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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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Duals of subhypergroups and quotients of commutative hypergroups *

Michael Voit

Mathematisches Institut, Technische Universität München,
Arcisstrasse 21, W-8000 München 2, Federal Republic of Germany

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Introduction

Hypergroups generalize locally compact groups. Therefore it is natural to ask for relationships of the dual spaces of subhypergroups and quotients of hypergroups with the dual of a given commutative hypergroup. Well-known examples show that the situation is not as clear as in the case of locally compact abelian groups (cf. Chap. 6 in Hewitt and Ross [8]). For instance, the dual space $\hat{K}$ of a commutative hypergroup often fails to bear a dual hypergroup structure (examples can be found in Dunkl [5] and Jewett [9]). Moreover, if $\hat{K}$ admits a dual hypergroup structure, then the bidual $K\hat{}\,\hat{}$ may be different from, i.e., bigger than, K. Examples for such phenomena were given recently by Zeuner [24, 25]. Furthermore, there exist (closed) subhypergroups H of some commutative hypergroups K such that K/H fails to be a quotient hypergroup or such that not each character on H extends to a character on K (see Zeuner [23, 25] and Sects. 2.14–2.17 below). On the other hand, some particular results are known for these problems when assuming some additional restrictions concerning the subhypergroup H (see Dunkl [5], Hauenschild et al. [7], Jewett [9], Ross [13] and Vrem [22]). The purpose of this paper is to generalize these results and to establish as complete as possible results for some important special cases. Complete results will be obtained for compact subhypergroups only; see the Theorems 2.5 and 2.6 below. Furthermore we shall describe the dual spaces of open subhypergroups whenever the hypergroup K fulfills a growth condition. This result can be considered as a first step only. However, some examples given in the end of the paper show that the growth condition is essential. We shall apply the results mentioned above to generalize a criterion of Lasser [10] concerning Pontryagin hypergroups such that our approach also covers the case of locally compact abelian groups.

This article is organized as follows: In Sect. 1 we first recapitulate some basic facts about hypergroups. Then we shall establish some natural results about hypergroup homomorphisms and normal subhypergroups of arbitrary

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hypergroups. This supplements investigations of Jewett [9]. In Sect. 2 we shall confine ourself to the commutative case and we shall study dual spaces of subhypergroups and quotients of hypergroups. To do this, we first collect some results about annihilators (see also Bloom [1], Bloom and Heyer [3], and Dunkl [5]). As mentioned above, we shall conclude this article with some examples in order to demonstrate the problems which may arise when studying duals of subhypergroups and quotient hypergroups.

1 Hypergroups and hypergroup homomorphisms

Throughout this paper we use the notation of Jewett's fundamental paper [9] about hypergroups. While most of our notation is standard and will not be introduced here, we want to clarify some – possibly ambiguous – notation and conventions.

1.1. General notation. To avoid complications, we assume that every locally compact space in this paper is a Hausdorff space. Let K be a locally compact space. Then by $M(K)$, $M_b(K)$, $M_b^+(K)$ and $M^1(K)$ we denote all Radon measures, the bounded ones, those that are bounded and nonnegative, and the probability measures on K respectively. $\delta_x \in M^1(K)$ is the point measure at $x \in K$.

Let K and L be locally compact spaces and $p: K \to L$ a continuous mapping. Then the associated mapping from $M_b(K)$ to $M_b(L)$ is denoted by p again. We say that p is proper whenever $p^{-1}(W)$ is compact for every compactum $W \subset L$.

The following observation will be useful for our subsequent considerations:

1.2. Lemma. *Let K and L be locally compact spaces and $p: K \to L$ continuous and bijective. Then p is a homeomorphism if and only if p is proper.*

Proof. The lemma is obvious if K and L are compact. If K and L are not compact, then consider the one-point compactifications $K \cup \{\infty_K\}$ and $L \cup \{\infty_L\}$. If p is proper, then the extension $p': K \cup \{\infty_K\} \to L \cup \{\infty_L\}$ of p with $p'(\infty_K) = \infty_L$ is a continuous bijection and hence a homeomorphism. This proves the if-part. The only-if-part is clear.

We next recapitulate the definition of a hypergroup.

1.3. Definition. Let K be a locally compact Hausdorff space and $*$ a bilinear, associative mapping on $M_b(K)$. $(K, *)$ (or, for short, K) is called a *hypergroup*, if the following conditions are satisfied:

(1) $\delta_x * \delta_y$ is a probability measure with compact support for all $x, y \in K$.
(2) The mapping $M_b^+(K) \times M_b^+(K) \to M_b^+(K)$, $(\mu, \nu) \mapsto \mu * \nu$, is continuous with respect to the weak topology.
(3) If $\mathscr{C}(K)$, the space of all nonvoid compact subsets of K, bears the Michael topology (i.e. the topology generated by the sets $U_{V,W} := \{L \in \mathscr{C}(K): L \cap V \neq \varnothing, L \subset W\}$ where V and W run through the open subsets of K; see Jewett [9]), then the mapping $K \times K \to \mathscr{C}(K)$, $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$, is continuous.
(4) There exists a neutral element $e \in K$ satisfying $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in K$, and there is a continuous involution $x \mapsto \bar{x}$ on K such that $e \in \operatorname{supp}(\delta_x * \delta_y)$ is equivalent to $x = \bar{y}$.

1.4. Further notation. (1) A closed nonempty set $H \subset K$ is called a subhypergroup if $H*H=H$ and $\bar{H}=H$, where $\bar{H}:=\{x\in K: \bar{x}\in H\}$. A subhypergroup is called normal if $x*H=H*x$ for all $x\in K$ and supernormal if $x*H*\bar{x}=H$ for all $x\in K$. It is obvious that every supernormal subhypergroup is normal (cf. Vrem [22]). $Z(K):=\{x\in K: \delta_x*\delta_{\bar{x}}=\delta_{\bar{x}}*\delta_x=\delta_e\}$ is the maximal subgroup of K and is called the center of K (see Ross [13]).
(2) A hypergroup is called commutative if the convolution $*$ is commutative. Moreover, it is said to be symmetric if the hypergroup involution is the identity mapping. Every symmetric hypergroup is commutative.
(3) Let K be a commutative hypergroup. Then K admits a Haar measure (see Spector [14]) which we denote by ω_K or, briefly, ω. The dual space $\hat{K}$ of K is given by

$$\hat{K}:=\{\alpha\in C_b(K): \alpha(x*\bar{y})=\alpha(x)\cdot\overline{\alpha(y)} \text{ for all } x, y\in K, \alpha\not\equiv 0\}$$

and equipped with the topology of uniform convergence on compact subsets. $\hat{K}$ is a locally compact space. The identity character is denoted by **1**. By Sect. 7.3 of Jewett [9] there exists a unique Plancherel measure π_K (or, briefly, π) on $\hat{K}$ which is associated with ω_K.

K is called a Pontryagin hypergroup whenever the dual $\hat{K}$ admits a dual hypergroup structure and the bidual $K\hat{}\,\hat{}$ agrees with K in the obvious way. In particular, every locally compact abelian group is a Pontryagin hypergroup. It should be, however, noticed that there exist commutative hypergroups which admit dual hypergroup structures, but which are not Pontryagin. Examples can be found in Zeuner [24, 25].
(4) Let $(K,*)$ and $(J,\cdot)$ be hypergroups. A continuous mapping $p\colon K\to J$ is said to be a hypergroup homomorphism if

$$\delta_{p(x)}\cdot\delta_{\overline{p(y)}}=p(\delta_x*\delta_{\bar{y}}) \quad \text{for all } x, y\in K.$$

It should be noted that, by our definition, a hypergroup homomorphism is not necessarily surjective or open. On the other hand, a hypergroup homomorphism is said to be an isomorphism if and only if it is a homeomorphism.

