u = f \ \text{ in } \Omega, \quad \mathscr{B}_1 u = g \ \text{ on } \Gamma_1, \quad \gamma_2 u = 0 \ \text{ on } \Gamma_2, \quad \gamma_3 u = h \ \text{ on } \Gamma_3.$$

Similar $w := \mathscr{S}(f, g)$ and $z := \mathscr{T} h$ solve uniquely:

$$(\lambda + \mathscr{A})w = f \ \text{ in } \Omega, \quad \mathscr{B}_1 w = g \ \text{ on } \Gamma_1, \quad \gamma_2 w = 0 \ \text{ on } \Gamma_2, \quad \gamma_3 w = 0 \ \text{ on } \Gamma_3$$

and

$$(\lambda + \mathscr{A})z = 0 \ \text{ in } \Omega, \quad \mathscr{B}_1 z = 0 \ \text{ on } \Gamma_1, \quad \gamma_2 z = 0 \ \text{ on } \Gamma_2, \quad \gamma_3 z = h \ \text{ on } \Gamma_3,$$

respectively.

The formal adjoint operators $\mathscr{R}^{\#}$, $\mathscr{S}^{\#}$ and $\mathscr{T}^{\#}$ are defined analogously. With these notations we are able to formulate the following a priori estimates for the operator $\mathscr{B}_3 \mathscr{T}$ and its formal adjoint $\mathscr{B}_3^{\#} \mathscr{T}^{\#}$:

Theorem 1.1 *There exist constants $c > 0$, $\mu_* > 0$ and $\beta \in (0, \pi/2)$ such that*

(a) $\mu + \mathscr{B}_3 \mathscr{T} \in \mathrm{Isom}(B_p^2(\Gamma_3), B_p^1(\Gamma_3)), \qquad \mu + \mathscr{B}_3^{\#} \mathscr{T}^{\#} \in \mathrm{Isom}(B_{p'}^2(\Gamma_3), B_{p'}^1(\Gamma_3)),$

(b)
$$|\mu| \, \|z\|_{3,1,p} + \|z\|_{3,2,p} \leqq c \, \|(\mu + \mathscr{B}_3 \mathscr{T})z\|_{3,1,p},$$
$$|\mu| \, \|\psi\|_{3,1,p'} + \|\psi\|_{3,2,p'} \leqq c \, \|(\mu + \mathscr{B}_3^{\#} \mathscr{T}^{\#})\psi\|_{3,1,p'},$$

for all $\mu \in S(\beta, \mu_*)$, $z \in B_p^2(\Gamma_3)$ and $\psi \in B_{p'}^2(\Gamma_3)$.

Proof. This follows by obvious modifications of the proof of Theorem 1.11 and of Example 1.2 in [18]. □

As an immediate consequence of Theorem 1.1 we have

Corollary 1.2 *$-\mathscr{B}_3 \mathscr{T}$ (resp. $-\mathscr{B}_3^{\#} \mathscr{T}^{\#}$) generates an analytic semigroup on $B_p^1(\Gamma_3)$ (resp. $B_{p'}^1(\Gamma_3)$).*

Proof. Using Theorem 1.1 (a), we see that $\mathscr{B}_3 \mathscr{T}$ is a closed operator in $B_p^1(\Gamma_3)$. *Since* $D(\mathscr{B}_3 \mathscr{T}) = B_p^2(\Gamma_3)$ is dense in $B_p^1(\Gamma_3)$, the assertion follows from Theorem 1.1 (b) and a well known characterization for generators of analytic semigroups (cf. Theorem 4.2.1 in [13]). □

Let E, F be Banach spaces and suppose that $L \in \mathscr{L}(E, F)$. Then we denote by $L^* \in \mathscr{L}(F', E')$ the dual operator of L. Suppose in addition that E is densely embedded in F. If we consider L as a – in general unbounded – linear operator in F, then its dual operator

$$L': D(L') \subset F' \to F' \quad \text{with}$$
$$D(L') := \{y' \in F'; \text{there is a } x' \in F' \text{ such that } \langle y', Lx \rangle = \langle x', x \rangle, \, x \in E\}, \; L' y' := x',$$

is also well defined.

Again, we consider a linear operator $L: D(L) \subset F \to F$ and we suppose that $E \hookrightarrow F$. Then the E-realization L_E is the linear operator in E, given by $D(L_E) := \{x \in D(L) \cap E; Lx \in E\}$ and $L_E x = Lx$, $x \in D(L_E)$. Finally, we denote in the following by $[\cdot, \cdot]_\theta$, $\theta \in (0, 1)$, the standard complex interpolation functor. We refer to [8], [19] or [27] for the general theory of interpolation. For interpolation in Hilbert spaces see also [22].

The next lemma shows that $(\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*$ restricts to a bounded linear operator from $B_p^2(\Gamma_3)$ to $B_p^1(\Gamma_3)$. This implies in particular that $(\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*$ is a morphism from the couple $(B_p^0(\Gamma_3), B_p^2(\Gamma_3))$ to the couple $(B_p^{-1}(\Gamma_3), B_p^1(\Gamma_3))$ of the category $\mathscr{C}_1$ of compatible couples in the sense of [8].

Lemma 1.3 $(\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*|B_p^2(\Gamma_3)=\mathscr{B}_3\mathscr{T}$.

Proof. Choose $z\in D(\mathscr{B}_3\mathscr{T})=B_p^2(\Gamma_3)\hookrightarrow B_p^0(\Gamma_3)=D((\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*)$ and $\psi\in B_{p'}^2(\Gamma_3)$ arbitrarily and put $u:=\mathscr{T}z\in H_p^2(\Omega)$, $\varphi:=\mathscr{T}^{\#}\psi\in H_{p'}^2(\Omega)$. Then we have $(\lambda+\mathscr{A},\mathscr{B}_1,\gamma_2)u=0$ and $(\lambda+\mathscr{A}^{\#},\mathscr{B}_1^{\#},\gamma_2^{\#})\varphi=0$. Thus the divergence theorem implies

$$\langle\psi,\mathscr{B}_3\mathscr{T}z\rangle_3=\langle\mathscr{B}_3^{\#}\mathscr{T}^{\#}\psi,z\rangle_3=\langle\psi,(\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*z\rangle_3.$$

Since $B_{p'}^2(\Gamma_3)$ is dense in $B_{p'}^0(\Gamma_3)=(B_p^1(\Gamma_3))'$, we obtain $\mathscr{B}_3\mathscr{T}z=(\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*z$. □

Now we define $\mathbb{B}_{-1/2}:=B_p^0(\Gamma_3)$-realization of $(\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*$ and $\mathbb{B}_{1/2}^{\#}:=\mathscr{B}_3^{\#}\mathscr{T}^{\#}$. Using the fact that $(B_p^s)'=B_{p'}^{1-s}$, $s\in\mathbb{R}$, cf. [8, Corollary 6.2.8] and the fact that the spaces B_p^s are stable under complex interpolation, i.e. $[B_p^s,B_p^t]_\theta=B_p^{s\theta+t(1-\theta)}$, $s,t\in\mathbb{R}$, $\theta\in(0,1)$, cf. [8, Theorem 6.4.5], we can prove the following

Lemma 1.4 *$(\mathbb{B}_{-1/2})'=\mathbb{B}_{1/2}^{\#}$ and $-\mathbb{B}_{-1/2}$ generates an analytic semigroup on $B_p^0(\Gamma_3)$.*

Proof. By the definition of $\mathbb{B}_{-1/2}$ it follows that

$$\langle\psi,\mathbb{B}_{-1/2}z\rangle_3=\langle\mathbb{B}_{1/2}^{\#}\psi,z\rangle_3,\ z\in D(\mathbb{B}_{-1/2}),\ \psi\in D(\mathbb{B}_{1/2}^{\#})$$

and consequently we have $(\mathbb{B}_{-1/2})'\supset\mathbb{B}_{1/2}^{\#}$. From Theorem 1.1(a) we know that

$$\mu_*+(\mathscr{B}_3^{\#}\mathscr{T}^{\#})^*\in\mathrm{Isom}(B_p^0(\Gamma_3),B_p^{-1}(\Gamma_3)).$$

Thus it follows from Lemma 1.3, Theorem 1.1(a) and by interpolation that

$$\mu_*+\mathbb{B}_{-1/2}\in\mathrm{Isom}(B_p^1(\Gamma_3),B_p^0(\Gamma_3)),$$

and therefore $(\mu_*+\mathbb{B}_{-1/2})'=\mu_*+(\mathbb{B}_{-1/2})'$ is also one to one. This, together with Theorem 1.1(a), implies that $(\mathbb{B}_{-1/2})'$ cannot be a proper extension of $\mathbb{B}_{1/2}^{\#}$, which proves the first assertion. Since $\mathbb{B}_{-1/2}$ is closed in $B_p^0(\Gamma_3)$, it follows that $\mathbb{B}_{-1/2}=(\mathbb{B}_{1/2}^{\#})'$. Now Corollary 1.2 and the reflexivity of $B_{p'}^1(\Gamma_3)$ imply the second assertion. □

In the following we denote by $\{(\mathbb{B}_{\alpha-1/2},E_\alpha);\alpha\in\mathbb{R}\}$ and by $\{(\mathbb{B}_{\alpha+1/2}^{\#},E_\alpha^{\#});\alpha\in\mathbb{R}\}$ the scale of generators constructed by Amann [4, 5] starting with $E:=B_p^0(\Gamma_3)$, $A:=\mathbb{B}_{-1/2}$ and $E^{\#}:=B_{p'}^1(\Gamma_3)$, $A^{\#}:=\mathbb{B}_{1/2}^{\#}$, respectively. For completeness, we list some of the most important properties of $(\mathbb{B}_{\alpha-1/2},E_\alpha)$ which will be used below.

Theorem 1.5 *Let $-\infty<\beta<\alpha<\infty$. Then*

(a) *$E_\alpha=B_p^\alpha(\Gamma_3)$ and $\mathbb{B}_{\alpha-1/2}$ is the $B_p^\alpha(\Gamma_3)$-realization of $\mathbb{B}_{\beta-1/2}$.*
(b) *$-\mathbb{B}_{\alpha-1/2}$ generates an analytic semigroup $\{e^{-t\mathbb{B}_{\alpha-1/2}};t\geqq 0\}$ on $B_p^\alpha(\Gamma_3)$ and $e^{-t\mathbb{B}_{\alpha-1/2}}=e^{-t\mathbb{B}_{\beta-1/2}}|B_p^\alpha(\Gamma_3)$.*
(c) *$\mu+\mathbb{B}_{\alpha-1/2}\in\mathrm{Isom}(B_p^{\alpha+1}(\Gamma_3),B_p^\alpha(\Gamma_3))$, $\mu\geqq\mu_*$.*
(d) *$(E_\alpha)'=E_{-\alpha}^{\#}$ and $(\mathbb{B}_\alpha)'=\mathbb{B}_{-\alpha}^{\#}$ (as unbounded operators).*

2 The parabolic fundamental system

In this section we return to the time dependent case. Let us first recall the concept of parabolic fundamental systems. To this end, we fix $T>0$ and put $T_\Delta := \{(t,s)\in\mathbb{R}^2; 0\leqq s\leqq t\leqq T\}$. Let E_0, E_1 be Banach spaces with $E_1 \hookrightarrow E_0$ and assume that $\{B(t); t\in[0,T]\}$ is a given family of closed linear operators in E_0 such that $D(B(t))=E_1$ for $t\in[0,T]$.

Then for $(f,x,s)\in C([0,T],E_0)\times E_0\times[0,T]$ we consider the following linear Cauchy problem in E_0:

$$\text{(C)} \qquad \dot z + B(t)z = f(t), \quad s<t\leqq T, \quad z(s)=x.$$

We say z is a solution of (C) iff $z\in C([s,T],E_0)\cap C^1((s,T],E_0)\cap C((s,T],E_1)$ and z satisfies (C) pointwise on $[s,T]$.

