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Weak continuity of holomorphic automorphisms in JB*-triples

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0 Introduction

Suppose E is a complex Banach space and $D \subset E$ is the open unit ball of E. Then the linear geometry of E is completely determined by the holomorphic structure of the open unit ball D, more precisely (compare [15]): Two complex Banach spaces are isometrically isomorphic if and only if the corresponding open unit balls are biholomorphically equivalent. In the linear theory it is standard to consider besides the norm topology on E other topologies like the weak topology w or w^* if E has a predual. The question arises which of the holomorphic transformations of the open unit ball D of E are also continuous in these other topologies.

In this note we study the special case of complex Banach spaces E where there are many biholomorphic automorphisms of the open unit ball, more precisely where the group $\mathrm{Aut}(D)$ of all biholomorphic automorphisms is transitive on D. It is known [13] that these Banach spaces can be algebraically characterized by a certain ternary-type structure (called JB*-triple) given by a (uniquely determined) Jordan triple product $\{xyz\}$. For instance, the underlying Banach space of every C*-algebra A is a JB*-triple and $\{xyz\}=(xy^*z+zy^*x)/2$ in this case.

In Sect. 3 of this paper we answer the above question in the following way (compare 3.6 and 3.7): *Denote by* $C := \mathrm{Cont}_w(E)$ *the set of all* $a \in E$ *such that the a-squaring map* $q_a: x \mapsto \{xax\}$ *on E is weakly continuous on bounded subsets. Then C is a closed characteristic* (*triple*) *ideal in E and* $g \in \mathrm{Aut}(D)$ *is weakly continuous if and only if* $g(0) \in C$ *holds.* In Sect. 2 we compute this space for various examples. It turns out that the elements of $\mathrm{Cont}_w(E)$ are closely related to compact operators on Hilbert space.

Notation. For every complex Banach space E denote by $\mathscr{L}(E)$ the Banach algebra of all bounded linear operators on E and by $\mathrm{GL}(E)$ the subgroup of all

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invertible operators. $\mathscr{K}(E)$ is the subalgebra of compact operators and $\mathscr{H}(E) \subset \mathscr{L}(E)$ is the $\mathbb{R}$-linear subspace of hermitian operators on E. By $\mathscr{L}(E, F)$ we denote the Banach space of all bounded operators $E \to F$. For every locally compact topological space S we denote by $\mathscr{C}_0(S)$ the space of all continuous complex-valued functions on S vanishing at ∞ endowed with the sup-norm. In case S is compact we simply write $\mathscr{C}(S)$. For every topological group G we denote by G^0 the connected identity component of G. Although we consider various topologies on Banach spaces the notion of boundedness always refers to the norm topology.

1 Preliminaries

We recall that a JB*-triple (compare [13]) is a complex Banach space E together with a continuous mapping (called Jordan triple product)

$$E \times E \times E \to E \qquad (x, y, z) \mapsto \{x y z\}$$

such that for all elements in E the following conditions (J_1)–(J_4) hold, where for every $x, y \in E$ the operator $x \square y$ on E is defined by $z \mapsto \{x y z\}$:

(J_1) $\{x y z\}$ is symmetric bilinear in the outer variables x, z and conjugate linear in the inner variable y,
(J_2) $\{ab\{xyz\}\} = \{\{abx\} yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$, (Jordan triple identity)
(J_3) $x \square x \in \mathscr{L}(E)$ is a hermitian operator with spectrum $\geqq 0$,
(J_4) $\|\{xxx\}\| = \|x\|^3$.

It is known [13, p. 523] that in this definition condition (J_4) can be replaced by $\|x \square x\| = \|x\|^2$ and that

$$\|x \square y\| \leqq \|x\| \cdot \|y\| \tag{1.1}$$

holds for all x, $y \in E$ (compare [8]). The simplest example of a non-trivial JB*-triple is the complex line $\mathbb{C}$ with the triple product $\{x y z\} = x \bar{y} z$. More generally, every C*-algebra A becomes a JB*-triple in the triple product $\{x y z\} = (x y^* z + z y^* x)/2$ – we denote it by A^{JT}.