According to Sect. 4.2 of Jewett [9] the convolution of a measure $\mu\in M_b(K)$ and a function $f\in C_c(K)$ is defined by

$$\mu*f(x):=\int_K f(\bar{y}*x)\,d\mu(y)=\int f(z)\,d(\mu^-*\delta_x)(z)$$

and

$$f*\mu(x):=\int_K f(x*\bar{y})\,d\mu(y)=\int f(z)\,d(\delta_x*\mu^-)(z).$$

The following lemma supplements Theorem 14.1 B of Jewett [9].

1.5. Lemma. *Let ω_N be the normalized Haar measure of a compact normal subhypergroup N of a hypergroup K. Then $\omega_N*\mu=\omega_N*\mu*\omega_N=\mu*\omega_N$ for all $\mu\in M_b(K)$, and $\omega_N*f=\omega_N*f*\omega_N=f*\omega_N$ for all $f\in C_c(K)$.*

Proof. It suffices to prove the desired result for all Dirac measures $\mu=\delta_x$, $x\in K$. Fix $x\in K$ and a real valued function $f\in C_c(K)$. Since $h:=\omega_N*f\in C_c(K)$ is real valued, there exists $z\in N*x$ such that $h(z)=\sup_{t\in N*x} h(t)$. Since $\delta_{\bar{t}}*h=h$ and thus

$h(t*z)=h(z)$ for all $t\in N$, we have $h(w)=h(z)$ for all $w\in\{t\}*\{z\}$ and $t\in N$. Hence, h is constant on $N*x$. $x*N=N*x$ and Lemma 4.2 of Jewett [9] now imply that

$$\int_K f\,d(\omega_N*\delta_x*\omega_N)=\int_K h\,d(\delta_x*\omega_N)=h(x)=\int_K f\,d(\omega_N*\delta_x).$$

Hence, $\omega_N*\delta_x*\omega_N=\omega_N*\delta_x$. A similar argument yields $\omega_N*\delta_x*\omega_N=\delta_x*\omega_N$.

The following theorem establishes the well known relation between normal subhypergroups and open hypergroup homomorphisms.

1.6. Theorem. *Let K and J be hypergroups, and $p\colon K\to J$ an open and surjective hypergroup homomorphism. Then $H:=p^{-1}(e)$ is a normal subhypergroup in K, $K/H:=\{x*H\colon x\in K\}$ is a locally compact space with respect to the quotient topology, and*

$$\delta_{x*H}*\delta_{y*H}:=\int_K \delta_{z*H}\,d(\delta_x*\delta_y)(z)\qquad (x,y\in K) \tag{1.1}$$

defines a hypergroup structure on K/H. This hypergroup is isomorphic with J.

Conversely, if H is a normal subhypergroup of a hypergroup $(K,*)$ such that Eq. (1.1) defines a hypergroup structure on K/H (which includes that the definition of the convolution of cosets is independent of the choice of the representatives x, $y\in K$), then the canonical projection $\pi\colon K\to K/H$, $x\mapsto x*H$, is an open hypergroup homomorphism. In particular, the assertion is true if H is a compact normal subhypergroup, a supernormal subhypergroup, or a normal subgroup.

Proof. Assume first that $p\colon K\to J$ is a homomorphism with kernel H. We claim that

$$x*H=p^{-1}(p(x))=H*x\qquad \text{for all } x\in K. \tag{1.2}$$

To show this, we fix $x\in K$ and take $z\in x*H$. Then we choose $y\in H$ such that $z\in\operatorname{supp}\delta_x*\delta_y$. It follows that $p(z)\in\operatorname{supp}(\delta_{p(x)}\cdot\delta_{p(y)})=\{p(x)\}$ and $z\in p^{-1}(p(x))$. If conversely $z\in K$ and $z\notin x*H$, then $(\{\bar{x}\}*\{z\})\cap H=\varnothing$ (Jewett [9, Lemma 4.1 B]) and $e\notin\operatorname{supp}(\delta_{\overline{p(x)}}\cdot\delta_{p(z)})$ which implies that $p(z)\neq p(x)$ and $z\notin p^{-1}(p(x))$. The first part of Eq. (1.2) now follows. The second part can be checked in a similar way.

Since $H=\bar{H}$ is closed, it follows that H is a subhypergroup. Moreover, using Eq. (1.2) again, the quotient topology on K/H, and Eq. (1.1), we obtain that

$$\tilde{p}\colon K/H\to J,\qquad x*H\mapsto p(x)$$

is a well-defined homeomorphism satisfying

$$\begin{aligned}\tilde{p}(\delta_{x*H}*\delta_{y*H})&=\int_K \delta_{\tilde{p}(z*H)}\,d(\delta_x*\delta_y)(z)=\int_K \delta_{p(z)}\,d(\delta_x*\delta_y)(z)\\&=p(\delta_x*\delta_y)=\delta_{p(x)}*\delta_{p(y)}=\delta_{\tilde{p}(x*H)}*\delta_{\tilde{p}(y*H)}\end{aligned}$$

for all x, $y\in K$. Therefore Eq. (1.1) defines a hypergroup on K/H isomorphic to J.

Since $\pi\colon K\to K/H$, $x\mapsto x*H$, is open and continuous (Jewett [9, Theorem 10.3 B]), the second assertion of the theorem is obvious. Moreover, if H is compact and normal, then, as Lemma 1.5 shows, K/H agrees with the double

coset hypergroup $K//H$ which has been studied by Jewett in Sect. 14 of [9]. If H is a normal subgroup, then the consistency of the convolution (1.1) on K/H can be proved by using the same methods as in the proof of Theorem 4.1 in Ross [13] where this result is formulated for the special case when K is commutative. Finally, for supernormal hypergroups, the assertion is shown in Vrem [22, Theorem 2.1].

Every continuous homomorphism from a σ-compact locally compact group onto a second countable locally compact group is open (see, for instance, Theorem 5.29 in Hewitt and Ross [8]). We do not know whether this well-known result carries over to hypergroups. The only result of this type which is known for hypergroups is the following.

1.7. Proposition. *Let K and L be hypergroups and $\pi: K \to L$ a surjective hypergroup homomorphism such that* $\ker \pi := \pi^{-1}(e)$ *is compact. Then π is open if and only if π is proper. In particular, if K is compact, then every surjective hypergroup homomorphism $\pi: K \to L$ is open.*

Proof. $\ker \pi$ is a compact normal subhypergroup of K (see the proof of Theorem 1.6), and the canonical projection $p: K \to K/\ker \pi$ is a continuous, open and surjective hypergroup homomorphism (see Jewett [9, Theorem 10.3 B]). Moreover, it follows from Lemma 13.1 C in Jewett [9] that p is proper. Therefore π is open or proper if and only if the induced mapping $\tilde{\pi}: K/\ker \pi \to L$ has the same property (for the existence of $\tilde{\pi}$ we again refer to the proof of Theorem 1.6). Thus, Proposition 1.7 is a consequence of Lemma 1.2.

It is well-known that – up to a multiplicative constant – there exists at most one left (and, in a similar way, right) Haar measure ω_K on a hypergroup K (see Jewett [9]). On the other hand, the existence of a left Haar measure is proved only for the special cases of commutative, compact, and discrete hypergroups (Spector [14] and Jewett [9]). Therefore it is interesting to compare the Haar measures on K and K/H when H is a normal subhypergroup of K.