A function V: $T_\Delta \to \mathscr{L}(E_0)$ is said to be a *parabolic fundamental system* for the family $\{B(t); t\in[0,T]\}$ if

$$(\mathrm{P}_1) \qquad V\in C(T_\Delta, \mathscr{L}_s(E_0))\cap C(T_\Delta, \mathscr{L}_s(E_1)) \text{ such that } \sup\{(t-s)\,\|B(t)\,V(t,s)\|_{\mathscr{L}(E_0)}; 0\leqq s<t\leqq T\}<\infty.$$

$$(\mathrm{P}_2) \qquad \text{If } (f,x,s)\in C([0,T],E_0)\times E_0\times[0,T] \text{ and } z \text{ is a solution of (C), then } z(t)=V(t,s)x+\int_s^t V(t,\tau)f(\tau)\,d\tau,\ t\in[s,T].$$

The space E_1 is called *regularity subspace* for V.

The main result of this section is Corollary 2.2, which shows that for $2\theta\in(1/p, 1+1/p)$ the family $\{\mathbb{B}_{2\theta-3/2}(t); t\in[0,T]\}$ possesses a unique parabolic fundamental system. For this purpose, we need some basic properties of the so called Lions-Magenes extension for nonhomogeneous elliptic boundary value problems. For a proof of the following facts see [21] or [5].

First we note that $\mathscr{A}(t)$ as well as $\mathscr{A}^\#(t)$, considered as unbounded operators in $L_p(\Omega)$ and $L_{p'}(\Omega)$ respectively, are closable. We denote their closures by $\bar{\mathscr{A}}(t)$ and $\bar{\mathscr{A}}^\#(t)$, respectively. Furthermore there exist extensions

$$(\bar\gamma_i(t), \bar{\mathscr{B}}_i(t))\in\mathscr{L}(D(\bar{\mathscr{A}}(t)), B_p^0(\Gamma_i)\times B_p^{-1}(\Gamma_i)),$$
$$(\bar\gamma_i^\#(t), \bar{\mathscr{B}}_i^\#(t))\in\mathscr{L}(D(\bar{\mathscr{A}}^\#(t)), B_{p'}^0(\Gamma_i)\times B_{p'}^{-1}(\Gamma_i))$$

of $(\gamma_i, \mathscr{B}_i(t))$ and $(\gamma_i^\#, \mathscr{B}_i^\#(t))$, $1\leqq i\leqq 3$, respectively, such that the following generalized Green formulas hold:

$$\begin{aligned}(2.1)\qquad &\langle\varphi, \bar{\mathscr{A}}(t)u\rangle + \langle\gamma_1^\#\varphi, \bar{\mathscr{B}}_1(t)u\rangle_1 - \langle\mathscr{B}_2^\#(t)\varphi, \bar\gamma_2(t)u\rangle_2 + \langle\gamma_3^\#\varphi, \bar{\mathscr{B}}_3(t)u\rangle_3\\ &= \langle\mathscr{A}^\#(t)\varphi, u\rangle + \langle\mathscr{B}_1^\#(t)\varphi, \bar\gamma_1(t)u\rangle_1 - \langle\gamma_2^\#\varphi, \bar{\mathscr{B}}_2(t)u\rangle_2\\ &\quad + \langle\mathscr{B}_3^\#(t)\varphi, \bar\gamma_3(t)u\rangle_3,\end{aligned}$$

$t\in[0,T]$, $\varphi\in H_{p'}^2(\Omega)$, $u\in D(\bar{\mathscr{A}}(t))$,

$$\begin{aligned}(2.2)\qquad &\langle\varphi, \mathscr{A}(t)u\rangle + \langle\bar\gamma_1^\#(t)\varphi, \mathscr{B}_1(t)u\rangle_1 - \langle\bar{\mathscr{B}}_2^\#(t)\varphi, \gamma_2(t)u\rangle_2 + \langle\bar\gamma_3^\#(t)\varphi, \mathscr{B}_3(t)u\rangle_3\\ &= \langle\bar{\mathscr{A}}^\#(t)\varphi, u\rangle + \langle\bar{\mathscr{B}}_1^\#(t)\varphi, \gamma_1(t)u\rangle_1 - \langle\bar\gamma_2^\#(t)\varphi, \mathscr{B}_2(t)u\rangle_2 + \langle\bar{\mathscr{B}}_3^\#(t)\varphi, \gamma_3(t)u\rangle_3,\end{aligned}$$

$t\in[0,T]$, $\varphi\in D(\bar{\mathscr{A}}^\#(t))$, $u\in H_p^2(\Omega)$.

Moreover, we have

(2.3) $$(\lambda+\bar{\mathscr{A}}(t), \bar{\mathscr{B}}_1(t), \bar{\gamma}_2(t), \bar{\gamma}_3(t)) \in \mathrm{Isom}(D(\bar{\mathscr{A}}(t)), L_p(\Omega) \times B_p^{-1}(\Gamma_1) \times B_p^0(\Gamma_2) \times B_p^0(\Gamma_3)),$$
$$(\lambda+\bar{\mathscr{A}}^{\#}(t), \bar{\mathscr{B}}_1^{\#}(t), \bar{\gamma}_2^{\#}(t), \bar{\gamma}_3^{\#}(t)) \in \mathrm{Isom}(D(\bar{\mathscr{A}}^{\#}(t)), L_{p'}(\Omega) \times B_{p'}^{-1}(\Gamma_1) \times B_{p'}^0(\Gamma_2) \times B_{p'}^0(\Gamma_3)),$$

for $t\in[0, T]$, $\lambda \geqq \lambda_*$.

Due to (1.6), (2.3) and $D(\bar{\mathscr{A}}(t)) \hookrightarrow L_p(\Omega)$, we obtain by interpolation (cf. [5, Theorem 6.3]):

(2.4) $$\mathscr{R} \in C^\rho([0, T], \mathscr{L}(L_p(\Omega) \times B_p^{2\theta-1}(\Gamma_1) \times B_p^{2\theta}(\Gamma_3), H_p^{2\theta}(\Omega))),$$
$$\mathscr{S} \in C^\rho([0, T], \mathscr{L}(L_p(\Omega) \times B_p^{2\theta-1}(\Gamma_1), H_p^{2\theta}(\Omega))),$$
$$\mathscr{T} \in C^\rho([0, T], \mathscr{L}(B_p^{2\theta}(\Gamma_3), H_p^{2\theta}(\Omega))), \theta\in[0, 1].$$

Analogous results hold for $\mathscr{R}^{\#}$, $\mathscr{S}^{\#}$ and $\mathscr{T}^{\#}$.

Next we define for $t\in[0, T]$, $2\theta\in(1/p, 1+1/p)$, $\varphi\in H_{p'}^{2(1-\theta)}(\Omega)$ and $u\in H_p^{2\theta}(\Omega)$ the following time dependent bilinear form:

$$a(t)(\varphi, u) := \int_\Omega [(\partial_j \varphi | a_{jk}(t, \cdot)\, \partial_k u) + (\varphi | a_j(t, \cdot)\, \partial_j u + a_0(t, \cdot) u)]\, dx + \int_{\Gamma_1} (\gamma_1^{\#} \varphi | b_1(t, \cdot)\, \gamma_1 u)\, d\sigma + \int_{\Gamma_3} (\gamma_3^{\#} \varphi | b_3(t, \cdot)\, \gamma_3 u)\, d\sigma.$$

The bilinear form $a(t)$ is called the *Dirichlet form* associated to the elliptic boundary value problem $(\mathscr{A}(t); \mathscr{B}_1(t); \gamma_2; \mathscr{B}_3(t))$, $t\in[0, T]$.

We have (cf. [5, Sect. 13]):

(2.5) $$a \in C^\rho([0, T], \mathscr{L}(H_{p'}^{2(1-\theta)}(\Omega), H_p^{2\theta}(\Omega); \mathbb{R}))$$

as well as

(2.6) $$a(t)(\varphi, u) = \langle \mathscr{A}^{\#}(t)\varphi, u\rangle,$$

for $t\in[0, T]$, $u\in H_p^2(\Omega)$ with $\gamma_2 u=0$ and $\varphi\in H_{p'}^2(\Omega)$ with $(\mathscr{B}_1^{\#}(t), \gamma_2^{\#}, \mathscr{B}_3^{\#}(t))\varphi=0$. Finally, we put $a_\lambda := a+\lambda\langle\cdot, \cdot\rangle$ for $\lambda\in\mathbb{R}$.

Lemma 2.1 *For $t\in[0, T]$, $2\theta\in(1/p, 1+1/p)$ and $\lambda\geqq\lambda_*$ we have*

(a) $a_\lambda(t)(\varphi, \mathscr{R}_\lambda(t)(f, g, z)) = \langle\varphi, f\rangle + \langle\gamma_1^{\#}\varphi, g\rangle_1 + \langle\gamma_3^{\#}\varphi, \mathbb{B}_{2\theta-3/2}(t) z + \mathscr{B}_3(t)\mathscr{S}_\lambda(t)(f, g)\rangle_3$, *for* $f\in L_p(\Omega)$, $g\in B_p^{2\theta-1}(\Gamma_1)$, $z\in B_p^{2\theta}(\Gamma_3)$ *and* $\varphi\in H_{p'}^{2(1-\theta)}(\Omega)$ *satisfying* $\gamma_2^{\#}\varphi=0$.

(b) $a_\lambda(t)(\mathscr{R}_\lambda^{\#}(t)(f, g, \psi), u) = \langle f, u\rangle + \langle g, \gamma_1 u\rangle_1 + \langle \mathbb{B}_{1/2-2\theta}^{\#}(t)\psi + \mathscr{B}_3^{\#}(t)\mathscr{S}_\lambda^{\#}(t)(f, g), \gamma_3 u\rangle_3$, *for* $f\in L_{p'}(\Omega)$, $g\in B_{p'}^{1-2\theta}(\Gamma_1)$, $\psi\in B_{p'}^{2(1-\theta)}(\Gamma_3)$ *and* $u\in H_p^{2\theta}(\Omega)$ *satisfying* $\gamma_2 u=0$.

Proof. Fix $t\in[0,T]$, $2\theta\in(1/p,1+1/p)$, $\lambda\geqq\lambda_*$, $f\in L_p(\Omega)$ and choose any $z\in B_p^2(\Gamma_3)$, $g\in B_p^1(\Gamma_1)$, $\varphi\in H_{p'}^2(\Omega)$ satisfying $(\mathscr{B}_1^\#(t),\gamma_2^\#,\mathscr{B}_3^\#(t))\varphi=0$. Since $\gamma_2\mathscr{R}_\lambda(t)(f,g,z)=0$, we conclude from (2.6) and the generalized Green formula (2.1):

$$\begin{aligned}a_\lambda(t)(\varphi,\mathscr{R}_\lambda(t)(f,g,z))&=\langle(\lambda+\mathscr{A}^\#(t))\varphi,\mathscr{R}_\lambda(t)(f,g,z)\rangle\\&=\langle\varphi,(\lambda+\bar{\mathscr{A}}(t))\mathscr{R}_\lambda(t)(f,g,z)\rangle+\langle\gamma_1^\#\varphi,\bar{\mathscr{B}}_1(t)\mathscr{R}_\lambda(t)(f,g,z)\rangle\\&\quad-\langle\mathscr{B}_2^\#(t)\varphi,\bar{\gamma}_2(t)\mathscr{R}_\lambda(t)(f,g,z)\rangle_2\\&\quad+\langle\gamma_3^\#\varphi,\bar{\mathscr{B}}_3(t)\mathscr{R}_\lambda(t)(f,g,z)\rangle_3.\end{aligned}$$

Thus by the definition of $\mathscr{R}_\lambda$, $\mathscr{S}_\lambda$, $\mathscr{T}$ and by Theorem 1.5(a) we obtain:

$$\begin{aligned}a_\lambda(t)(\varphi,\mathscr{R}_\lambda(t)(f,g,z))=\langle\varphi,f\rangle+\langle\gamma_1^\#\varphi,g\rangle_1+\langle\gamma_3^\#\varphi,\,&\mathbb{B}_{2\theta-3/2}(t)z\\&+\mathscr{B}_3(t)\mathscr{S}_\lambda(t)(f,g)\rangle_3.\end{aligned}$$

Now let $X^\#(t):=\ker((\mathscr{B}_1^\#(t),\gamma_2^\#,\mathscr{B}_3^\#(t))|H_{p'}^2(\Omega))$. Then by well known results of Seeley [23] we have:

$$X^\#(t)\overset{d}{\hookrightarrow}[L_{p'}(\Omega),X^\#(t)]_{1-\theta}=\ker(\gamma_2^\#|H_{p'}^{2(1-\theta)}(\Omega)).$$

Hence the first assertion follows from (2.5), (2.4), $B_p^1(\Gamma_1)\overset{d}{\hookrightarrow}B_p^{2\theta-1}(\Gamma_1)$, $B_p^2(\Gamma_3)\overset{d}{\hookrightarrow}B_p^{2\theta}(\Gamma_3)$, the trace theorem and Theorem 1.5(c). Assertion (b) is proven analogously. □

Using Lemma 2.1 it is now easy to prove

Corollary 2.2 *For each $2\theta\in(1/p,1+1/p)$ there is a unique parabolic fundamental system $U_{2\theta-3/2}$ for the family $\{\mathbb{B}_{2\theta-3/2}(t);t\in[0,T]\}$ possessing $B_p^{2\theta}(\Gamma_3)$ as regularity subspace. If, in addition, $1/p<2\theta<2\eta<1+1/p$ then $U_{2\eta-3/2}=U_{2\theta-3/2}|B_p^{2\eta-1}(\Gamma_3)$ and*

$$\sup\{(t-s)^{1+2\theta-2\eta}\|U_{2\theta-3/2}(t,s)\|_{\mathscr{L}(B_p^{2\eta-1}(\Gamma_3),B_p^{2\theta}(\Gamma_3))};0\leqq s<t\leqq T\}<\infty.$$

Similar results hold for the family $\{\mathbb{B}_{1/2-2\theta}^\#(t);t\in[0,T]\}$.