A linear subspace $I \subset E$ is called a *subtriple* if $\{III\} \subset I$ and an *ideal* in E if $\{EEI\} + \{EIE\} \subset I$. For every closed ideal $I \subset E$ the quotient Banach space E/I is again a JB*-triple in the obvious triple product and the canonical projection is a homomorphism. Here by a homomorphism h: $E \to F$ of JB*-triples we understand just a linear mapping h satisfying

$$h\{x y z\} = \{(hx)(hy)(hz)\}$$

for all x, y, z in E. Every homomorphism $E \to \mathbb{C}$ is called a *character* on E. By [13, p. 505] every character is continuous.

Like in the C*-algebra case, those JB*-triples are of particular interest in which the points can be separated by characters. By definition the JB*-triple E is called *abelian* or *commutatitve* if $E \square E \subset \mathscr{L}(E)$ is a commutative set of operators. Clearly, A^{JT} is a commutative JB*-triple for every commutative C*-algebra A. On the other hand, to every JB*-triple E and every $a \in E$ there is a closed abelian subtriple of E containing a. By [13, p. 507] the abelian JB*-triples are, up to isomorphism, exactly the spaces

$$\mathscr{C}_0^{\mathbb{T}}(S) := \{f \in \mathscr{C}_0(S) \colon f(ts) = tf(s) \text{ for all } t \in \mathbb{T}\}$$

where $\mathbb{T} := \{t \in \mathbb{C} \colon |t| = 1\}$ and S is a locally compact $\mathbb{T}$-principal fibre bundle. Furthermore the characters on $\mathscr{C}_0^{\mathbb{T}}(S)$ are just the point evaluations $h(f) = f(s)$ at points $s \in S$. From this, the following result is an easy consequence (realize that the homomorphic image of an abelian subtriple is contained in a closed abelian subtriple and get the statement for abelian triples by lifting back characters – compare also [1]):

(1.2) **Proposition.** *Let $h \colon E \to F$ be a homomorphism of* JB*-*triples. Then h is a contraction, i.e. $\|h(x)\| \leqq \|x\|$ for all $x \in E$. In particular, h is continuous. Furthermore the image of h is closed in F and h induces an isometry $E/I \cong h(E)$.*

(1.3) **Corollary.** *Let E be a* JB*-*triple and I, $F \subset E$ closed subtriples. Then also $I + F$ is a closed subtriple if I is an ideal in E.*

Proof. Apply (1.2) to the canonical projection $E \to E/I$. □

Let E be a JB*-triple and denote by $\mathrm{Aut}(E) \subset \mathrm{GL}(E)$ the subgroup of all triple automorphisms of E. The elements of $\mathrm{Aut}(E)$ are precisely the isometries in $\mathrm{GL}(E)$. $\mathrm{Aut}(E)$ is a real algebraic subgroup of $\mathrm{GL}(E)$ in the sense of [9] and in particular a real Banach Lie group. The Lie algebra $\mathfrak{aut}(E)$ of $\mathrm{Aut}(E)$ can be identified with the space of all derivations of E, i.e. of all linear mappings $\delta \colon E \to E$ with

$$\delta\{xyz\} = \{(\delta x)\, yz\} + \{x(\delta y)z\} + \{xy(\delta z)\}$$

for all x, y, $z \in E$. All derivations on E are automatically continuous (compare [2]). By [13, p. 523] we have

$$\mathfrak{aut}(E) = i\mathscr{H}(E). \tag{1.4}$$

Notice that, by polarization, the Jordan triple identity J_2 is equivalent to $ia\square a$ being a derivation of the triple product for every $a \in E$. For every x, $y \in E$ in particular the operator $\exp(x \square y - y \square x)$ is in $\mathrm{Aut}(E)$. The subgroup generated by all such elements is denoted by $\mathrm{Int}(E)$, the group of all *inner automorphisms* of E. It is a connected normal subgroup of $\mathrm{Aut}(E)$.