1.8. Theorem. *Let ω_N be the normalized Haar measure of a compact normal subhypergroup N of a hypergroup K. Then $L := K/N$ admits a left Haar measure if and only if K admits a left Haar measure. In particular, if $\pi: K \to L$ is the canonical projection, then $\pi(\omega_K)$ is a left Haar measure on L for a left Haar measure ω_K on K. Conversely, if ω_L is a left Haar measure on L and ω_N the normalized Haar measure of N, then*

$$\omega_K(f) := \omega_L(\tilde{\pi}^{-1}(\omega_N * f)) \qquad (f \in C_c(K)) \tag{1.3}$$

defines a left Haar measure on K with $\pi(\omega_K) = \omega_L$ where $\tilde{\pi}^{-1}$ stands for the inverse mapping of the isomorphism

$$\tilde{\pi}: C_c(K/N) \to \{g \in C_c(K): g = \omega_N * g\}, \qquad h \mapsto h \circ \pi.$$

The corresponding assertions are true also for right Haar measures. Furthermore, K is unimodular if and only if L is unimodular.

Proof. We first note that Theorem 14.2 E of Jewett [9] and Lemma 1.5 imply that $\tilde{\pi}$ is in fact an isomorphism. Since $\omega_N * \delta_x = \delta_x * \omega_N$ for all $x \in K$ by Lem-

ma 1.5, for $x, y \in K$ and $f \in C_c(K)$ it follows from Lemma 4.2 of Jewett [9] and $\omega_N = \omega_N^-$ that

$$\begin{aligned}\tilde{\pi}^{-1}(\omega_N *_x f)(y*N) = (\omega_N *_x f)(y) &= \int_N {}_x f(\bar{z}*y)\, d\omega_N(z) \\ &= \delta_x * \omega_N * \delta_y(f) = \omega_N * \delta_x * \delta_y(f) \\ &= \int_K \omega_N * f(u)\, d(\delta_x * \delta_y)(u) \\ &= {}_{x*N}\tilde{\pi}^{-1}(\omega_N * f)(y*N).\end{aligned}$$

Thus, for a given Haar measure ω_L on L,

$$\begin{aligned}\omega_K({}_x f) &= \omega_L(\tilde{\pi}^{-1}(\omega_N *_x f)) = \omega_L({}_{x*N}\tilde{\pi}^{-1}(\omega_N * f)) \\ &= \omega_L(\tilde{\pi}^{-1}(\omega_N * f)) = \omega_K(f).\end{aligned}$$

Thus, if ω_L is a left Haar measure of L, and if ω_K is given by Eq. (1.3), then ω_K is a left Haar measure. Since N is compact and therefore unimodular (Jewett [9, Theorem 7.2A]), the remaining assertions follow easily.

2 Duals of quotients and subhypergroups of commutative hypergroups

We first introduce some notation and generalize annihilators:

2.1. Definition. Let K be a commutative hypergroup, and $\hat{K}$ its dual. For a set $L \subset K$ and a character $\beta \in \hat{K}$ we define the *generalized annihilator* by

$$A(\hat{K}, L, \beta) := \{\alpha \in \hat{K} : \alpha(x) = \beta(x) \text{ for all } x \in L\}.$$

Likewise, for $W \subset \hat{K}$ and $z \in K$, we put

$$A(K, W, z) := \{x \in K : \alpha(x) = \alpha(z) \text{ for all } \alpha \in W\}.$$

Defining $A(\hat{K}, L) := A(\hat{K}, L, \mathbf{1})$ and $A(K, W) := A(K, W, e)$ for sets $L \subset K$ and $W \subset \hat{K}$, we obtain the usual annihilators (cf. Bloom and Heyer [3]).

We next list some known results about the dual $\hat{K}$ of a commutative hypergroup K for the sake of convenience.

2.2. Facts. Let K be a commutative hypergroup and π_K the Plancherel measure on $\hat{K}$.

(1) A character $\alpha \in \hat{K}$ is contained in $\operatorname{supp} \pi_K$ if and only if there exists a net $(f_\alpha)_{\alpha \in A} \subset C_c(K)$ of positive definite functions which tend to α uniformly on compact subsets of K (see Corollary 6 in Voit [19]).
(2) Let H be a subhypergroup of K. If $\alpha \in \operatorname{supp} \pi_K$, then the restriction $\alpha|_H$ of α is contained in the support of the Plancherel measure of H. This is an immediate consequence of part (1).
(3) $\operatorname{supp} \pi_K$ contains exactly one positive character (see Theorem 2.11 of Voit [17]). Moreover, if $\mathbf{1} \in \operatorname{supp} \pi$, then $\mathbf{1}$ is the only positive character (Corollary 2.9 in Voit [17]).

(4) If $\mu \in M_b(\hat{K})$ is a positive measure such that its inverse Fourier transform $\check{\mu}$ is nonnegative, then $\operatorname{supp}\mu$ contains a positive character (see Corollary 1.3 of Voit [21]).
(5) Assume $\hat{K}$ is a hypergroup. Then K is a Pontryagin hypergroup if and only if the support of the Plancherel measure in $K^{\wedge\wedge}$ is equal to $K^{\wedge\wedge}$ (see Theorem 12.4 of Jewett [9]).

Next we generalize Proposition 3.16 of Dunkl [5] and the Theorems 5 and 6 of Bloom [1].

2.3. Proposition. *Let H be a subhypergroup of a commutative hypergroup K which admits a dual hypergroup $\hat{K}$. If $\alpha, \beta \in \hat{K}$, then $\alpha|_H = \beta|_H$ is equivalent to $\beta \in \alpha * A(\hat{K}, H)$.*

Proof. Let $p\colon \hat{K} \to \hat{H}$, $\chi \mapsto \chi|_H$, be the restriction map of characters. Take $\alpha \in \hat{K}$ and $\gamma \in A(\hat{K}, H)$. Then $\alpha(x)\,\gamma(x) = \int_{\hat{K}} \chi(x)\, d(\delta_\alpha * \delta_\gamma)(\chi)$ for all $x \in K$, and therefore

$$\alpha|_H(x) = \alpha|_H(x)\,\gamma|_H(x) = \int_{\hat{H}} \varphi(x)\, d(p(\delta_\alpha * \delta_\gamma))(\varphi) \quad \text{for } x \in H.$$

In other words, $\alpha|_H$ is the inverse Fourier transform of $p(\delta_\alpha * \delta_\gamma) \in M^1(\hat{H})$. It follows from the injectivity of the Fourier transform (Jewett [9, Theorem 12.2A]) that $p(\delta_\alpha * \delta_\gamma) = \delta_{\alpha|_H}$. Thus $\operatorname{supp}(\delta_\alpha * \delta_\gamma) \subset \{\chi \in \hat{K} : \chi|_H = \alpha|_H\}$. In summary, we have shown that $\beta \in \alpha * A(\hat{K}, H)$ implies that $\alpha|_H = \beta|_H$.

To prove the converse implication, fix $\alpha, \beta \in \hat{K}$ such that $\alpha|_H = \beta|_H$. Since

$$\bar{\alpha}(x)\,\beta(x) = \int_{\hat{K}} \gamma(x)\, d(\delta_{\bar{\alpha}} * d_\beta)(\gamma)$$

for all $x \in K$, we have $(\bar{\alpha}\beta)|_H = (p(\delta_{\bar{\alpha}} * \delta_\beta))^\vee$ ($^\vee$ being the symbol for the inverse Fourier transform on H). Since we may apply fact 2.2(4) to $p(\delta_{\bar{\alpha}} * \delta_\beta)$, we conclude that $\operatorname{supp}(p(\delta_{\bar{\alpha}} * \delta_\beta))$ contains a positive character $\alpha_0 \in \hat{H}$. Since $\hat{K}$ is a hypergroup, we have $\mathbf{1} \in \operatorname{supp}\pi_K$. Therefore, by fact 2.2(2), $\mathbf{1} \in \operatorname{supp}\pi_H$, π_H being the Plancherel measure of H. Fact 2.2(3) now yields that $\alpha_0 = \mathbf{1}$. In summary, we see that $\mathbf{1} \in \operatorname{supp}(p(\delta_{\bar{\alpha}} * \delta_\beta))$. Therefore

$$A(\hat{K}, H) \cap \operatorname{supp}(\delta_{\bar{\alpha}} * \delta_\beta) \neq \varnothing.$$

It follows from Lemma 4.1B in Jewett [9] that $\beta \in \alpha * A(\hat{K}, H)$ which finishes the proof.