Proof. Due to Theorem 1.5 and well known results of Sobolevskii [25], Tanabe [26] and Amann [6], it is sufficient to prove that

$$\mathbb{B}_{2\theta-3/2}\in C^\tau([0,T],\mathscr{L}(B_p^{2\theta}(\Gamma_3),B_p^{2\theta-1}(\Gamma_3)))$$

for some $\tau\in(0,1)$.

Choose $\psi\in B_{p'}^{2(1-\theta)}(\Gamma_3)$, $z\in B_p^{2\theta}(\Gamma_3)$ arbitrarily and put $\varphi:=\mathscr{T}^\#(t)\psi\in H_{p'}^{2(1-\theta)}(\Omega)$. Since $\mathscr{R}(t)(0,0,z)=\mathscr{T}(t)z$, $\gamma_3^\#\varphi=\psi$ and $\gamma_2^\#\varphi=0$, it follows from Lemma 2.1(a) that

$$a_\lambda(t)(\mathscr{T}^\#(t)\psi,\mathscr{T}(t)z)=\langle\psi,\mathbb{B}_{2\theta-3/2}(t)z\rangle_3.$$

Now define $\hat{a}(t)(\psi,z):=a_\lambda(t)(\mathscr{T}^\#(t)\psi,\mathscr{T}(t)z)$ for $t\in[0,T]$. Then by (2.4) and (2.5) it follows that

$$\hat{a}\in C^\rho([0,T],\mathscr{L}(B_{p'}^{2(1-\theta)}(\Gamma_3),B_p^{2\theta}(\Gamma_3);\mathbb{R})).$$

On the other side, we also have

$$\|\hat{a}(t)\| = \|\mathbb{B}_{2\theta-3/2}(t)\|, \qquad t\in[0, T],$$

which proves the assertion. □

We mention another consequence of Lemma 2.1, which we need in the following section.

Corollary 2.3 *For* $2\theta\in(1/p, 1+1/p)$ *we have*

$$\mathscr{B}_3(\cdot)\mathscr{S}(\cdot)\in C^\rho([0, T], \mathscr{L}(L_p(\Omega)\times B_p^{2\theta-1}(\Gamma_1), B_p^{2\theta-1}(\Gamma_3))),$$
$$\mathscr{B}_3^{\#}(\cdot)\mathscr{S}^{\#}(\cdot)1\in C^\rho([0, T], \mathscr{L}(L_{p'}(\Omega)\times B_{p'}^{1-2\theta}(\Gamma_1), B_{p'}^{1-2\theta}(\Gamma_3))).$$

Proof. For $t\in[0, T]$, $\psi\in B_{p'}^{2(1-\theta)}(\Gamma_3)$ and $(f, g)\in L_p(\Omega)\times B_p^{2\theta-1}(\Gamma_1)$ we define

$$b(t)(\psi, (f,g)) := a_\lambda(t)(\mathscr{T}^{\#}(t)\psi, \mathscr{S}(t)(f,g)) - \langle\mathscr{T}^{\#}(t)\psi, f\rangle - \langle\gamma_1^{\#}\mathscr{T}^{\#}(t)\psi, g\rangle_1.$$

Then it follows from (2.4) and (2.5) that

$$b\in C^\rho([0, T], \mathscr{L}(B_{p'}^{2(1-\theta)}(\Gamma_3), [L_p(\Omega)\times B_p^{2\theta-1}(\Gamma_1)]; \mathbb{R})),$$

and from Lemma 2.1(a) we obtain

$$b(t)(\psi, (f, g)) = \langle\psi, \mathscr{B}_3(t)\mathscr{S}(t)(f, g)\rangle_3,$$

which proves the first assertion, since again $\|b(t)\| = \|\mathscr{B}_3(t)\mathscr{S}(t)\|$ for $t\in[0, T]$. The second assertion can be obtained analogously. □

3 Weak, mild, and classical solutions

In this section we introduce the notion of weak and mild solutions. While weak solutions take account of the variational structure of the problem under consideration, mild solutions are of some advantage in the study of nonlinear problems. However, we show that weak and mild solutions are in fact the same.

Throughout this section we assume that

(3.1) $$1\leq 2\theta < 2\eta < 1+1/p, \qquad z_0\in B_p^{2\theta}(\Gamma_3),$$
$$(f, g)\in C([0, T]\times H_p^{2\theta}(\Omega), \quad L_p(\Omega)\times B_p^{2\eta-1}(\Gamma_1)),$$

(3.2) $$h\in C([0, T]\times B_p^{2\theta}(\Gamma_3), B_p^{2\eta-1}(\Gamma_3)).$$

Then we consider the following abstract initial value problem

$$(\mathrm{P}_\lambda)\qquad \left.\begin{aligned} (\lambda+\mathscr{A}(t))u &= f(t, u)\\ \mathscr{B}_1(t)u &= g(t, u)\\ \gamma_2 u &= 0\\ (\gamma_3 u)^{\cdot} + \mathscr{B}_3(t)u &= h(t, \gamma_3 u)\end{aligned}\right\}\ 0<t\leq T,$$
$$(\gamma_3 u)(0) = z_0.$$

We say u is a *weak solution on* J of (P_λ) iff $u \in C(J, H_p^{2\theta}(\Omega))$, where J is a nontrivial subinterval of $[0, T]$ containing 0, such that $\gamma_2 u = 0$, $(\gamma_3 u)(0) = z_0$ and

$$\int_0^{T'} \{-\langle(\gamma_3^\# \varphi)^{\cdot}, \gamma_3 u\rangle_3 + a_\lambda(\cdot)(\varphi, u)\}\, dt$$

$$= \int_0^{T'} \{\langle\varphi, f(\cdot, u)\rangle + \langle\gamma_1^\# \varphi, g(\cdot, u)\rangle_1 + \langle\gamma_3^\# \varphi, h(\cdot, \gamma_3 u)\rangle_3\}\, dt + \langle(\gamma_3^\# \varphi)(0), z_0\rangle_3$$

for every $T' \in J \setminus \{0\}$ and every

$$\varphi \in C_{\theta, T'} := \{\psi \in C([0, T'], H_{p'}^{2(1-\theta)}(\Omega));\ \gamma_2^\# \psi = 0, (\gamma_3^\# \psi)(T') = 0 \text{ and } \gamma_3^\# \psi \in C^1([0, T'], B_{p'}^{1-2\theta}(\Gamma_3))\}.$$

Formally, we obtain this identity by multiplying the equations in (P_λ) with appropriate test functions and by integrating by parts with respect to time.

Suppose now that $0 \leqq s < T' \leqq T$, $z_0 \in B_p^{2\theta}(\Gamma_3)$ and $w \in C([s, T'], H_p^{2\theta}(\Omega))$. Then we define

$$z(s, z_0, w, t) := U(t, s)\, z_0 + \int_s^t U(t, \tau)\, F(\tau, w(\tau))\, d\tau, \qquad t \in [s, T'],$$

where $U := U_{2\theta - 3/2}$ and

$$F(t, w(t)) := h(t, \gamma_3 w(t)) - \mathscr{B}_3(t)\, \mathscr{S}(t)(f(t, w(t)), g(t, w(t))), \qquad t \in [s, T'].$$

Further, we put for $t \in [s, T']$:

$$K(s, z_0, w)(t) := \mathscr{R}(t)(f(t, w(t)), g(t, w(t)), z(s, z_0, w, t)).$$

Then u is said be a *mild solution on* J of (P_λ) iff $u \in C(J, H_p^{2\theta}(\Omega))$ and u satisfies

$$u(t) = K(0, z_0, u)(t), \qquad t \in J.$$

Finally, we say that u is a *classical solution on* J of (P_λ) iff $u \in C(J, C^2(\bar{\Omega}))$ such that $\gamma_3 u \in C^1(J \setminus \{0\}, C^1(\Gamma_3))$ and u satisfies (P_λ) pointwise on J.

A given solution u is said to be a *maximal* solution of (P_λ) if there is no proper extention of u. If u is a maximal solution on J of (P_λ), then J is the *maximal interval of existence.*

Proposition 3.1 (a) *u is a weak solution on J of* (P_λ) iff *u is a mild solution on J of* (P_λ).

(b) *Every classical solution of* (P_λ) *is a weak solution.*

Proof. (a) Suppose first that u is a mild solution of (P_λ) on J. Observe that due to (3.1), (3.2) and Corollary 2.3 we have

$$F(\cdot, u) \in C([0, T'], B_p^{2\eta - 1}(\Gamma_3)), \qquad T' \in J \setminus \{0\}. \tag{3.3}$$

Therefore [5, Theorem 9.3] and Corollary 2.2 imply that $z := z(0, z_0, \cdot)$ is the solution of the linear Cauchy problem

$$\dot{z} + \mathbb{B}_{2\theta - 3/2}(t) z = F(t, u), \qquad 0 < t \leqq T', \qquad z(0) = z_0. \tag{3.4}$$

In particular, z satisfies

$$(3.5)\quad \int_0^{T'} \{-\langle(\gamma_3^\# \varphi)^\cdot, z\rangle_3 + \langle\gamma_3^\# \varphi, \mathbb{B}_{2\theta-3/2}(\cdot)z\rangle_3\}\,dt = \int_0^{T'} \langle\gamma_3^\# \varphi, F(\cdot, u)\rangle_3\,dt + \langle(\gamma_3^\# \varphi)(0), z_0\rangle_3,$$

for all $T' \in J\setminus\{0\}$ and all $\varphi \in C_{\theta, T'}$.

On the other hand, we conclude from Lemma 2.1(a) and the definition of mild solutions:

$$(3.6)\quad \langle\gamma_3^\# \varphi, \mathbb{B}_{2\theta-3/2}(\cdot)z\rangle_3 = a_\lambda(\cdot)(\varphi, u) - \langle\varphi, f(\cdot, u)\rangle - \langle\gamma_1^\# \varphi, g(\cdot, u)\rangle_1 - \langle\gamma_3^\# \varphi, \mathscr{B}_3(\cdot)\mathscr{S}(\cdot)(f(\cdot, u), g(\cdot, u))\rangle_3.$$

Since $\gamma_3 u = \gamma_3 K(0, z_0, u) = z$, it follows from (3.5) and (3.6) that u is a weak solution on J.