A JB*-triple E is called a JBW*-triple if E is a dual Banach space (compare [11, 3]). In that case it has a unique predual E_* and we refer to $w^* := \sigma(E, E_*)$ as the weak*-topology on E. Every automorphism of a JBW*-triple is $w^* - w^*$-continuous. The bidual E^{**} of any JB*-triple E is a JBW*-triple whose triple product extends that of E and is separately w^*-continuous. For every norm closed ideal $I \subset E$ the w^*-closure J of I in E^{**} is an ideal in E^{**} with $J \cap E = I$. Often this can be used to reduce the study of ideals in JB*-triples to the special case of JBW*-triples. We give some examples:
First recall the definition of a JB*-algebra (sometimes also called a Jordan C*-algebra). This is a complex Jordan algebra B with a complete norm and

conjugate-linear involution $*$ such that the Jordan product $x \circ y$ and the derived triple product

$$\{x y z\} = x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*) \tag{1.5}$$

satisfy the conditions (J_4)–(J_6) where

(J_5) $\|x \circ y\| \leqq \|x\| \cdot \|y\|$,
(J_6) $(x \circ y)^* = y^* \circ x^*$.

Every C*-algebra A with product xy becomes a JB*-algebra A^J in the Jordan product $x \circ y = (xy + yx)/2$ and every JB*-algebra B becomes a JB*-triple B^T in the triple product (1.5). The identity $A^{JT} = (A^J)^T$ is easily verified. In case the JB*-algebra B has a unit e, the Jordan product and involution on B are recovered from the Jordan triple product on B^T by $x \circ y = \{x e y\}$ and $x^* = \{e x e\}$. For every JB*-algebra B, the bidual B^{**} is a JB*-algebra with unit.

For all three structures (i.e. C*-algebras, JB*-algebras, JB*-triples) the notion of ideal exists and behaves very well by passing from one structure to the other, more precisely

(1.6) **Lemma.** *Let B be a* JB*-*algebra, $E = B^T$ the corresponding* JB*-*triple and $I \subset B$ a closed linear subspace. Then the following two conditions are equivalent:*

(1) *I is a (Jordan algebra) ideal in B, i.e. $I \circ B \subset I$,*
(2) *I is a (triple) ideal in E, i.e. $\{EEI\} + \{EIE\} \subset I$.*

Each of these two conditions implies

(3) *I is $*$-invariant, i.e. $I^* = I$.*

In case $B = A^J$ for a C-algebra A, condition (1) is also equivalent to*

(4) *I is a (two-sided associative) ideal in A, i.e. $IA + AI \subset I$.*

Proof. (1)⇒(3) follows from [18, p. 98] and also gives (1)⇒(2). The implication (2)⇒(1) can easily be shown by passing to the bidual. Finally, (1)⇒(4) follows from [18, p. 107]. □

The essential part of the following proposition (i.e. the direction (1)⇒(3)) can be found in [7, p. 330]. We provide an alternative proof:

(1.7) **Proposition.** *Let E be a* JB*-*triple and $I \subset E$ a closed linear subspace. Then the following conditions are equivalent:*

(1) *I is invariant under all operators $x \square x$, $x \in E$,*
(2) *I is invariant under the group* Int(E) *of all linear automorphisms,*
(3) *I is an ideal.*

Proof. (1)⇒(2) is obvious from the definition of Int(E) and the fact that (1) implies $\{EEI\} \subset I$ by polarization.

(2)⇒(1) Suppose I is Int(E)-invariant. Int(E) contains the real one-parameter subgroup $\exp(it x \square x)$ for every $x \in E$. Differentiating with respect to t implies that I is invariant under the operator $i x \square x$ and hence under $x \square x$.