The following proposition supplements results of Bloom and Heyer [3, Proposition 3.1] about annihilators.

2.4. Proposition. *Let K be a commutative hypergroup.*

(1) *If $L \subset K$ is open, and if $\beta \in \hat{K}$ satisfies $\beta|_L \not\equiv 0$, then $A(\hat{K}, L, \beta)$ is compact.*
(2) *If $W \subset \hat{K}$ is open, and if $z \in K$ satisfies $\pi(W \cap \{\alpha \in \hat{K} : \alpha(z) \neq 0\}) > 0$, then $A(K, W, z)$ is compact.*
(3) *If L is a compact subhypergroup of K, and $\beta \in \hat{K}$ satisfies $\inf_{x \in K} |\beta(x)| > 0$, then $A(\hat{K}, L, \beta)$ is open.*
(4) *Assume $\hat{K}$ is a hypergroup. If L is a subhypergroup of K, then $A(\hat{K}, L, \beta) = A(\hat{K}, L) * \{\beta\}$ for all $\beta \in \hat{K}$.*

(5) *Assume $\hat{K}$ is a hypergroup. If L is a compact subhypergroup of K, then $A(\hat{K}, L, \beta) = A(\hat{K}, L) * \{\beta\}$ is open for all $\beta \in \hat{K}$.*
(6) *Assume $\hat{K}$ is a hypergroup. If W is a subhypergroup of $\hat{K}$, then $A(K, W, z) = A(K, W) * \{z\}$ for all $z \in K$.*
(7) *Assume $\hat{K}$ is a hypergroup. If W is a compact subhypergroup of $\hat{K}$, then $A(K, W, z) = A(K, W) * \{z\}$ is open for all $z \in K$.*

Proof. (1) Take $f \in C_c(K)$ with $f|_{K-L} \equiv 0$, $f|_L \geqq 0$ and $\beta f \not\equiv 0$. Then $\beta f \in C_c(K)$, $(\beta \cdot f)^\wedge \in C_0(\hat{K})$ (Jewett [9, Theorem 7.3H]) and $(\beta f)^\wedge|_{A(\hat{K}, L, \beta)} \equiv c$, $c > 0$ being a constant. As $A(\hat{K}, L, \beta)$ is closed, the proof is complete.
(2) This follows exactly as part (1) by using that $h^\vee \in C_0(K)$ for all $h \in C_c(\hat{K})$ (cf. Bloom and Heyer [2, Theorem 2.4.1]).
(3) This can be shown exactly as Proposition 3.1(e) in Bloom and Heyer [3].
(4) This is an immediate consequence of Proposition 2.3.
(5) This follows from part (4) and part (3) for $\beta = \mathbf{1}$.
(6) $A(K, W, z) \subset A(K, W) * \{z\}$ can be shown as the corresponding part of Proposition 2.3 by using the injectivity of the inverse Fourier transform (Jewett [9, Theorem 12.2A]). To establish the converse inclusion, fix $u \in A(K, W, z)$. Since $\varphi_u = \varphi_z$, the function $\varphi_u \cdot \varphi_{\bar{z}}$ is nonnegative on W and satisfies $\varphi_u \cdot \varphi_{\bar{z}} = (p(\delta_u * \delta_{\bar{z}}))^\vee$ ($^\vee$ being the inverse Fourier transform on W). Therefore, by fact 2.2(4), there exists a positive character $\alpha_0 \in \hat{W}$ contained in $\operatorname{supp}(p(\delta_u * \delta_{\bar{z}}))$. $\hat{K}$ is a hypergroup, and the Plancherel measure $\pi_{\hat{K}}$ on its dual satisfies $\mathbf{1} \in \operatorname{supp} \pi_{\hat{K}}$ (in fact, by Theorem 12.4 in Jewett [9], $\operatorname{supp} \pi_{\hat{K}}$ agrees with K when K is embedded into its bidual in the natural way). Fact 2.2(2) now shows that $\mathbf{1} \in \operatorname{supp} \pi_W$, π_W being the Plancherel measure of W. Therefore, by fact 2.2(3), $\alpha_0 = \mathbf{1}$. In summary, it follows that $\mathbf{1} \in \operatorname{supp}(p(\delta_u * \delta_{\bar{z}}))$. Thus, $A(K, W) \cap \operatorname{supp}(\delta_u * \delta_{\bar{z}}) \neq \varnothing$. It follows from Lemma 4.1B in Jewett [9] that $u \in A(K, W) * \{z\}$ which finishes the proof.
(7) After embedding K into $K^{\wedge\wedge}$ in the canonical way, we conclude from part (3) that $A(K, W)$ is open. Consequently, (7) follows from (6).

Let $p: K \to J$ be an open homomorphism from a commutative hypergroup K onto a hypergroup J. We are interested in a description of $\hat{J}$ in terms of $\hat{K}$. Because of Theorem 1.6 we can assume that $J = K/H$, H being a subhypergroup of K. This case is treated in the following theorem parts of which are shown also in Zeuner [25].

2.5. Theorem. *Let H be a subhypergroup of a commutative hypergroup K such that K/H is a hypergroup. Let $p: K \to K/H$ be the canonical projection. Then the mapping $\hat{p}: (K/H)^\wedge \to \hat{K}$, $\alpha \mapsto \alpha \circ p$, has the following properties:*

(1) *$\hat{p}$ is a homeomorphism from $(K/H)^\wedge$ onto $A(\hat{K}, H)$.*
(2) *If $\hat{K}$ is a hypergroup, then $(K/H)^\wedge$ is a hypergroup isomorphic to $A(\hat{K}, H)$ via $\hat{p}$.*
(3) *Assume H is compact (in which case $A(\hat{K}, H)$ is open in $\hat{K}$). Let ω be a Haar measure on K and π the associated Plancherel measure on $\hat{K}$. Then $p(\omega)$ is a Haar measure on K/H, and its associated Plancherel measure π_H on $(K/H)^\wedge$ satisfies*

$$\hat{p}(\pi_H) = \pi|_{A(\hat{K}, H)}.$$

Proof. (1) It is clear that $\hat{p}((K/H)^\wedge) \subset A(\hat{K}, H)$ and that $\hat{p}$ is injective and, in view of the topologies on $\hat{K}$ and $(K/H)^\wedge$, continuous.

In order to show that $\hat{p}$ is onto, we take $\alpha \in A(\hat{K}, H)$. Then α is constant on each H-coset. Since p is open, it follows that there exists $\tilde{\alpha} \in C_b(K/H)$ such that $\alpha = \tilde{\alpha} \circ p$. Since p is a homomorphism, we have $\tilde{\alpha} \in (K/H)^\wedge$. Since $\hat{p}(\tilde{\alpha}) = \alpha$, we see that $\hat{p}$ is surjective as claimed.

It remains to show that $\hat{p}$ is open, i.e. that for each $\alpha_0 \in (K/H)^\wedge$, $\varepsilon > 0$, and each compactum $W \subset K/H$ the set $R := \{\alpha \circ p : \alpha \in (K/H)^\wedge;\ |\alpha - \alpha_0| < \varepsilon \text{ on } W\}$ is open in $A(\hat{K}, H)$. Since p is open and continuous, we find a compactum $L \subset K$ such that $p(L) = W$. It follows that

$$R = \{\gamma \in A(\hat{K}, H) : |\gamma - \alpha_0 \circ p| < \varepsilon \text{ on } L\}$$

is open as claimed.