If $u \in C(J, H_p^{2\theta}(\Omega))$ is a weak solution on J, we fix $T' \in J\setminus\{0\}$ and define $w(t) := K(0, z_0, u)(t)$, $t \in [0, T']$. Then $\gamma_3 w$ satisfies again (3.5) for each $\varphi \in C_{\theta, T'}$. Now let $v := u - w$. Then we deduce from (3.5), Lemma 2.1(a) and the definition of weak solutions the following identity

$$(3.7)\quad \int_0^{T'} \{-\langle(\gamma_3^\# \varphi)^\cdot, \gamma_3 v\rangle_3 + a_\lambda(\cdot)(\varphi, v)\}\,dt = 0, \qquad \varphi \in C_{\theta, T'}.$$

Choose any $r \in L_{p'}(\Omega)$, $k \in C([0, T'], \mathbb{R})$ and put

$$H(t) := -k(T'-t)\,\mathscr{B}_3^\#(T'-t)\,\mathscr{S}^\#(T'-t)(r, 0), \qquad t \in [0, T'].$$

Then $H \in C([0, T'], B_{p'}^{\frac{1}{p'}}(\Gamma_3))$, by the formal adjoint analogue of (1.6). Thus the linear Cauchy problem

$$\dot{\psi} + \mathbb{B}_{1/2-2\theta}^\#(T'-t)\psi = H, \qquad 0 < t \leqq T', \qquad \psi(0) = 0$$

possesses a unique solution

$$\psi \in C([0, T'], B_{p'}^{2(1-\theta)}(\Gamma_3)) \cap C^1([0, T'], B_{p'}^{\frac{1}{p'}-2\theta}(\Gamma_3)).$$

Now define

$$\varphi(t) := k(t)\,\mathscr{S}^\#(t)(r, 0) + \mathscr{T}^\#(t)\,\psi(T'-t), \qquad t \in [0, T'].$$

Then $\varphi \in C_{\theta, T'}$ and

$$(\gamma_3^\# \varphi)^\cdot(t) = \mathbb{B}_{1/2-2\theta}^\#(t)\,\gamma_3^\# \varphi(t) - H(T'-t), \qquad t \in [0, T'].$$

This last identity implies, together with (3.7) and Lemma 2.1(b), that

$$\int_0^{T'} k\langle r, v\rangle\,dt = 0$$

for all $k \in C([0, T'], \mathbb{R})$ and all $r \in L_{p'}(\Omega)$. Consequently, $v = u - K(0, z_0, u) = 0$, which means that u is a mild solution of (P_λ) on $[0, T']$.

(b) Assume that $u \in C(J, C^2(\bar{\Omega}))$ is a classical solution of (P_λ). Fix $T' \in J \setminus \{0\}$ and let $z := \gamma_3 u$, $w(t) := \mathscr{S}(t)(f(t, u(t)), g(t, u(t)))$, $t \in [0, T']$. Then it follows from (P_λ) that $u = w + \mathscr{T}(\cdot) z$. Further we have

$$(\gamma_3 u)^{\cdot} + \mathscr{B}_3(\cdot) u = \dot{z} + \mathscr{B}_3(\cdot) \mathscr{T}(\cdot) z + \mathscr{B}_3(\cdot) w = h(\cdot, \gamma_3 u).$$

Thus it follows from Theorem 1.5(a) that

$$\dot{z} + \mathbb{B}_{2\theta - 3/2}(t) z = F(t, u), \quad 0 < t \leqq T', \quad z(0) = z_0. \tag{3.8}$$

Since the linear Cauchy problem (3.8) possesses the unique solution $z(0, z_0, u, \cdot)$, we have $z = z(0, z_0, u, \cdot)$. Now it follows from $u = w + \mathscr{T}(\cdot) z$ that u is a mild solution of (P_λ). □

4 Existence and uniqueness results

Throughout this section we assume that

$$p \geqq 2, \quad 1 \leqq 2\theta < 2\eta < 1 + 1/p, \quad z_0 \in B_p^{2\theta}(\Gamma_3). \tag{4.1}$$

$$(f, g) \in C^{0,1}([0, T] \times H_p^{2\theta}(\Omega), L_p(\Omega) \times B_p^1(\Gamma_1)) \quad \text{with} \tag{4.2}$$
$$|\partial_2 f(t, u)| + |\partial_2 g(t, u)| \leqq \hat{\lambda}, \quad (t, u) \in [0, T] \times H_p^{2\theta}(\Omega).$$

$$h \in C^{0,1}([0, T] \times B_p^{2\theta}(\Gamma_3), B_p^{2\eta - 1}(\Gamma_3)) \quad \text{and} \quad h(t, \cdot) \tag{4.3}$$

is bounded on bounded subsets of $B_p^{2\theta}(\Gamma_3)$, uniformly in t.

There exists a $\bar{\lambda}_0 \in \mathbb{R}$ such that

$$\begin{aligned} & v \in H_p^2(\Omega), u \in H_p^{2\theta}(\Omega), t \in [0, T], \lambda > \bar{\lambda}_0 \quad \text{and} \\ & (\lambda - \partial_2 f(t, u) + \mathscr{A}(t) - \partial_2 g(t, u) + \mathscr{B}_1(t), \gamma_2, \gamma_3) v = 0 \\ & \text{imply} \quad v = 0. \end{aligned} \tag{4.4}$$

We first prove the following *local* existence, uniqueness and continuity result for problem (P_λ).

Proposition 4.1 *Suppose that $A \subset B_p^{2\theta}(\Gamma_3)$ is compact. Then there exists a $\tilde{\lambda}_0 \geqq \lambda_*$ (not depending on A) and a bounded closed neighbourhood V in $B_p^{2\theta}(\Gamma_3)$ of A with the following property:*

For each $s_0 \in [0, T)$ and each $\lambda \geqq \tilde{\lambda}_0$ there is a $\delta \in (0, T - s_0)$ such that $K(s, \tilde{z}, \cdot)$ possesses for each $s \in [0, s_0]$ and each $\tilde{z} \in V$ a unique fixed point $u(\cdot, \tilde{z}, s, \lambda)$ in $C([s, s + \delta], H_p^{2\theta}(\Omega))$. Furthermore, there is a positive constant c such that

$$|u(t, \tilde{z}, s, \lambda) - u(t, \hat{z}, s, \lambda)|_{2\theta, p} \leqq c \, \|\tilde{z} - \hat{z}\|_{3, 2\theta, p}$$

for all $s \in [0, s_0]$, $t \in [s, s + \delta]$, $\tilde{z}, \hat{z} \in V$.

Proof. From (4.2) we deduce the existence of $L_1 > 0$ with

$$|(f, g)(t, u) - (f, g)(t, v)|_{L_p(\Omega) \times B_p^1(\Gamma_1)} \leqq L_1 \, |u - v|_{2\theta, p} \tag{4.5}$$

for all $u, v \in H_p^{2\theta}(\Omega)$ and $t \in [0, T]$.

Further it follows from (1.7) that

$$\lambda|\mathcal{S}_\lambda(t)(\hat{f},\hat{g})|_p+|\mathcal{S}_\lambda(t)(\hat{f},\hat{g})|_{2,p}\leqq c(|\hat{f}|_p+(1+\lambda)^{\frac{1}{2p'}}\|\hat{g}\|_{1,1,p}),$$

for $(\hat{f},\hat{g})\in L_p(\Omega)\times B^1_p(\Gamma_1)$, $\lambda\geqq\lambda_*$ and $t\in[0,T]$. Thus by interpolation we obtain

$$|\mathcal{S}_\lambda(t)|_{\mathcal{L}(L_p(\Omega)\times B^1_p(\Gamma_1),H^{2\theta}_p(\Omega))}\leqq c(1+\lambda)^{\frac{1}{2p'}}\lambda^{\theta-1},$$

$t\in[0,T]$, $\theta\in[0,1]$, $\lambda\geqq\lambda_*$.

Let $\beta\in(0,1)$ be given. Since $2\theta<1+1/p=2-1/p'$, there exists a $\tilde{\lambda}_0\geqq\lambda_*$ such that

$$|\mathcal{S}_\lambda(t)|_{\mathcal{L}(L_p(\Omega)\times B^1_p(\Gamma_1),H^{2\theta}_p(\Omega))}\leqq\frac{1-\beta}{L_1}, \tag{4.6}$$

$t\in[0,T]$, $\lambda\geqq\tilde{\lambda}_0$.

Now we fix $\lambda\geqq\tilde{\lambda}_0$ and $s_0\in[0,T)$. From (4.3) and the compactness of A it follows that there are constants $L_2>0$ and $r>0$ with

$$\|F(t,u_1)-F(t,u_2)\|_{3,2\eta-1,p}\leqq L_2|u_1-u_2|_{2\theta,p}, \tag{4.7}$$

for all $t\in[0,T]$ and all $u_i\in H^{2\theta}_p(\Omega)$ satisfying $\gamma_3 u_i\in\bar{\mathbb{B}}_{B^{2\theta}_p(\Gamma_3)}(A,3r)$, $i=1,2$.

Now we observe that $U\in C(T_\Delta,\mathcal{L}_s(B^{2\theta}_p(\Gamma_3)))$ and the uniform boundedness principle imply that $\sup\{\|U(t,s)\|_{\mathcal{L}(B^{2\theta}_p(\Gamma_3))};\ (t,s)\in T_\Delta\}<\infty$. Using this fact and again $U\in C(T_\Delta,\mathcal{L}_s(B^{2\theta}_p(\Gamma_3)))$, the compactness of A ensures the existence of a $\delta_1\in(0,T-s_0)$ and of a bounded closed neighbourhood V in $B^{2\theta}_p(\Gamma_3)$ of A such that

$$V\subset\bar{\mathbb{B}}_{B^{2\theta}_p(\Gamma_3)}(A,r), \tag{4.8}$$

$$\|U(t,s)\tilde{z}-\tilde{z}\|_{3,2\theta,p}\leqq r,\quad s\in[0,s_0],\quad t\in[s,s+\delta_1],\quad \tilde{z}\in V. \tag{4.9}$$

Next we define $\alpha:=\max\{\|\mathcal{T}(t)\|_{\mathcal{L}(B^{2\theta}_p(\Gamma_3),H^{2\theta}_p(\Omega))};\ t\in[0,T]\}<\infty$,

$$k:=\frac{1-\beta}{\beta}(2\alpha r+\sup\{\alpha\|\tilde{z}\|_{3,2\theta,p}+|f(t,\ 0)|_p/L_1+\|g(t,0)\|_{1,1,p}/L_1;\quad t\in[0,T],$$
$$\tilde{z}\in V\})<\infty,$$

$$\mathcal{M}:=\bigcup_{\tilde{z}\in V}\{u\in H^{2\theta}_p(\Omega);\ |u-\mathcal{T}(t)\tilde{z}|\leqq 2\alpha r+k,\ t\in[0,T]\}.$$

Note that $\mathcal{M}$ is bounded in $H^{2\theta}_p(\Omega)$. Thus it follows from (4.3), (4.5) and Corollary 2.3 that $M:=\sup\{\|F(t,u)\|_{3,2\eta-1,p};\ (t,u)\in[0,T]\times\mathcal{M}\}$ is finite. By Corollary 2.2 there is a $\delta_2\in(0,T-s_0)$ such that

(4.10)
$$\int_s^t\|u(t,\tau)\|_{\mathcal{L}(B^{2\eta-1}_p(\Gamma_3),B^{2\theta}_p(\Gamma_3))}\,d\tau\leqq\min\left\{\frac{\beta}{2\alpha L_2},\frac{r}{M}\right\},\qquad s\in[0,s_0],\qquad t\in[s,s+\delta_2].$$

Now let $\delta:=\delta_1\wedge\delta_2$ and define for $s\in[0,s_0]$ and $\tilde{z}\in V$:

$$X:=\{u\in C([s,s+\delta],H^{2\theta}_p(\Omega));\ \|(\gamma_3 u)(t)-U(t,s)\tilde{z}\|_{3,2\theta,p}\leqq r,\ |u(t)-(\mathcal{T}\gamma_3 u)(t)|_{2\theta,p}$$
$$\leqq k,\ t\in[s,s+\delta]\}.$$