(1)⇒(3) Suppose $I \subset E$ satisfies (1) and hence $\{EEI\} \subset I$. We have the show that $\{EIE\} \subset I$. After passing to the bidual of E we may assume without loss of generality that E is a JBW*-triple. But then the closed unit ball of E has an extreme point e. By [16, p. 190] the element e is a complete tripotent in

E, i.e. $\{eee\}=e$ and $E=E_1 \oplus E_2$, where E_k is the k-eigenspace of the hermitian operator $2e\square e$ for $k=1, 2$. From (1) we get $I=I_1 \oplus I_2$ with $I_k:=I\cap E_k$. Then E is a Jordan algebra in the product $x\circ y:=\{xey\}$ and clearly I is an algebra ideal in E. The subalgebra $E_2 \subset E$ is a JB*-algebra with unit e and involution $a\mapsto a^*=\{eae\}$. By 1.6 the algebra ideal $I_2\subset E_2$ is $*$-invariant. For $y\in E_2$ the formula (1.5) with y replaced by y^* is nothing but the Jordan triple identity, i.e. $\{EI_2E\}\subset I$. Because of $\{E_2 E_1 E_2\}=0$ we only have to show that $\{E_1 I_1 E\}\subset I$. For every x, $y\in E_1$ the element $F(x,y):=2\{xye\}\in E_2$ satisfies $F(x,y)=F(y,x)^*$. This implies for every $y\in I_1$ the inclusion $F(x,y)\in I_2$ and in particular $\{xyx\}=2F(x,y)\circ x\in I_1$ (compare [17, p. 76] and also [14, p. 469]), i.e. $\{E_1 I_1 E_1\}\subset I_1$. The only missing inclusion $\{E_1 I_1 E_2\}\subset I_2$ now is obtained in the following way: For every $x\in E_1$, $y\in I_1$, $a\in E_2$ the first three terms in the equation

$$\{ea^*\{xye\}\}=\{\{ea^*x\}\,ye\}-\{x\{a^*ey\}\,e\}+\{xya\}$$

belong to I by what has been proved so far, i.e. $\{xya\}\in I_2$. $\square$

In referring to [18, p. 107] in the proof of 1.6 we implicitely used already the notion of an M-ideal, which is defined for arbitrary Banach spaces X as follows: The closed linear subspace $Y\subset X$ is called an M-ideal in X, if the dual Banach space X^* admits a direct sum decomposition $X^*=N\oplus M$ into closed subspaces such that $N=\{\lambda\in X^*\colon\ \lambda(Y)=0\}$ is the annihilator of Y and $\|\nu+\mu\|=\|\nu\|+\|\mu\|$ holds for all $\nu\in N$ and $\mu\in M$.

(1.8) **Proposition.** *Let E be a* JB*-*triple. Then for every closed linear subspace $I\subset E$ the following conditions are equivalent*:

(1) *I is invariant under all hermitian operators on E,*
(2) *I is invariant under all derivations of E,*
(3) *I is an* M-*ideal in the underlying Banach space of E,*
(4) *I is an ideal in E.*

Proof. (1)$\Leftrightarrow$(2) follows from 1.4, (3)$\Rightarrow$(1) is contained in [18, p. 107] and (1)$\Rightarrow$(4) follows from 1.7 since every operator $x\square x$ is hermitian on E. It remains to show (4)$\Rightarrow$(3): Suppose that I is an ideal in E. Then the w*-closure J of I in the bidual E^{**} is an ideal in E^{**} with $J\cap E=I$. By [11, p. 128] the ideal I admits a complementary w*-closed ideal in E^{**} i.e., J is an M-summand of E^{**} in the sense of [6]. By [6, p. 305] also I is an M-ideal in E. $\square$

(1.9) **Definition.** A linear subspace $I\subset E$ is called *characteristic* if it is invariant under all automorphisms of E.

Notice that by (1.7) every characteristic closed linear subspace in E automatically is an ideal in E.

(1.10) *Example.* Let S be a locally compact topological space and $E:=\mathscr{C}_0(S)$. Then the closed ideals $I\subset E$ are well known to be precisely the subsets $I=\{a\in E\colon\ a|T=0\}$ with $T\subset S$ a closed subset. The ideal I is characteristic in E if and only if T is invariant under all homeomorphisms of S.