(2) Let $(K/H)^\wedge$ be equipped with the hypergroup structure which turns $\hat{p}$ into a hypergroup isomorphism. We have still to check that this hypergroup structure is consistent with pointwise multiplication. For this, we take $\alpha, \beta \in (K/H)^\wedge$ and $x \in K$. Then

$$\begin{aligned} \alpha(x * H)\, \beta(x * H) = \hat{p}(\alpha)(x)\, \hat{p}(\beta)(x) &= \int_{A(\hat{K}, H)} \gamma(x)\, d(\delta_{\hat{p}(\alpha)} * \delta_{\hat{p}(\beta)})(\gamma) \\ &= \int_{(K/H)^\wedge} \chi(x * H)\, d\hat{p}^{-1}(\delta_{\hat{p}(\alpha)} * \delta_{\hat{p}(\beta)})(\chi). \end{aligned}$$

(3) Assume H is compact. Then $A(\hat{K}, H)$ is open in $\hat{K}$ by Proposition 2.4(3). It is obvious that $p(\omega)$ is a Haar measure on K/H.

We still have to prove that $\hat{p}^{-1}(\pi|_{A(\hat{K}, H)})$ is the Plancherel measure on $(K/H)^\wedge$ associated with $p(\omega)$. To do this, fix $f \in C_c(K/H)$. Then $f \circ p \in C_c(K)$. Using the symbol $^\wedge$ to describe the Fourier transform of functions on K and K/H respectively, for $\beta \in (K/H)^\wedge$ we have $\beta \circ p \in \hat{K}$ and

$$(2.1) \qquad (f \circ p)^\wedge(\beta \circ p) = \int_K (f \circ p)(x)\, \overline{(\beta \circ p)(x)}\, d\omega(x) = \int_{K/H} f(y)\, \overline{\beta(y)}\, dp(\omega)(y) = \hat{f}(\beta).$$

Now take a character $\alpha \in \hat{K}$ with $\alpha|_H \not\equiv 1$ (i.e. α cannot be written as $\alpha = \beta \circ p$, $\beta \in (K/H)^\wedge$). Since $f \circ p$ is constant on each coset of H, and since characters on the compact hypergroup H are orthogonal with respect to the Haar measure ω_H, we conclude that

$$\begin{aligned} (2.2) \qquad (f \circ p)^\wedge(\alpha) &= \int_K (f \circ p)(x)\, \overline{\alpha(x)}\, d\omega(x) \\ &= \int_K \left(\int_K (f \circ p)(x * z)\, d\omega_H(z) \right) \cdot \alpha(\bar{x})\, d\omega(x) \\ &= \int_K \int_K (f \circ p)(x * z)\, \alpha(\bar{x})\, d\omega(x)\, d\omega_H(z) \\ &= \int_K \int_K (f \circ p)(\bar{x}) \cdot \alpha(x) \cdot \alpha(z)\, d\omega(x)\, d\omega_H(z) \\ &= \int_K \left((f \circ p)(\bar{x}) \cdot \alpha(x) \cdot \int_K \alpha(z)\, d\omega_H(z) \right) d\omega(x) = 0. \end{aligned}$$

It follows from (2.1), (2.2) and the Plancherel formula (Jewett [9, Theorem 7.3]) that

$$\begin{aligned}(2.3)\qquad \int_{K/H} |f(x)|^2\, dp(\omega)(x) &= \int_K |f\circ p(y)|^2\, d\omega(y) = \int_{\hat{K}} |(f\circ p)^\wedge(\alpha)|^2\, d\pi(\alpha)\\ &= \int_{(K/H)^\wedge} |(f\circ p)^\wedge(\beta\circ p)|^2\, d(\hat{p}^{-1}(\pi|_{A(\hat{K},H)}))(\beta)\\ &= \int_{(K/H)^\wedge} |\hat{f}(\beta)|^2\, d(\hat{p}^{-1}(\pi|_{A(\hat{K},H)}))(\beta).\end{aligned}$$

Since (2.3) is true for all $f\in C_c(K/H)$, and since this formula determines the Plancherel measure uniquely (Jewett [9, Theorem 7.3]), the proof is complete.

We next consider the problem to describe the dual of a subhypergroup H in terms of the dual of K. Since in general there may exist characters on H which do not extend to characters on K (see 2.14 and 2.17), we cannot give a general solution. For compact or open subhypergroups, however, the problem can be treated in a satisfying way. The following theorem generalizes Theorem 3.17 of Dunkl [5] and deals with the compact case:

2.6. Theorem. *Let K be a commutative hypergroup having a dual hypergroup $\hat{K}$. Let H be a compact subhypergroup of K. Then $\hat{H}$ is a hypergroup, $\hat{K}/A(\hat{K},H)$ is a hypergroup with respect to the convolution defined by Eq.* (1.1), *and $\hat{H}$ and $\hat{K}/A(\hat{K},H)$ are isomorphic.*

Proof. In order to define a hypergroup structure on the discrete space $\hat{H}$, we consider the mapping

$$p\colon \hat{K}\to\hat{H},\qquad \alpha\mapsto\alpha|_H.$$

p is surjective (see Ross [13, Lemma 2.2]) and continuous. For given $\alpha,\beta\in\hat{K}$, we define the convolution

$$\delta_{p(\alpha)} * \delta_{p(\beta)} := p(\delta_\alpha * \delta_\beta).$$

This convolution is obviously consistent with respect to pointwise multiplication of characters on H. Moreover, the injectivity of the Fourier transform on H (Jewett [9, 12.2B]) shows that this convolution on $\hat{H}$ is well-defined. Furthermore, it is clear that for $\alpha,\beta\in\hat{K}$, $p(\delta_\alpha * \delta_\beta)$ is a probability measure with compact, and thus finite, support. It is straightforward to check that $*$ defines in fact a commutative hypergroup structure on $\hat{H}$. It follows from Proposition 2.3 that the mapping

$$\tilde{p}\colon \hat{K}/A(\hat{K},H)\to\hat{H},\qquad \alpha * A(\hat{K},H)\mapsto\alpha|_H,$$

is a well-defined bijection. Since $A(\hat{K},H)$ is open in $\hat{K}$ by Proposition 2.4(3), it follows that $\hat{K}/A(\hat{K},H)$ is discrete, and $\tilde{p}$ a homeomorphism. Now Theorem 1.6 shows that $\hat{K}/A(\hat{K},H)$ is a hypergroup isomorphic to $\hat{H}$ via $\tilde{p}$.

2.7. Theorem. *Let H be an open subhypergroup of a commutative hypergroup K. Then the following statements hold:*

(1) *The mapping $\hat{p}\colon \hat{K}\to\hat{H}$, $\alpha\mapsto\alpha|_H$, is continuous, proper, and closed.*
(2) *Let ω be a Haar measure on K and π its associated Plancherel measure on $\hat{K}$. Then $\omega|_H$ is a Haar measure on H, and its associated Plancherel measure π_H satisfies $\pi_H=\hat{p}(\pi)$. In particular,* $\operatorname{supp}\pi_H=\hat{p}(\operatorname{supp}\pi)$.