Then X is a complete metric space and for $u \in X$ we have $z(s, \tilde{z}, u, \cdot) \in C([s, s+\delta], B_p^{2\theta}(\Gamma_3))$, due to (3.3) and Theorem 9.3 in [5]. Consequently, it follows from (2.4) that

$$K(s, \tilde{z}, u) \in C([s, s+\delta], H_p^{2\theta}(\Omega)).$$

Furthermore, we have for $u \in X$:

$$|u(t) - \mathscr{T}(t)\tilde{z}|_{2\theta,p} \leqq 2\alpha r + k, \qquad t \in [s, s+\delta], \tag{4.11}$$

since (4.9) implies that $|(\mathscr{T}\gamma_3 u)(t) - \mathscr{T}(t)\tilde{z}|_{2\theta,p} \leqq 2\alpha r$. Now using the identity $\gamma_3 K(s, \tilde{z}, u) = z(s, \tilde{z}, u, \cdot)$ it follows from (4.10) and the definition of M:

$$\|\gamma_3 K(s, \tilde{z}, u)(t) - U(t,s)\tilde{z}\|_{3,2\theta,p} \leqq r, \qquad t \in [s, s+\delta].$$

Next we note that $|u(t)|_{2\theta,p} \leqq k + \alpha(2r + \|\tilde{z}\|_{3,2\theta,p})$, $t \in [s, s+\delta]$, by (4.11). Thus it follows from (4.5) and (4.6) that

$$\begin{aligned}
|K(s, \tilde{z}, u)(t) - (\mathscr{T}\gamma_3 K(s, \tilde{z}, u))(t)|_{2\theta,p} &= |\mathscr{S}(t)(f(t, u(t)), g(t, u(t)))|_{2\theta,p} \\
&\leqq (1-\beta)(|u(t)|_{2\theta,p} + |f(t,0)|_p/L_1 + \|g(t,0)\|_{1,1,p}/L_1) \\
&\leqq (1-\beta)k + (1-\beta)(2\alpha r + \alpha\|\tilde{z}\|_{3,2\theta,p} + |f(t,0)|_p/L_1 + \|g(t,0)\|_{1,1,p}/L_1) \leqq k,
\end{aligned}$$

$t \in [s, s+\delta]$. Hence we have $K(s, \tilde{z}, X) \subset X$.

Finally, take $u_1, u_2 \in X$. Then (4.9) shows that

$$\|(\gamma_3 u_i)(t) - \tilde{z}\|_{3,2\theta,p} \leqq 2r, \qquad t \in [s, s+\delta], \qquad i = 1, 2.$$

Therefore (4.6), (4.7), (4.8) and (4.10) imply

$$\begin{aligned}
&|K(s, \tilde{z}, u_1)(t) - K(s, \tilde{z}, u_2)(t)|_{2\theta,p} \\
&\leqq (1-\beta)|u_1(t) - u_2(t)|_{2\theta,p} + \alpha \int_s^t \|U(t,\tau)\| \, \|F(\tau, u_1(\tau)) - F(\tau, u_2(\tau))\|_{3,2\eta-1,p} \, d\tau \\
&\leqq \left(1 - \frac{\beta}{2}\right) \|u_1 - u_2\|_X, \qquad t \in [s, s+\delta].
\end{aligned}$$

Now the first assertion follows from Banach's fixed point theorem.

To prove the second assertion, let $s \in [0, s_0]$, $\tilde{z}_i \in V$ be given and denote by $v_i := u(\cdot, \tilde{z}_i, s, \lambda)$, $i = 1, 2$ the unique fixed points of $K(s, \tilde{z}, \cdot)$ in X. Since $\gamma_3 v_i(t) \in \mathbb{B}_{B_p^{2\theta}(\Gamma_3)}(A, 3r)$ for $t \in [s, s+\delta]$, it follows from (4.5)–(4.7), the definition of $z(s, \tilde{z}, v_i, \cdot)$ and Corollary 2.2 that for each $t \in [s, s+\delta]$:

$$\begin{aligned}
|v_1(t) - v_2(t)|_{2\theta,p} &= |K(s, \tilde{z}_1, v_1)(t) - K(s, \tilde{z}_2, v_2)(t)|_{2\theta,p} \\
&\leqq |\mathscr{S}(t)(f(v_1(t)) - f(v_2(t)), g(v_1(t)) - g(v_2(t)))|_{2\theta,p} \\
&\quad + |\mathscr{T}(t)(z(s, \tilde{z}_1, v_1, t) - z(s, \tilde{z}_2, v_2, t))|_{2\theta,p} \\
&\leqq (1-\beta)|v_1(t) - v_2(t)|_{2\theta,p} + \alpha\|U(t,s)(\tilde{z}_1 - \tilde{z}_2)\|_{3,2\theta,p} \\
&\quad + \alpha L_2 \int_s^t (t-\tau)^{2(\eta-\theta)-1} |v_1(\tau) - v_2(\tau)|_{2\theta,p} \, d\tau.
\end{aligned}$$

Consequently, we find constants $c_1, c_2>0$ such that for $t\in[s, s+\delta]$

$$|v_1(t)-v_2(t)|_{2\theta,p} \leqq c_1 \|\tilde{z}_1-\tilde{z}_2\|_{3,2\theta,p}+c_2 \int_s^t (1-\tau)^{2(\eta-\theta)-1} |v_1(\tau)-v_2(\tau)|_{2\theta,p}\, d\tau.$$

Since $\eta>\theta$, the assertion now follows from a generalized Gronwall inequality, cf. Lemma 3.3 in [3]. □

Theorem 4.2 *There is a* $\lambda_0 \geqq \lambda_*$ *such that* (P_λ) *possesses for each* $z_0\in B_p^{2\theta}(\Gamma_3)$ *and each* $\lambda\geqq\lambda_0$ *a unique maximal weak solution* $u(\cdot, z_0, \lambda)\in C(J(z_0,\lambda), H_p^{2\theta}(\Omega))$ *and a maximal interval of existence is right open in* $[0, T]$.

Proof. Define $\lambda_0 := \max\{\tilde{\lambda}_0, \bar{\lambda}_0\}$, where $\tilde{\lambda}_0$ and $\bar{\lambda}_0$ are the constants from (4.4) and Proposition 4.1 respectively, and fix $\lambda\geqq\lambda_0$. Then by Proposition 4.1 there is a $t_1>0$ and a unique $u_1\in C([0,t_1], H_p^{2\theta}(\Omega))$ satisfying

$$(4.12) \qquad u_1(t)=K(0, z_0, u_1)(t), \qquad t\in[0, t_1].$$

If $t_1<T$, define $z_1 := z(0, z_0, u_1, t_1)\in B_p^{2\theta}(\Gamma_3)$. Then, again by Proposition 4.1, there is a $t_2>t_1$ and a unique $u_2\in C([t_1,t_2], H_p^{2\theta}(\Omega))$ such that

$$(4.13) \qquad u_2(t)=K(t_1, z_1, u_2)(t), \qquad t\in[t_1, t_2].$$

Now define u: $[0, t_2]\to H_p^{2\theta}(\Omega)$ by $u(t):=u_1(t)$ if $t\in[0,t_1)$ and by $u(t):=u_2(t)$ if $t\in[t_1,t_2]$ and let $\tilde{u}_i := u_i(t_1)$, $i=1, 2$. Then using (4.12), (4.13) and the definition of $K(s, z_0, w)$ we obtain:

$$(\lambda+\mathscr{A}(t_1), \mathscr{B}_1(t_1), \bar{\gamma}_2(t_1), \bar{\gamma}_3(t_1))\,\tilde{u}_i=(f(t_1,\tilde{u}_i), g(t_1,\tilde{u}_i), 0, z_1), \qquad i=1,2.$$

Since $(f(t_1,\tilde{u}_i), g(t_1,\tilde{u}_i))\in L_p(\Omega)\times B_p^1(\Gamma_1)$, we conclude from (1.6) that $v:=\tilde{u}_1-\tilde{u}_2\in H_p^2(\Omega)$. Due to assumption (4.2), it follows from the mean value theorem that for some $w\in H_p^{2\theta}(\Omega)$

$$(\lambda-\partial_2 f(t_1, w)+\mathscr{A}(t_1), -\partial_2 g(t_1, w)+\mathscr{B}_1(t_1), \gamma_2, \gamma_3)\, v=0$$

and therefore $v=0$, by (4.4). This shows that $u\in C([0,t_2], H_p^{2\theta}(\Omega))$. Further, it follows from basic properties of the parabolic fundamental system U and from Proposition 4.1 that u is the unique fixed point of $K(0, z_0, \cdot)$ in $C([0,t_2], H_p^{2\theta}(\Omega))$. Now define

$$J(z_0,\lambda):=\cup\{[0,t]\subset[0,T];\, K(0,z_0,\cdot) \text{ has a fixed point in } C([0,t], H_p^{2\theta}(\Omega))\}.$$

Then $J(z_0,\lambda)$ is a nontrivial subinterval of $[0, T]$ containing 0 and Propositions 4.1 and 3.1 show that there is a unique weak solution on $J(z_0,\lambda)$. Furthermore, $J(z_0,\lambda)$ is the maximal interval of existence and $J(z_0,\lambda)$ is right open in $[0, T]$, since otherwise Proposition 4.1, applied to the right endpoint, would give a contradiction. □

Remark. It should be observed that an essential point in the proof of Theorem 4.2 is the fact that the constant $\bar{\lambda}$ in Proposition 4.1 does not depend on $s_0\in[0, T)$. This is a consequence of the second assumption in (4.2). However, if one is only interested in local solutions, a carful check of the proof of Proposition

4.1 shows that the second assumption in (4.2) can be dropped. Note also that assumption (4.4) was only used for the proof of Theorem 4.2, but not for the proof of Proposition 4.1. □

For $\lambda \geqq \lambda_0$ we define

$$\mathscr{D}(\lambda) := \{(t, \tilde{z}) \in [0, T) \times B_p^{2\theta}(\Gamma_3);\, t \in J(\tilde{z}, \lambda)\}.$$

Then we prove the following continuity result:

Theorem 4.3 *$\mathscr{D}(\lambda)$ is open in $[0, T) \times B_p^{2\theta}(\Gamma_3)$ and $u(\cdot, \cdot, \lambda) \in C^{0,1-}(\mathscr{D}(\lambda), B_p^{2\theta}(\Gamma_3))$.*

Proof. For a given $(t_0, z_0) \in \mathscr{D}(\lambda)$ let $\tilde{A} := \{u(t, z_0, \lambda);\, t \in [0, t_0]\}$. Since $\tilde{A}$ is a compact in $H_p^{2\theta}(\Omega)$, it follows from the trace theorem that $A := \gamma_3(\tilde{A})$ is compact in $B_p^{2\theta}(\Gamma_3)$. Hence by Proposition 4.1 there is a $\delta \in (0, T - t_0)$ and an $\varepsilon > 0$ such that for each $s \in [0, t_0]$ and each $\tilde{z} \in V := \mathbb{B}_{B_p^{2\theta}(\Gamma_3)}(A, \varepsilon)$ the operator $K(s, \tilde{z}, \cdot)$ possesses a unique fixed point $u(\cdot, \tilde{z}, s, \lambda)$ in $C([s, s+\delta], H_p^{2\theta}(\Omega))$. Moreover, there is a $c > 0$ with

$$|u(t, \tilde{z}, s, \lambda) - u(t, \hat{z}, s, \lambda)|_{2\theta, p} \leqq c\, \|\tilde{z} - \hat{z}\|_{3, 2\theta, p} \tag{4.14}$$

for $s \in [0, t_0]$, $t \in [s, s+\delta]$ and $\tilde{z}, \hat{z} \in V$.

Now suppose that $t_0 \in [0, \delta)$. Then put $s = 0$ in (4.14) and the assertion follows by observing that $u(\cdot, \tilde{z}, \lambda) = u(\cdot, \tilde{z}, 0, \lambda)$ and that $u(\cdot, \tilde{z}, \lambda) \in C(J(\tilde{z}, \lambda), H_p^{2\theta}(\Omega))$ for each $\tilde{z} \in V$.