2 Weak continuity and squaring in JB*-triples

In the following let E be a JB*-triple. For every $a\in E$ then E becomes a Jordan algebra with the product $x\circ y:=\{xay\}$. This product depends on the choice of the element a and is uniquely determined by the corresponding squaring mapping q_a: $E \to E$ defined by $q_a(x) := \{x a x\}$.

With v the norm topology on E, we call a locally convex topology $\mathfrak{t}$ on E *admissible* if it is coarser than v, i.e. $\mathfrak{t} \leqq v$. We are mainly interested in the weak topology $w = \sigma(E, E^*)$ on E, which clearly is admissible.

In case E is a JBW*-triple we get further examples of admissible topologies on E by $w^* = \sigma(E, E_*)$, τ^* the Mackey topology associated to the duality $\langle E, E_* \rangle$ and the strong* topology s^* as defined in [2, 19]. These satisfy $w^* \leqq s^* \leqq \tau^* \leqq v$ and all are invariant under $\mathrm{Aut}(E)$.

Suppose E is a JBW*-triple and $\mathfrak{t}$ is a linear topology on E with $w^* \leqq \mathfrak{t} \leqq v$. Then in [20] the notion of a $\mathfrak{t}$-compact element was introduced – this is an element $a \in E$ such that q_a is $w^* - \mathfrak{t}$-continuous on bounded subsets of E. By this – essentially for $\mathfrak{t} = w^*$ – the compactness of arbitrary elements $a \in E$ can be characterized in terms of the triple product structure without realizing a as an operator on a Hilbert space. Here we modify the notion of $\mathfrak{t}$-compactness slightly and extend it to arbitrary JB*-triples E: For every admissible topology $\mathfrak{t}$ on E, we denote by $\mathrm{Cont}_{\mathfrak{t}}(E)$ the set of all $a \in E$ such that q_a is $\mathfrak{t} - \mathfrak{t}$-continuous on bounded subsets of E. Then the following statement is obvious

(2.1) **Lemma.** *For every closed subtriple $F \subset E$*

$$F \cap \mathrm{Cont}_{\mathfrak{t}}(E) \subset \mathrm{Cont}_{\mathfrak{t}}(F)$$

holds, where the restriction of $\mathfrak{t}$ *to F is again denoted by* $\mathfrak{t}$.

In the same way as in [20, p. 175] one can show

(2.2) **Lemma.** $\mathrm{Cont}_{\mathfrak{t}}(E)$ *is a norm closed inner ideal in E, i.e. a linear subspace C of E with* $\{CEC\} \subset C$.

As an application of (1.7) we get furthermore

(2.3) **Proposition.** *Let E be a* JB*-*triple and* $\mathfrak{t}$ *an admissible topology on E which is invariant under the group* $\mathrm{Aut}(E)$ *– for instance,* $\mathfrak{t} = w$, *the weak topology or, in case E is a* JBW*-*triple,* $\mathfrak{t}$ *one of* w^*, s^*, τ^*. *Then* $\mathrm{Cont}_{\mathfrak{t}}(E)$ *is a closed characteristic ideal in E.*

For the rest of this section we restrict $\mathfrak{t}$ to the weak topology $\mathfrak{t} = w$. Since for every JB*-triple E the triple product is w-continuous in every variable separately, an element $a \in E$ is in $\mathrm{Cont}_w(E)$ if and only if the following is true: For every bounded net $(x_\alpha)_{\alpha \in A}$ converging weakly to 0, also the net $(y_\alpha)_{\alpha \in A}$ with $y_\alpha := \{x_\alpha\, a\, x_\alpha\}$ converges weakly to 0.

(2.4) **Lemma.** *Let be a* JB*-*algebra with unit e. Then the following three conditions are equivalent*

(1) $\mathrm{Cont}_w(E) = E$,

(2) *For every bounded self-adjoint net* (x_α) *in E converging weakly to* 0, *the same is true for the net* (x_α^2),

(3) *For every bounded net* (x_α) *in E converging weakly to* 0, *the same is true for the net* $(x_\alpha^* \circ x_\alpha)$.