(3) *Assume that* $\hat{K}$ *is a hypergroup and that* $\operatorname{supp}\pi_H=\hat{H}$. *Then* $\hat{H}$ *is a hypergroup,* $\hat{K}/A(\hat{K},H)$ *is a hypergroup with respect to the convolution defined by Eq.* (1.1), *and* $\hat{H}$ *and* $\hat{K}/A(\hat{K},H)$ *are isomorphic.*

Proof. (1) $\hat{p}$ is obviously continuous. Let ω be a Haar measure of K. Since H is open in K, $\omega|_H$ is a (nontrivial) Haar measure of H, and $L^1(H):=L^1(H,\omega|_H)$ may be identified with the closed $*$-subalgebra $\{f\in L^1(K): f|_{K-H}=0\}$ of $L^1(K)$ in a canonical way. Let B_H and B_K be the spaces of all multiplicative hermitian linear functionals on $L^1(H)$ and $L^1(K)$ respectively (where 0 is contained in both spaces). Both spaces are compact with respect to the weak $*$-topology. Consequently, $\tilde{p}: B_K\to B_H, f\mapsto f|_{L^1(H)}$, is proper and closed. For any commutative hypergroup M, $\tau_M: \hat{M}\to\Delta(L^1(M)):=B_M-\{0\}$, $\alpha\mapsto\varphi_\alpha$, $\varphi_\alpha(f):=\int_H \alpha f\,d\omega_M$, is a natural homeomorphism (see Sect. 6.3 of Jewett [9]). Since $\hat{p}=\tau_H^{-1}\circ\tilde{p}|_{\Delta(L^1(K))}\circ\tau_K$, and since $\tilde{p}(\varphi)=0$ is equivalent to $\varphi=0$ for $\varphi\in B_K$, it follows that $\hat{p}$ is proper and closed.

(2) To prove that $\hat{p}(\pi)$ is the Plancherel measure associated with $\omega|_H$, take $f\in C_c(H)$ and define $\tilde{f}\in C_c(K)$ as its trivial extension to K. Then, denoting the Fourier transform with respect to H and K by $\hat{}$ and using $\hat{f}(\alpha|_H)=\tilde{f}\hat{}(\alpha)$, we have

$$\int_H |f|^2\,d\omega|_H=\int_K |\tilde{f}|^2\,d\omega=\int_{\hat{K}} |\tilde{f}\hat{}|^2\,d\pi=\int_{\hat{H}} |\hat{f}|^2\,d\hat{p}(\pi).$$

Since the Plancherel measure on $\hat{H}$ is determined uniquely by this equation (Jewett [9, Theorem 7.3]), $\pi_H=\hat{p}(\pi)$ follows. In particular, we have

$$\hat{p}(\operatorname{supp}\pi)\subset\operatorname{supp}\pi_H\subset\operatorname{cl}(\hat{p}(\operatorname{supp}\pi)).$$

Since $\hat{p}$ is closed by part (1), it follows that $\hat{p}(\operatorname{supp}\pi)=\operatorname{supp}\pi_H$.

(3) $A(\hat{K},H)$ is a compact subhypergroup of $\hat{K}$ by Proposition 2.4(1). Therefore, $\hat{K}/A(\hat{K},H)$ bears a quotient hypergroup structure. Let $\rho: \hat{K}\to\hat{K}/A(\hat{K},H)$ be the canonical homomorphism. Applying Proposition 2.3, we conclude that there exists a unique bijective and continuous mapping $\tilde{p}: \hat{K}/A(\hat{K},H)\to\hat{H}$ such that $\hat{p}=\tilde{p}\circ\rho$. Since $\hat{p}$ is closed by part (1), $\tilde{p}$ is closed and hence a homeomorphism. Let $\hat{H}$ be equipped with the hypergroup structure that turns $\tilde{p}$ into a hypergroup isomorphism. The methods of the proof of Theorem 2.6 show that this hypergroup structure on $\hat{H}$ is consistent with pointwise multiplication of characters. This completes the proof.

If in Theorem 2.7 $\hat{K}$ bears a hypergroup structure, then $\hat{K}=\operatorname{supp}\pi$. We do not know whether this assumption also implies that $\hat{p}(\hat{K})=\hat{H}$ or $\hat{H}=\operatorname{supp}\pi_H$. On the other hand, we can give a condition on K which ensures that $\operatorname{supp}\pi_H=\hat{H}=\hat{p}(\hat{K})$ and which is satisfied by many examples. This condition is the symmetry of the L^1-algebra $L^1(K)$ which means that every continuous multiplicative linear functional on $L^1(K)$ is hermitian (see Rickart [12] or Naimark [11]). Symmetry conditions for $L^1(K)$ are discussed in Vogel [15, 16]. Here we note the following criterion.

2.8. Proposition. *Let* H *be an open subhypergroup of a commutative hypergroup* K *whose* L^1*-algebra* $L^1(K,\omega)$ *is symmetric. Then* $L^1(H)$ *is symmetric, and* $\hat{H}=\{\alpha|_H: \alpha\in\hat{K}\}$.

If in addition $\operatorname{supp}\pi_K=\hat{K}$, *then* $\operatorname{supp}\pi_H=\hat{H}=\{\alpha|_H: \alpha\in\hat{K}\}$.

Proof. The L^1-algebra $L^1(H)$ is isometrically isomorphic to a closed $*$-subalgebra of $L^1(K)$ and therefore symmetric (see, for example, Rickart [27, Corollary 4.7.7]). $\hat{H}=\{\alpha|_H: \alpha\in\hat{K}\}$ is a consequence of Theorem 4.7.11 of Rickart [12] and the fact that the dual space $\hat{M}$ of a commutative hypergroup M can be identified with the space of all nontrivial multiplicative hermitian linear functionals on $L^1(M)$ (see Jewett [20, Sect. 6.3]). The last conclusion follows from fact 2.2(2).

The following result is a consequence of Theorems 2.5, 2.6 and 2.7:

2.9. Corollary. *Let H be a subhypergroup of a Pontryagain hypergroup K. If H is compact, or if H is open and satisfies* $\operatorname{supp}\pi_H=\hat{H}$, *then H is a Pontryagin hypergroup and $H=A(K, A(\hat{K}, H))$.*

Proof. Applying Theorem 2.6 and 2.7 respectively, we see that $\hat{H}$ is a hypergroup isomorphic to $\hat{K}/A(\hat{K}, H)$. Theorem 2.5 shows that $(\hat{K}/A(\hat{K}, H))^\wedge$ is a hypergroup which is isomorphic to $A(K^{\wedge\wedge}, A(\hat{K}, H))$. Since K is a Pontryagin hypergroup, we know that $A(K^{\wedge\wedge}, A(\hat{K}, H))$ is isomorphic to $A(K, A(\hat{K}, H))$ in the obvious way. The bidual of H is a hypergroup which is, by Theorem 12.4D of Jewett [9], isomorphic to H in the obvious way. It follows that $H=A(K, A(\hat{K}, H))$ as claimed.

We collect further results about relations between annihilators and quotient hypergroups.

2.10. Lemma. *Let H be a subhypergroup of a commutative hypergroup K such that Eq.* (1.1) *defines a hypergroup structure on K/H. Then $H=A(K, A(\hat{K}, A))$.*

Proof. The inclusion $\subset$ is obvious. To prove the converse inclusion, take $x\in K-H$. Then there is a character $\rho\in(K/H)^\wedge$ such that $\rho(x*H)\neq 1$ (see, for instance, Ross [13, Lemma 2.1]). Thus, denoting the canonical projection from K onto K/H by π, we obtain $\rho\circ\pi\in A(\hat{K}, H)$ and $\rho\circ\pi(x)\neq 1$ which yields the inclusion.

2.11. Corollary. *Let H be an open subhypergroup of a Pontryagin hypergroup K. Then K/H is a hypergroup if and only if $H=A(K, A(\hat{K}, H))$. Moreover, if K/H is a hypergroup, then it is a Pontryagin hypergroup.*

Proof. The only-if-part follows from Lemma 2.10, and the if-part from Theorem 2.6 applied to the compact subhypergroup $A(\hat{K}, H)$. Furthermore, if K/H is a hypergroup, then its dual is a hypergroup isomorphic to $A(\hat{K}, H)$. Therefore, by Theorem 2.6, K/H is a Pontryagin hypergroup.