Next suppose that $t_0 \in [\delta, 2\delta)$. Clearly, we can assume that $2\delta < T$. Define $\varepsilon_0 := \dfrac{\varepsilon}{c\, \|\gamma_3\|} \wedge \varepsilon$ and fix $\tilde{z} \in B := \mathbb{B}_{B_p^{2\theta}(\Gamma_3)}(z_0, \varepsilon_0)$. Then $u(\cdot, \tilde{z}, 0, \lambda)$ is well defined on $[0, \delta]$ and for $\tilde{z}_1 := \gamma_3\, u(\delta, \tilde{z}, 0, \lambda)$ we have by (4.14):

$$\|\tilde{z}_1 - \gamma_3\, u(\delta, z_0, 0, \lambda)\|_{3, 2\theta, p} \leqq c\, \|\gamma_3\|\, \|\tilde{z} - z_0\|_{3, 2\theta, p} \leqq \varepsilon.$$

Consequently, $\tilde{z}_1 \in V$ and therefore $u(\cdot, \tilde{z}_1, \delta, \lambda)$ is well defined on $[\delta, 2\delta]$. By a similar argument as in the proof of Theorem 4.2 we find that $u(\cdot, \tilde{z}, \lambda)$ is well defined on $[0, 2\delta]$. Hence $[0, 2\delta] \subset J(\tilde{z}, \lambda)$ and since $[0, 2\delta] \times B$ is a neighbourhood of (t_0, z_0) we find that $\mathscr{D}(\lambda)$ is open in $[0, T) \times B_p^{2\theta}(\Gamma_3)$. Finally, take $\tilde{z}, \hat{z} \in B$. Observe that $B \subset V$. Thus we know from (4.14) that

$$|u(t, \tilde{z}, \lambda) - u(t, \hat{z}, \lambda)|_{2\theta, p} \leqq c\, \|\tilde{z} - \hat{z}\|_{3, 2\theta, p}, \qquad t \in [0, \delta]$$

as well as

$$\begin{aligned} |u(t, \tilde{z}, \lambda) - u(t, \hat{z}, \lambda)|_{2\theta, p} &= |u(t, \gamma_3\, u(\delta, \tilde{z}, 0, \lambda), \delta, \lambda) - u(t, \gamma_3\, u(\delta, \hat{z}, 0, \lambda), \delta, \lambda)|_{2\theta, p} \\ &\leqq c\, \|\gamma_3\|\, |u(\delta, \tilde{z}, 0, \lambda) - u(\delta, \hat{z}, 0, \lambda)|_{2\theta, p} \\ &\leqq c^2\, \|\gamma_3\|\, \|\tilde{z} - \hat{z}\|_{3, 2\theta, p}, \qquad t \in [\delta, 2\delta). \end{aligned}$$

This proves the assertion for $t_0 \in [\delta, 2\delta)$.

We now obtain the general case by iterating the arguments above. □

5 Global solutions

In this section we assume that $(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_3)$ and (f, g, h) do not depend on t. In other words, we consider the following autonomous problem

$$(\mathrm{A}_\lambda)\qquad \left.\begin{aligned}(\lambda+\mathscr{A})u&=f(u)\\ \mathscr{B}_1 u&=g(u)\\ \gamma_2 u&=0\\ (\gamma_3 u)^\cdot+\mathscr{B}_3 u&=h(\gamma_3 u)\end{aligned}\right\} t>0,\qquad (\gamma_3 u)(0)=z_0.$$

Furthermore, we assume that (4.1)–(4.4) hold. We fix $\lambda \geqq \lambda_0$ and suppress it in our notation. For $z_0 \in B_p^{2\theta}(\Gamma_3)$ we denote by $u := u(\cdot, z_0)$ the unique maximal weak solution of (A) on $J := J(z_0)$. Moreover, we put $t^+ := \sup J$ and $z := z(0, z_0, u, \cdot)$. Note that due to Proposition 3.1(a) we have $z(t) = \gamma_3 u(t)$, $t \in J$. Finally, we define $\mathscr{D} := \{(t, \tilde{z}) \in [0, \infty) \times B_p^{2\theta}(\Gamma_3); t \in J(\tilde{z})\}$ and φ: $\mathscr{D} \to B_p^{2\theta}(\Gamma_3)$, $\varphi(t, \tilde{z}) := \gamma_3 u(t, \tilde{z})$.

Theorem 5.1 *$\mathscr{D}$ is open in $[0, \infty) \times B_p^{2\theta}(\Gamma_3)$ and $\varphi \in C^{0,1-}(\mathscr{D}, B_p^{2\theta}(\Gamma_3))$ defines a local semiflow on $B_p^{2\theta}(\Gamma_3)$ such that bounded orbits are relatively compact.*

Proof. The fact that $\mathscr{D}$ is open in $[0, \infty) \times B_p^{2\theta}(\Gamma_3)$ and that φ defines a local semiflow on $B_p^{2\theta}(\Gamma_3)$ follows from Theorem 4.3 and Theorem 4.2, respectively. Following the lines of the proof of Theorem 12.3 in [5] we easily obtain the third assertion, since $B_p^\alpha(\Gamma_3)$ is compactly embedded in $B_p^\beta(\Gamma_3)$ for $\alpha > \beta$. □

Lemma 5.2 *There exists a positive constant c such that*

$$|\mathscr{S}(f(u(t))) - f(u(s)), g(u(t)) - g(u(s)))|_{2\theta, p} \leqq c\, \|z(t) - z(s)\|_{3, 2\theta, p}, \qquad s, t \in J.$$

Proof. Denoting by $L_1 > 0$ the Lipschitz constant of (f, g) it follows from (4.6) that there is a $\beta \in (0, 1)$ such that

$$\|\mathscr{S}\|_{\mathscr{L}(L_p(\Omega) \times B_p^1(\Gamma_1), H_p^{2\theta}(\Omega))} \leqq \frac{1-\beta}{L_1}.$$

Therefore we obtain for $s, t \in J$:

$$\begin{aligned}&|\mathscr{S}(f(u(t)) - f(u(s)), g(u(t)) - g(u(s)))|_{2\theta, p}\\ &\quad\leqq (1-\beta)|u(t) - u(s)|_{2\theta, p} = (1-\beta)|K(0, z_0, u)(t) - K(0, z_0, u)(s)|_{2\theta, p}\\ &\quad\leqq (1-\beta)|\mathscr{S}(f(u(t)) - f(u(s)), g(u(t)) - g(u(s)))|_{2\theta, p} + (1-\beta)|\mathscr{T}(z(t) - z(s))|_{2\theta, p}.\end{aligned}$$

Now the assertion follows by observation of (2.4). □

Theorem 5.3 (a) *Assume that*

$$\sup\{\|(\gamma_3 u)(t)\|_{3, 2\theta, p}; t \in [0, t^+ \wedge T)\} < \infty \tag{5.1}$$

for every $T > 0$. Then u is a global solution, i.e. $t^+ = \infty$.

(b) *Suppose there is a constant $c \geqq 0$ such that*

$$\|h((\gamma_3 u)(t))\|_{3, 2\eta-1, p} \leqq c(1 + \|(\gamma_3 u)(t)\|_{3, 2\theta, p}), \qquad t \in J. \tag{5.2}$$

Then u exists globally.

(c) *If* $t^+<\infty$, *then the solution blows up as* $t\to t^+$, *that is*

$$\limsup_{t\to t^+}|u(t)|_{2\theta,p}=\infty \quad \textit{and} \quad \lim_{t\to t^+}|u(t)|_{2\xi,p}=\infty \quad \textit{for each } \xi>\theta.$$

Proof. (a) Suppose that $t^+<\infty$. Then by (5.1), (4.3) and Lemma 5.2 we find that $F(u(\cdot))\in L_\infty((0,t^+), B_p^{2\eta-1}(\Gamma_3))$. Consequently, Proposition 1.3 in [3] implies that $z\in C([0,t^+], B_p^{2\theta}(\Gamma_3))$. Hence, by using $u=K(0,z_0,u)$ on J and again Lemma 5.2, it follows that $u\in BUC([0,t^+), H_p^{2\theta}(\Omega))$. Now define $\bar{u}$: $[0,t^+]\to H_p^{2\theta}(\Omega)$ by $\bar{u}(t):=u(t)$ if $t\in[0,t^+)$ and $\bar{u}(t^+):=\lim_{t\to t^+}u(t)$. Then $\bar{u}\in C([0,t^+], H_p^{2\theta}(\Omega))$ and $\bar{u}=K(0,z_0,\bar{u})$ on $[0,t^+]$. Thus $\bar{u}$ is a weak solution of (A_λ) on $[0,t^+]$ extending u. But this contradicts the maximality of u since $t^+\notin J$.

(b) Let $T>0$ be given. Assumption (5.2) and Lemma 5.2 show that there is a $c>0$ such that

$$\|F(u(t))\|_{3,2\eta-1,p}\leqq c(1+\|z(t)\|_{3,2\theta,p}), \qquad t\in J. \tag{5.3}$$

Thus Corollary 2.2 implies that

$$\|z(t)\|_{3,2\theta,p}\leqq c\left(1+\int_0^t (t-\tau)^{2(\eta-\theta)-1}\|z(\tau)\|_{3,2\theta,p}\,d\tau\right), \qquad t\in[0,t^+\wedge T)$$

and therefore, by a generalized Gronwall inequality (cf. [3, Lemma 3.3]),

$$\sup\{\|z(t)\|_{3,2\theta,p}\,;\,t\in[0,t^+\wedge T)\}<\infty.$$

Now the assertion follows from (a).

(c) If the first assertion is not true, if follows from the trace theorem that $\sup\{\|z(t)\|_{3,2\theta,p}\,;\,t\in[0,t^+)\}<\infty$, which gives the same contradiction as in (a).

Finally, assume that the second assertion is false. Then there is a $R>0$ and a sequence $(t_k)_{k\in\mathbb{N}}\subset[0,t^+)$ such that $|u(t_k)|_{2\xi,p}\leqq R$ for all $k\in\mathbb{N}$ and such that $t_k\to t^+$ as $k\to\infty$. Since $H_p^{2\xi}(\Omega)$ is compactly embedded in $H_p^{2\theta}(\Omega)$ for $\xi>\theta$, it follows that $A:=\{\gamma_3 u(t_k);\,k\in\mathbb{N}\}$ is relatively compact in $B_p^{2\theta}(\Gamma_3)$. Consequently Proposition 4.1 implies the existence of a positive δ such that the solution u exists on $[0,t_k+\delta]$ for all $k\in\mathbb{N}$. This contradicts the maximality of u, since $t_k\to t^+$ as $k\to\infty$. □

6 Applications

In this section we apply the results of Sects. 4 and 5 to the case where the nonlinearities are induced by local functions. It should be observed that the abstract results also apply to nonlocal operators, as they appear, for example, in control theory.

Throughout this section we assume that

$$f\in C^1(\bar{\Omega}\times[0,T]\times\mathbb{R}^N\times\mathbb{R}^{nN},\mathbb{R}^N),$$
$$g\in C^1(\Gamma_1\times[0,T]\times\mathbb{R}^N,\mathbb{R}^N), \qquad h\in C^1(\Gamma_3\times[0,T]\times\mathbb{R}^N,\mathbb{R}^N).$$

For a given $z_0 \in B_p^s(\Gamma_3)$, where $s \in [1, 1/p)$ is fixed, we consider the following nonlinear elliptic boundary value problem with a *dynamic boundary condition on* Γ_3:

$$(\mathrm{Q}_\lambda) \qquad \begin{aligned} \lambda u - \partial_j(a_{jk}\partial_k u) + a_j\partial_j u + a_o u &= f(\cdot,\cdot,u,\partial u) && \text{in } \Omega\times(0,T],\\ a_{jk}\nu^j\gamma_1\partial_k u + b_1\gamma_1 u &= g(\cdot,\cdot,\gamma_1 u) && \text{on } \Gamma_1\times(0,T],\\ \gamma_2 u &= 0 && \text{on } \Gamma_2\times(0,T],\\ \partial_t(\gamma_3 u) + a_{jk}\nu^j\gamma_3\partial_k u + b_3\gamma_3 u &= h(\cdot,\cdot,\gamma_3 u) && \text{on } \Gamma_3\times(0,T],\\ (\gamma_3 u)(\cdot,0) &= z_0 && \text{on } \Gamma_3. \end{aligned}$$

We denote by (A_λ) the autonomous version of problem (Q_λ), i.e. in (A_λ) all the coefficients a_{jk}, a_j, a_0, b_1, b_3 as well as the nonlinearities f, g, h are independent of $t \in [0, T]$.