2.12. Lemma. *Let K be a commutative hypergroup with a dual hypergroup $\hat{K}$. Then*

(1) $H=A(\hat{K}, A(K, H))$ for every compact subhypergroup H of $\hat{K}$, and
(2) every compact open subhypergroup H of $\hat{K}$ is a Pontryagin hypergroup whose dual is isomorphic to $K/A(K, H)$.

Proof. (1) The inclusion $\subset$ is obvious. To prove the converse inclusion, take $\alpha\in\hat{K}-H$. Then there exists $\rho\in(\hat{K}/H)^\wedge$ with $\rho(\alpha*H)\neq 1$ such that ρ is contained in the support of the Plancherel measure on $(\hat{K}/H)^\wedge$ (this is a consequence of the fact that the Fourier transform $M_b(L)\to C_b(\operatorname{supp}\pi_L)$, $\mu\mapsto\hat{\mu}|_{\operatorname{supp}\pi_L}$ is injective for every commutative hypergroup L). Therefore, by Theorem 2.5, $\tilde{\rho}(\beta):=\rho(\beta*H)$ $(\beta\in\hat{K})$ defines a character on $\hat{K}$ which is contained in the support of the Plancherel measure on $K^{\wedge\wedge}$. By Jewett [9, Theorem 12.4A], there exists

$x \in K$ with $\tilde{\rho}(\beta) = \beta(x)$ for all $\beta \in \hat{K}$. Thus, $\beta(x) = 1$ for all $\beta \in H$ and $\alpha(x) \neq 1$ which establishes $\supset$.

(2) Since $A(K, H)$ is a compact open subhypergroup of K (see Proposition 2.4), Theorem 2.5 and part (1) imply that $K/A(K, H)$ is a hypergroup having a dual hypergroup which is isomorphic to $A(\hat{K}, A(K, H)) = H$. The assertion now follows from Theorem 12.4 of Jewett [9] and the fact that $\operatorname{supp} \pi_H = \hat{H}$.

The following theorem generalizes Proposition 2 of Lasser [10]:

2.13. Theorem. *Let K be a commutative hypergroup whose maximal subgroup $Z(K)$ is open. If $\hat{K}$ is a hypergroup, then K is a Pontryagin hypergroup.*

Proof. By Theorem 2.7(3), $Z(K)^\wedge$ is isomorphic to $\hat{K}/A(\hat{K}, Z(K))$ where $A(\hat{K}, Z(K))$ is a compact subhypergroup of $\hat{K}$. Since $\hat{K}/A(\hat{K}, Z(K))$ is a locally compact abelian group, it has polynomial growth (see, for instance, Emerson and Greenleaf [6]). Since $A(\hat{K}, Z(K))$ is compact, it follows that $\hat{K}$ is polynomially growing. Therefore, by Theorem 2.17 of Voit [17], the support of the Plancherel measure on $K^{\wedge\wedge}$ is equal to $K^{\wedge\wedge}$. Consequently, by Theorem 12.4 of Jewett [9], $K = K^{\wedge\wedge}$.

2.14. Example. (1) If we define a convolution $*$ on $K = [0, \infty[$ by

$$\delta_x * \delta_y = \frac{\cosh(x-y)}{2 \cosh x \cosh y} \delta_{|x-y|} + \frac{\cosh(x+y)}{2 \cosh x \cosh y} \delta_{x+y} \qquad (x, y \in [0, \infty[), \tag{2.4}$$

then $(K, *)$ becomes a symmetric hypergroup. Following Zeuner [23], it is called the *cosh hypergroup*. $H := \mathbb{N}_0$ is obviously a closed subhypergroup. This subhypergroup has the following pathological properties:

(a) Equation (1.1) does not define a quotient hypergroup on K/H.

This has been observed by Zeuner [23] and can be proved as follows: Since $A(\hat{K}, \mathbb{N}_0) = \{\mathbf{1}\}$ ($\hat{K}$ is given in [23]) and therefore $A(K, A(\hat{K}, \mathbb{N}_0)) = K$, Lemma 2.10 yields that $K/\mathbb{N}_0$ does not admit a quotient convolution.

(b) There exist characters on H which do not extend to characters on K.

For example, $\beta(n) := (-1)^n$ ($n \in \mathbb{N}_0$) is such a character on $\mathbb{N}_0$.

It should be noticed that the cosh hypergroup appears as deformation of the cosine hypergroup on $[0, \infty[$ whose convolution is generated by

$$\delta_x \cdot \delta_y := (\delta_{|x-y|} + \delta_{x+y})/2 \qquad (x, y \in [0, \infty[)$$

(see Voit [17, Example 4.5]). The hypergroup $(K, \cdot)$ does not have the pathological properties (a) and (b).

(2) Using the cosh hypergroup of part (1), we easily find discrete hypergroups which have the properties (a) and (b). For this we take K as the discrete subhypergroup $\mathbb{N}_0$ of the hypergroup considered in (1). Then $6 \cdot \mathbb{N}_0$ is a subhypergroup of $\mathbb{N}_0$ which has the properties (a) and (b). In fact, (a) follows from $A(\hat{K}, \mathbb{N}_0) = \{\mathbf{1}\}$, and (b) is a consequence of the fact that $\beta(6n) := (-1)^n$ ($n \in \mathbb{N}_0$) defines a character on $6 \cdot \mathbb{N}_0$ which does not extend to $\mathbb{N}_0$.

The example discussed in 2.14(2) is a polynomial hypergroup associated with a sequence of symmetric orthogonal polynomials. For such polynomial hypergroups on $\mathbb{N}_0$ we can describe the set of all characters on the subhyper-

group $2\cdot\mathbb{N}_0$ which do not extend to $\mathbb{N}_0$. We first recapitulate some basic facts about polynomial hypergroups.

2.15. Polynomial hypergroups. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}\subset\mathbb{R}$ be sequences satisfying a_n, $c_n>0$, $b_n\geqq 0$ and $a_n+b_n+c_n=1$ for all $n\in\mathbb{N}$. Assume that $\alpha:=\lim_{n\to\infty} a_n\in]0,1[$, $\beta:=\lim_{n\to\infty} b_n\in[0,1[$ and $\gamma:=\lim_{n\to\infty} c_n\in]0,1[$ exist. By Favard's Theorem (see Chihara [4, Chap. 1.4]), the recurrence formula

$$P_0=1,$$

$$P_1(x)=2\sqrt{\alpha\gamma}\cdot x+\beta \quad\text{and}\quad P_{n+1}=\frac{1}{a_n}((P_1-b_n)P_n-c_nP_{n-1}) \qquad (n\in\mathbb{N}) \tag{2.5}$$

defines a sequence $(P_n)_{n\in\mathbb{N}_0}$ of orthogonal polynomials. Assume that the linearization coefficients $g_{m,n,k}$ determined uniquely by $P_m\cdot P_n=\sum_{k=0}^{m+n} g_{m,n,k}P_k$ are nonnegative for all $m, n, k\in\mathbb{N}_0$. Then

$$\delta_m * \delta_n=\sum_{k=0}^{m+n} g_{m,n,k}\,\delta_k \qquad (m,n\in\mathbb{N}_0)$$

defines a hypergroup structure K on $\mathbb{N}_0$ which is called a polynomial hypergroup. For details and examples see Lasser [10] and Voit [18, 20]. For $x\in\mathbb{R}$ we define $\alpha_x\colon \mathbb{N}_0\to\mathbb{R}$ by $\alpha_x(n):=P_n(x)$. The dual space $\hat{K}$ and $\operatorname{supp}\pi$ are given by

$$\hat{K}=\left\{\alpha_x\colon x\in\left[-\frac{1-\beta}{2\sqrt{\alpha\gamma}},\frac{1-\beta}{2\sqrt{\alpha\gamma}}\right]\cup A\right\} \quad\text{and}\quad \operatorname{supp}\pi=\{\alpha_x\colon x\in[-1,1]\cup A\} \tag{2.6}$$

where $A\subset]-\infty,-1[$ is at most countable and -1 is its only possible cluster point (see Voit [20, Theorem 8.2(2)]). If $b_n=0$ for all $n\in\mathbb{N}$, then the polynomials P_n are symmetric, $A=\varnothing$ (see Theorem 8.2(2) in [20]), and the following statements hold:

2.16. Proposition. *Assume that $b_n=0$ for all $n\in\mathbb{N}$. Then $H:=2\cdot\mathbb{N}_0$ is a subhypergroup of K which is isomorphic to the polynomial hypergroup which is associated with the parameters*

$$\tilde{a}_n:=\frac{a_n a_{n+1}}{a_1}, \qquad \tilde{b}_n=\frac{a_n c_{n+1}+c_n a_{n-1}-c_1}{a_1}, \qquad \tilde{c}_n:=\frac{c_n c_{n-1}}{a_1} \quad (n\in\mathbb{N}).$$

Let $(\tilde{P}_n)_{n\in\mathbb{N}_0}$ be the associated sequence of orthogonal polynomials according to (2.5). Let $\tilde{\alpha}_z\colon H\to\mathbb{R}$ be given by $\tilde{\alpha}_z(2n):=\tilde{P}_n(z)$. Then

$$\hat{H}=\{\tilde{\alpha}_z\colon z\in[1-1/(2\alpha\gamma),\,1/(2\alpha\gamma)-1]\} \quad\textit{and}\quad \operatorname{supp}\pi_H=\{\tilde{\alpha}_z\colon z\in[-1,1]\}.$$

For $\alpha_x\in\hat{K}$, we have $\alpha_x|_H=\tilde{\alpha}_{2x^2-1}$. In particular, $\tilde{\alpha}_z\in\hat{H}$ can be extended to a character on $\hat{K}$ if and only if $z\in[-1,1/(2\alpha\gamma)-1]$.

Proof. It follows from Eq. (2.5) that $P_2(x)=\frac{1}{a_1}(4\alpha\gamma x^2-1+a_1)$ and

$$P_2(x)\,P_{2n}(x)=\tilde{a}_n\,P_{2n+2}(x)+\tilde{b}_n\,P_{2n}(x)+\tilde{c}_n\,P_{2n-2}(x) \qquad (n\in\mathbb{N}).$$

Defining

$$\tilde{P}_1(x):=2\lim_{n\to\infty}\sqrt{\tilde{a}_n\,\tilde{c}_n}\cdot x+\lim_{n\to\infty}\tilde{b}_n$$

according to Eq. (2.5), we conclude that H is isomorphic to the polynomial hypergroup associated with the parameters $\tilde{a}_n$, $\tilde{b}_n$ and $\tilde{c}_n$. Induction shows that the associated orthogonal polynomials $\tilde{P}_n$ satisfy $\tilde{P}_n(2x^2-1)=P_{2n}(x)$. The remaining assertions are obvious.

2.17. There exist many examples of polynomial hypergroups which satisfy $\alpha>\gamma$ and $b_n=0$ for all $n\in\mathbb{N}$ (see, for instance, Lasser [10] and Voit [18]). All these hypergroups have $2\cdot\mathbb{N}_0$ as subhypergroup such that not every character on $2\cdot\mathbb{N}_0$ extends to $\mathbb{N}_0$.

In particular, there exists a family of discrete Gelfand pairs (G, H) such that the double coset hypergroups $K=G//H$ are isomorphic to polynomial hypergroups having the properties above. To give examples, one has to take G as the automorphism group of the homogeneous tree of valency $d\geqq 3$ and H as the stabilizer of a fixed vertex of this tree. For details we refer to Sect. 5 of Voit [18].

References

1. Bloom, W.: Idempotent measures on commutative hypergroups. In: Heyer, H. (ed.) Probability Measures on Groups VIII. Proc. Conf., Oberwolfach 1985. (Lect. Notes Math., vol. 1210, pp. 13–23) Berlin Heidelberg New York: Springer 1986
2. Bloom, W., Heyer, H.: The Fourier transformation of probability measures on hypergroups. Rend. Math. **2**, 315–334 (1982)
3. Bloom, W., Heyer, H.: Convolution semigroups and resolvent families of measures on hypergroups. Math. Z. **188**, 449–474 (1985)
4. Chihara, T.S.: An Introduction to Orthogonal Polynomials. New York: Gordon and Breach 1978
5. Dunkl, C.F.: The measure algebra of a locally compact hypergroup. Trans. Am. Math. Soc. **179**, 331–348 (1973)
6. Emerson, W.R., Greenleaf, F.P.: Asymptotic behaviour of products $C^p=C+\ldots+C$ in locally compact abelian groups. Trans. Am. Math. Soc. **145**, 171–204 (1969)
7. Hauenschild, W., Kaniuth, E., Kumar, A.: Harmonic Analysis on central hypergroups and induced representations. Pac. J. Math. **110**, 83–112 (1984)
8. Hewitt, E., Ross, K.A.: Abstract Harmonic Analysis I. Berlin Heidelberg New York: Springer 1979
9. Jewett, R.I.: Spaces with an abstract convolution of measures. Adv. Math. **18**, 1–101 (1975)
10. Lasser, R.: Orthogonal polynomials and hypergroups. Rend. Mat. Appl. **2**, 185–209 (1983)
11. Naimark, M.A.: Normed Algebras, 3rd American ed. Groningen: Wolters-Noordhoff 1972
12. Rickart, C.E.: General Theory of Banach Algebras. New York: Van Nostrand 1960
13. Ross, K.A.: Centers of hypergroups. Trans. Am. Math. Soc. **243**, 251–269 (1978)
14. Spector, R.: Mesures invariantes sur les hypergroupes. Trans. Am. Math. Soc. **239**, 147–166 (1978)
15. Vogel, M.: Spectral synthesis on algebras of orthogonal polynomial series. Math. Z. **194**, 99–116 (1987)

16. Vogel, M.: Harmonic Analysis and spectral synthesis in central hypergroups. Math. Ann. **281**, 369–385 (1988)
17. Voit, M.: Positive characters on commutative hypergroups and some applications. Math. Z. **198**, 405–421 (1988)
18. Voit, M.: Central limit theorems for random walks on $\mathbb{N}_0$ that are associated with orthogonal polynomials. J. Multivariate Anal. **34**, 290–322 (1990)
19. Voit, M.: On the dual space of a commutative hypergroup. Arch. Math. **56**, 380–385 (1991)
20. Voit, M.: Factorization of probability measures on symmetric hypergroups. J. Aust. Math. Soc., Ser. A **50**, 417–467 (1991)
21. Voit, M.: On the Fourier transform of positive, positive definite measures on commutative hypergroups, and dual convolution structures. Manuscr. Math. **72**, 141–153 (1991)
22. Vrem, R.: Connectivity and supernormality results for hypergroups. Math. Z. **195**, 419–428 (1987)
23. Zeuner, Hm.: Properties of the cosh hypergroup. In: Heyer, H. (ed.) Probability Measures on Groups IX. Proc. Conf., Oberwolfach 1988. (Lect. Notes Math., vol. 1379, pp. 425–434) Berlin Heidelberg New York: Springer 1989
24. Zeuner, Hm.: Polynomial hypergroups in several variables. Arch. Math. (to appear)
25. Zeuner, Hm.: Duality of commutative hypergroups. In: Probability Measures on Groups X. Proc. Conf., Oberwolfach 1990. Plenum Press, pp. 467–488. New York London: Plenum Press 1991