Furthermore we introduce the following growth and structural conditions for f, g and h:

There is a positive constant M_1 *such that*

$$(6.1) \qquad \begin{aligned} |\partial_3 f(x,t,\xi,\eta)| + |\partial_4 f(x,t,\xi,\eta)| &\leqq M_1, && (x,t,\xi,\eta)\in\bar\Omega\times[0,T]\times\mathbb{R}^N\times\mathbb{R}^{nN},\\ |\partial_3 g(y,t,\xi)| &\leqq M_1, && (y,t,\xi)\in\Gamma_1\times[0,T]\times\mathbb{R}^N. \end{aligned}$$

h is independent of t and there is a constant $M_2 \geqq 0$ *such that*

$$(6.2) \qquad |h(y,\xi)| \leqq M_2(1+|\xi|), \qquad (y,\xi)\in\Gamma_3\times\mathbb{R}^N.$$

h is independent of t and if $n \geqq 2$ *there are constants* $\alpha \in \left[1, \frac{n}{n-2}\right)$, $M_3 \geqq 0$ *with*

$$(6.3) \qquad |\partial_2 h(y,\xi)| \leqq M_3(1+|\xi|^{\alpha-1}), \qquad (y,\xi)\in\Gamma_3\times\mathbb{R}^N.$$

h is independent of t and there is a $M_4 \geqq 0$ *and a* $\varphi \in C^{0,2}(\Gamma_3\times\mathbb{R}^N, \mathbb{R})$ *such that*

$$(6.4) \quad (h(y,\xi)|\xi) \leqq M_4, \qquad \varphi(y,0)=0 \text{ and } \partial_2\varphi(y,\xi) = h(y,\xi), \qquad (y,\xi)\in\Gamma_3\times\mathbb{R}^N.$$

Further we define for $(t,u) \in [0,T]\times H_p^1(\Omega)$ and $z \in B_p^1(\Gamma_3)$ the following substitution operators:

$$\begin{aligned} \hat f(t,u)(x) &:= f(x,t,u(x),\partial u(x)), && x\in\Omega,\\ \hat g(t,u)(y) &:= g(y,t,(\gamma_1 u)(y)), && y\in\Gamma_1,\\ \hat h(t,z)(y) &:= h(y,t,z(y)), && y\in\Gamma_3. \end{aligned}$$

Lemma 6.1 (a) *Suppose that* (6.1) *holds. Then* $(\hat f, \hat g)$ *satisfies assumption* (4.2) *and* (4.4) *for every* $\theta \in [1/2, 1/2+1/2p)$.

(b) *If* $p > n$, *then for every* $\theta \in [1/2, 1/2+1/2p)$ *there is a* $\eta > \theta$ *such that* (4.3) *hold for* $\hat h$. *Furthermore* (6.2) *implies that* (5.2) *is fulfilled.*

(c) *Suppose that* $n \geqq 2$, $p=2$ *and that* (6.3) *holds. Then there is a* $\eta > 1/2$ *such that* $\hat h$ *satisfies* (4.3) *with* $\theta = 1/2$.

Proof. (a) For $u, v \in H_p^1(\Omega)$ and $t \in [0, T]$ we define

$$M(t, u, v)(x) := \partial_3 f(x, t, u(x), \partial u(x))\, v(x) + \partial_4 f(x, t, u(x), \partial u(x))\, \partial v(x), \qquad x \in \Omega,$$
$$N(t, u, v)(y) := \partial_3\, g(y, t, (\gamma_1 u)(y))(\gamma_1 v)(y), \qquad y \in \Gamma_1.$$

Then using the mean value theorem and the fact that $H_p^{2\theta}(\Omega) \times (H_p^{2\theta-1}(\Omega))^n \hookrightarrow H_p^1(\Omega) \times (L_p(\Omega))^n$ we deduce from (6.1) that for each $(u, t) \in H_p^{2\theta}(\Omega) \times [0, T]$ we have:

$$(6.5) \qquad |\hat{f}(t, u+h) - \hat{f}(t, u) - M(t, u, h)|_p = o(|h|_{2\theta, p}) \quad \text{as} \quad |h|_{2\theta, p} \to 0,$$
$$(u \mapsto M(t, u, \cdot)) \in C(H_p^{2\theta}(\Omega), \mathscr{L}(H_p^{2\theta}(\Omega), L_p(\Omega))).$$

Since the continuity with respect to time is obvious, (6.5) shows that $\hat{f} \in C^{0,1}([0, T] \times H_p^{2\theta}(\Omega), L_p(\Omega))$ and that $\partial_2 \hat{f}(t, u) = M(t, u, \cdot)$.

Observe further, that by (6.1) there is a constant $c \geqq 0$ such that

$$|M(t, u, v)|_p \leqq c\, |v|_{2\theta, p} \qquad \text{for } t \in [0, T], u, v \in H_p^{2\theta}(\Omega).$$

Hence $\hat{f}$ satisfies assumption (4.2).

To prove the assertion for $\hat{g}$, let us first recall that there exists a regular localization system $\{(U_j, \varphi_j, \pi_j); 1 \leqq j \leqq n\}$ for Ω of class C^∞. This means that there exists an integer m, open subsets U_j of $\mathbb{R}^n$, $1 \leqq j \leqq m$ and functions φ_j, π_j, $1 \leqq j \leqq m$ such that $\bigcup_{j=1}^{m} U_j \supset \bar{\Omega}$, φ_j is a C^∞-diffeomorphism of U_j onto $\mathbb{B}^n$ if $\partial\Omega \cap U_j = \emptyset$ and of $U_j \cap \Omega$ onto $\mathbb{B}^n \cap \mathbb{H}^n$ if $\partial\Omega \cap U_j \neq \emptyset$, $1 \leqq j \leqq m$ and such that $\{\pi_j; 1 \leqq j \leqq m\}$ is a C^∞-partition of unity on $\bar{\Omega}$ subordinate to $\{U_j; 1 \leqq j \leqq m\}$.

With these notations we define for $s \in \mathbb{R}^+ \setminus (\mathbb{N} + 1/p)$ and $z \in B_p^s(\Gamma_1)$,

$$|||z|||_{1,[s-1/p],p} := \Big(\sum_{\Gamma_1} \sum_{|\alpha| \leqq [s-1/p]} \|\partial^\alpha \pi_j z\|_{1,p}^p\Big)^{1/p},$$

$$[z]_{1,s,p} := \left(\sum_{\Gamma_1} \sum_{|\alpha| \leqq [s-1/p]} \int_{\Gamma_1} \int_{\Gamma_1} \frac{|(\partial^\alpha \pi_j z)(x) - (\partial^\alpha \pi_j z)(y)|^p}{|x-y|^{n-1-p(s-1/p-[s-1/p])}}\, d\sigma(x)\, d\sigma(y)\right)^{1/p}.$$

Here, $[r]$ stands for the integral part of $r \in \mathbb{R}$ and $\sum_{\Gamma_1}$ denotes summation over those $j \in \{1, \ldots, m\}$ for which $U_j \cap \Gamma_1 \neq \emptyset$. Now we put $|||z|||_{1,s,p} := |||z|||_{1,[s-1/p],p} + [z]_{1,s,p}$, $z \in B_p^s(\Gamma_3)$. Then one can show that $|||\cdot|||_{1,s,p}$ defines an equivalent norm for $B_p^s(\Gamma_1)$, provided $s \in \mathbb{R}^+ \setminus (\mathbb{N} + 1/p)$. (Note that the statement is wrong if $s \in \mathbb{N} + 1/p$, $p \neq 2$, see for example Theorem 2.12 in [27].)

Using this equivalent norm on $B_p^{2\theta}(\Gamma_1)$ for $2\theta \in [1, 1+1/p)$, the trace theorem, the mean value theorem and assumption (6.1) it follows similarly to (6.5) that $\hat{g} \in C^{0,1}([0, T] \times H_p^{2\theta}(\Omega), B_p^1(\Gamma_1))$ with

$$\partial_2\, \hat{g}(t, u) = N(t, u, \cdot), \qquad (t, u) \in [0, T] \times H_p^{2\theta}(\Omega),$$

and that there is a $c \geqq 0$ such that

$$|||N(t, u, v)|||_{1,1,p} \leqq c\, |v|_{2\theta, p}, \qquad v \in H_p^{2\theta}(\Omega).$$

Consequently, $\hat{g}$ satisfies assumption (4.2), too.

Now let $u \in H_p^{2\theta}(\Omega)$, $v \in H_p^2(\Omega)$ and $\lambda \in \mathbb{R}$ be given and assume that

$$(6.6) \qquad (\lambda - \partial_2 \hat{f}(t,u) + \mathscr{A}(t),\ -\partial_2 \hat{g}(t,u) + \mathscr{B}_1(t), \gamma_2, \gamma_3)v = 0.$$

Using (6.6), the fact that $H_p^2(\Omega) \hookrightarrow H_2^2(\Omega)$ and Gauss' theorem, we find that

$$(6.7) \qquad \lambda |v|_2^2 + \int_\Omega [(a_{jk}(t,\cdot)\,\partial_k v \,|\, \partial_j v) + (a_j(t,\cdot)\,\partial_j v + a_0(t,\cdot)v - \partial_2 \hat{f}(t,u)v \,|\, v)]\,dx$$
$$+ \int_{\Gamma_1} (b_1(t,\cdot)\,\gamma_1 v - \partial_2 \hat{g}(t,u)v \,|\, \gamma_1 v)\,d\sigma = 0.$$

Next we note that due to the main result in [10] there are positive constants $\tilde{\lambda}$, c_1 such that

$$(6.8) \qquad \tilde{\lambda}|v|_2^2 + \int_\Omega [(a_{jk}(t,\cdot)\,\partial_k v \,|\, \partial_j v) + (a_j(t,\cdot)\,\partial_j v + a_0(t,\cdot)v \,|\, v)]\,dx \geqq c_1 |v|_{1,2}^2.$$

Furthermore we fix $\alpha \in (1/2, 1)$. Then $\gamma_1 \in \mathscr{L}(H_2^\alpha(\Omega), L_2(\Gamma_1))$. Thus by (6.1) we find a constant $c_2 > 0$, which does not depend on u, such that

$$(6.9) \qquad -\int_{\Gamma_1} (b_1(t,\cdot)\,\gamma_1 v + \partial_2 \hat{g}(t,u)v \,|\, \gamma_1 v)\,d\sigma \leqq c_2 |v|_{\alpha,2}^2.$$

Since the first embedding in $H_2^1(\Omega) \hookrightarrow H_2^\alpha(\Omega) \hookrightarrow L_2(\Omega)$ is compact, an abstract version of Ehring's Lemma, cf. [28, Satz I.7.3], implies that there is a $c_3 > 0$ with

$$(6.10) \qquad |v|_{\alpha,2}^2 \leqq \frac{c_1}{c_2} |v|_{1,2}^2 + c_3 |v|_2^2.$$

Finally, again by (6.1), there is a $c_4 > 0$ (not depending on u) such that

$$(6.11) \qquad \int_\Omega (\partial_2 \hat{f}(t,u)v \,|\, v)\,dx \leqq c_4 |v|_2^2.$$

Now define $\tilde{\lambda}_0 := \tilde{\lambda} + c_3 + c_4$ and choose $\lambda > \tilde{\lambda}_0$. Then it follows from (6.7)–(6.11) that we must have $v = 0$. Hence $(\hat{f}, \hat{g})$ satisfies assumption (4.4).

(b) Let $\theta, \eta \in [1/2, 1/2 + 1/2p)$ with $\eta > \theta$ be given. Then by a well known Sobolev embedding theorem we have

$$B_p^{2\theta}(\Gamma_3) \hookrightarrow C(\Gamma_3) \hookrightarrow B_p^{2\eta-1}(\Gamma_3).$$

Using these embeddings the assumption follows again from the mean value theorem.

(c) This is shown in [12, Lemma 4.1 (a)]. □

It is now easy to prove the following

Theorem 6.2 *Assume that* $p > n$, $s \in [1, 1 + 1/p)$ *and that* (6.1) *holds.*

(a) *Then there is a* $\lambda_0 \geqq 0$ *such that* (Q_λ) *possesses for each* $z_0 \in B_{pp}^{s-1/p}(\Gamma_3)$ *and each* $\lambda \geqq \lambda_0$ *a unique maximal weak solution* $u \in C(J(z_0,\lambda), H_p^s(\Omega))$ *and the maximal interval of existence,* $J(z_0,\lambda)$*, is right open in* $[0, T]$.

(b) *Consider the autonomous problem* (A_λ). *If* $\sup\{\|\gamma_3 u)(t)\|_{3,s,p};$ $t\in J(z_0,\lambda)\cap[0,T]\}<\infty$ *for every* $T>0$, *then* u *is a global solution, i.e.* $u\in C([0,\infty), H_p^s(\Omega))$.

If $t^+ := \sup J(z_0,\lambda)<\infty$, *then u blows up in finite time, that is*

$$\limsup_{t\to t^+} |u(t)|_{s,p}=\infty \quad \text{and} \quad \lim_{t\to t^+} |u(t)|_{r,p}=\infty \qquad \text{for each } r>s.$$

(c) *Suppose finally that* (6.2) *holds. Then for each* $(z_0,\lambda)\in B_{pp}^{s-1/p}(\Gamma_3)\times[\lambda_0,\infty)$ *the solution u of the autonomous problem* (A_λ) *exists globally.*

Proof. This follows from Lemma 6.1(a), (b), Theorem 4.2 and Theorem 5.3. □

Remark. It should be observed that in the first part of Theorem 6.2 we only assumed growth conditions for f and g but *no* growth restrictions for h. □

Finally, we apply some Hilbert space techniques to problem (A_λ) to obtain a priori estimates for $\gamma_3 u$ in $B_{22}^{1/2}(\Gamma_3)$. For this we assume that

(6.12)
$$p=2, \quad \theta=1/2 \quad \text{and}$$
$$a_{jk}=a_{jk}^T, \quad a_j=0, \quad 1\leqq j,k\leqq n, \quad a_0=a_0^T, \quad b_1=b_1^T, \quad b_3=b_3^T.$$

Observe that assumption (6.12) implies that the boundary value problem $(\mathscr{A},\mathscr{B}_1,\gamma_2,\mathscr{B}_3)$ is formally self-adjoint.

Theorem 6.3 *Assume that* (6.1), (6.3) *and* (6.4) *hold. Then for each* $(z_0,\lambda)\in B_{22}^{1/2}(\Gamma_3)\times[\lambda_0,\infty)$ *the solution u of the autonomous problem* (A_λ) *exists globally, i.e.* $u\in C([0,\infty), H_2^1(\Omega))$.

Proof. It follows from Lemma 6.1(a), (c) and Theorem 4.2 that there exists a unique weak solution $u\in C(J(z_0,\lambda), H_2^1(\Omega))$ for $(z_0,\lambda)\in B_{22}^{1/2}(\Gamma_3)$. We put $z:=\gamma_3 u$ and

$$P(z(t)):=\int_0^1 \langle z(t), \hat{h}(sz(t))\rangle_3\, ds, \qquad t\in J(z_0,\lambda).$$

Note that P defines a potential for $\hat{h}$ in the sense of [12], thanks to the structural condition (6.4), and that P is bounded above on $J(z_0,\lambda)$.

Next define $F_2(v):=-\mathscr{B}_3\mathscr{S}(\hat{f}(v),\hat{g}(v))$ for $v\in H_2^1(\Omega)$. Then we have due to (1.8) and (4.2):

(6.13)
$$F_2(v)\in L_2(\Gamma_3), \qquad v\in H_2^1(\Omega).$$

Again, from (1.8), (4.2) and (2.4), it follows that there is a $c>0$ with

$$\begin{aligned}|\mathscr{S}(\hat{f}(u(t)),\hat{g}(u(t)))|_{2,2} &\leqq c(1+|u(t)|_{1,2})=c(1+|K(0,z_0,u)(t)|_{1,2})\\ &\leqq c(1+|\mathscr{S}(\hat{f}(u(t)),\hat{g}(u(t)))|_{1,2}\\ &\quad +\|(\gamma_3 u)(t)\|_{3,1,2}), \qquad t\in J(z_0,\lambda).\end{aligned}$$

Thus Lemma 5.2 implies that there is a $c>0$ such that

$$|\mathscr{S}(\hat{f}(u(t)),\hat{g}(u(t)))|_{2,2}\leqq c(1+\|(\gamma_3 u)(t)\|_{3,1,2}),$$

and consequently we find a constant $c>0$ with

(6.14) $$\|F_2(u(t))\|_{3,2} \leqq c(1+\|(\gamma_3 u)(t)\|_{3,1,2}), \qquad t\in J(z_0,\lambda).$$

Now considering the equation $\dot{z}+\mathbb{B}_{-1/2}z=\hat{h}(z)+F_2(u)$ and observing (6.13) and (6.14), it follows along the lines of the proof of Theorem 2.7 in [12] that for each $T>0$ we have:

$$\sup\{\|(\gamma_3 u)(t)\|_{3,1,2};\, t\in[0,t^+\wedge T)\}<\infty.$$

Hence, Theorem 5.3(a) gives the assertion. □

Remark. Typical situations where assumption (6.4) is satisfied are:

(a) $h(y,\xi)=-q(y)|\xi|^{\alpha-1}\xi$ with $q\in C(\Gamma_3,\mathbb{R}^+)$.

(b) $N=1$, $h(y,\xi)\xi\leqq 0$ and $\varphi(y,\xi):=\int_0^\xi h(y,s)\,ds$, $(y,\xi)\in\Gamma_3\times\mathbb{R}$.

Appendix

A maximum principle

Throughout this section we assume that $N=1$ and $p>n$. We fix $t\in[0,T]$ and suppress it in our notation. Further we assume that $-\lambda_*\leqq\min\{a_0(x);x\in\bar{\Omega}\}$ and we choose $\lambda\geqq\lambda_*$.

Suppose that $(E,\leqq)$ is an ordered vector space. We denote by $E^+:=\{x\in E;x\geqq 0\}$ the positive cone of $(E,\leqq)$. Furthermore $C([0,T],E)$ is always given the natural ordering induced by E.

It is well known that $L_p(\Omega)$ is an ordered Banach space with positive cone $L_p^+(\Omega):=\{u\in L_p(\Omega);\, u(x)\geqq 0$ for almost all $x\in\Omega\}$. Since $H_p^s(\Omega)\hookrightarrow L_p(\Omega)$, for $s\geqq 0$, cf. [8, Theorem 6.2.3], we find that $H_p^s(\Omega)$ is also an ordered Banach space given by the natural ordering. Similar statements hold for $B_p^s(\Gamma_i)\hookrightarrow L_p(\Gamma_i)$, $s>1/p$, $i=1,2,3$, cf. [8, Theorem 6.3.2].

The following lemma is an immediate consequence of Bony's maximum principle [9].

Lemma 1 *Let $u\in H_p^2(\Omega)$ satisfy $(\lambda+\mathscr{A},\mathscr{B}_1,\gamma_2)u\geqq 0$. Then*

(a) *there is a $y_1\in\Gamma_3$ with $(\gamma_3 u)(y_1)<0$ and $(\mathscr{B}_3 u)(y_1)<0$, provided $(\gamma_3 u)(y_0)<0$ for some $y_0\in\Gamma_3$.*

(b) *$\mathscr{B}_3 u\leqq 0$, provided $\gamma_3 u=0$.*

Lemma 2 *$-\mathbb{B}_{\alpha-1/2}$ is for each $\alpha>1/p$ a resolvent positive operator.*

Proof. (a) Assume first that $\alpha\geqq 1$. Note that $\mathbb{B}_{\alpha-1/2}z=\mathbb{B}_{1/2}z=\mathscr{B}_3\mathscr{T}z$ for $z\in B_p^{\alpha+1}(\Gamma_3)$, due to Theorem 1.5(a). Now denote by $\mu_*\geqq 0$ the constant of Theorem 1.1 and let $\mu\geqq\mu_*$ as well as $\hat{z}\in B_p^{1+\alpha}(\Gamma_3)$ be given. Assume that

(A.1) $$(\mu+\mathscr{B}_3\mathscr{T})\hat{z}\geqq 0.$$

We have to show that $\hat{z} \geqq 0$. Suppose there is a $y_0 \in \Gamma_3$ with $\hat{z}(y_0) < 0$. Defining $v := \mathscr{T}\hat{z} \in H_p^2(\Omega)$, we have

$$(\lambda + \mathscr{A}, \mathscr{B}_1, \gamma_2)v = 0 \quad \text{and} \quad \gamma_3 v = \hat{z}.$$

Thus, by Lemma 1(a) there is a $y_1 \in \Gamma_3$ such that $\hat{z}(y_1) < 0$ and $(\mathscr{B}_3 v)(y_1) = (\mathscr{B}_3 \mathscr{T}\hat{z})(y_1) < 0$. Consequently, we have $\mu\hat{z}(y_1) + (\mathscr{B}_3 \mathscr{T}\hat{z})(y_1) < 0$, which contradicts assumption (A.1). Therefore $\hat{z} \geqq 0$.

(b) Now assume that $\alpha \in (1/p, 1)$ and that $\hat{z} \in B_p^{1+\alpha}(\Gamma_3)$ satisfies $(\mu + \mathbb{B}_{\alpha-1/2})\hat{z} \in B_p^{\alpha}(\Gamma_3)^+$. We choose a sequence (w_n) in $B_p^1(\Gamma_3)^+$ such that $w_n \to (\mu + \mathbb{B}_{\alpha-1/2})\hat{z}$ in $B_p^{\alpha}(\Gamma_3)^+$ as $n \to \infty$, and we define $z_n := (\mu + \mathbb{B}_{1/2})^{-1} w_n$, $n \in \mathbb{N}$. By (a) we have $z_n \in B_p^2(\Gamma_3)^+ \hookrightarrow B_p^{1+\alpha}(\Gamma_3)^+$. Since $z_n \to \hat{z}$ in $B_p^{1+\alpha}(\Gamma_3)$ as $n \to \infty$ and since $B_p^{1+\alpha}(\Gamma_3)^+$ is closed, we find that $\hat{z} \in B_p^{1+\alpha}(\Gamma_3)^+$. □

Corollary 3 *Let*

$$(f, g, h) \in C^{\varepsilon}(\mathbb{R}^+, L_p(\Omega) \times B_p^1(\Gamma_1) \times B_p^1(\Gamma_3))^+, \quad \varepsilon > 0 \quad \textit{and} \quad z_0 \in B_p^2(\Gamma_3)^+$$

be given. Denote by u the unique solution of

$$\begin{aligned} (\lambda + \mathscr{A})u &= f && \textit{in } \Omega \times (0, \infty), \\ \mathscr{B}_1 u &= g && \textit{on } \Gamma_1 \times (0, \infty), \\ \gamma_2 u &= 0 && \textit{on } \Gamma_2 \times (0, \infty), \\ (\gamma_3 u)^{\cdot} + \mathscr{B}_3 u &= h && \textit{on } \Gamma_3 \times (0, \infty), \\ (\gamma_3 u)(\cdot, 0) &= z_0 && \textit{on } \Gamma_3. \end{aligned}$$

Then $u \in C(\mathbb{R}^+, H_p^2(\Omega))^+$.

Proof. Due to (1.8) and Lemma 1(b) we know that $-\mathscr{B}_3 \mathscr{S}(f, g) \in C(\mathbb{R}^+, B_p^1(\Gamma_3))^+$ and therefore $F := h - \mathscr{B}_3 \mathscr{S}(f, g) \in C(\mathbb{R}^+, B_p^1(\Gamma_3))^+$. Now put

$$z(t) := e^{-t\mathbb{B}_{1/2}} z_0 + \int_0^t e^{-(t-s)\mathbb{B}_{1/2}} F(s)\, ds, \qquad t \geqq 0.$$

Then we have $z \in C(\mathbb{R}^+, B_p^2(\Gamma_3))^+$, due to Lemma 2. Note that, again by Bony's maximum principle, $\mathscr{R}$ is a positive operator. Since $u = \mathscr{R}(f, g, z)$, the assertion follows. □

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