Proof. Since $\mathrm{Cont}_w(E)$ is an ideal in E, condition (1) is equivalent to $e \in \mathrm{Cont}_w(E)$. For the proof of the remaining equivalences write $x_\alpha = u_\alpha + i v_\alpha$ with u_α, v_α self-adjoint. Then $x_\alpha^* \circ x_\alpha = u_\alpha^2 + v_\alpha^2$. $\square$

Statement (3) in 2.4 is similar to a condition in [5, p. 60] characterizing the Dunford-Pettis property of C*-algebras: By definition a Banach space E has the Dunford-Pettis property if for all weakly null-sequences (x_n) and (λ_n) in E and E^* respectively, $\lim_{n\to\infty} \lambda_n(x_n)=0$ is true. One of the conditions occuring in [5] in case of a C*-algebra A is the following: For every weakly null-sequence (x_n) in A also the sequence $(x_n^* \circ x_n)$ is weakly null, where $\circ$ is the Jordan product. Two differences to (3) in 2.4 seem to occur here: 1. Instead of nets there are sequences. This can be formulated in terms of sequential continuity. 2. No boundedness is required. This is not a difference, since by the principle of uniform boundedness the following holds: *Every weakly convergent sequence in a Banach space is norm bounded.*

(2.5) **Definition.** For every JB*-triple E, let $\mathrm{CONT}_w(E)$ be the set of all $a \in E$ such that the squaring map $q_a\colon E \to E$ is sequentially $w-w$-continuous on E or, equivalently, such that for every weakly null-sequence (x_n) in E also the sequence $(\{x_n\, a x_n\})$ is weakly null.

It is easily verified that $\mathrm{CONT}_w(E)$ is a closed linear subspace of E that is invariant under all automorphisms of E. With 1.7 we conclude

(2.6) **Proposition.** *For every* JB**-triple E the space* $\mathrm{CONT}_w(E)$ *is a closed characteristic ideal in E containing* $\mathrm{Cont}_w(E)$.

(2.7) **Proposition.** *Let A be a* C**-algebra with unit and let $E=A^{JT}$ be the underlying* JB**-triple. Then E has the Dunford-Pettis property if and only if* $\mathrm{CONT}_w(E)=E$.

In [5] all W*-algebras with the Dunford-Pettis property have been classified. These are precisely the l^∞-sums of W*-algebras of type I_{n_k} with $\sup_k(n_k)<\infty$. Furthermore the Dunford-Pettis property is inherited to C*-subalgebras. So in particular all commutative C*-algebras have the Dunford-Pettis property – a result originating from Grothendieck. As a consequence we get

(2.8) **Proposition.** $\mathrm{CONT}_w(E)=E$ *for every commutative* JB**-triple E.*

Let us present also a direct proof. Since E is of the form $\mathscr{C}_0^{\mathbb{T}}(S)$ by Sect. 1, the statement holds if it is true for the bigger JB*-triple $\mathscr{C}_0(S)$. So we may assume without loss of generality that $E=\mathscr{C}(\Omega)$ for some compact space Ω. It is enough to show:
The algebra product $(f,g)\mapsto fg$ is sequentially weakly continuous on $E\times E$.

Proof. We identify E^* with the space $\mathscr{M}(\Omega)$ of all complex regular Borel measures on Ω (compare [10, p. 364]). Suppose the statement is false. Then there is an $\varepsilon>0$, a regular Borel measure $\mu \geqq 0$ on Ω and a weakly null-sequence (f_n) of real-valued continuous functions on Ω such that for all n

$$\|f_n\| \leqq 1 \quad \text{and} \quad \int_\Omega f_n^2\, d\mu \geqq \varepsilon.$$

The conditions $\mu(\Omega)<\infty$ and $\lim_{n\to\infty} f_n(\omega)=0$ for all $\omega \in \Omega$ imply by Egorov's theorem (compare [10, p. 158]) the existence of a Borel set $A \subset \Omega$ such that the sequence (f_n) converges uniformly on A to 0 and such that $\mu(B)<\varepsilon/2$ holds for $B:=\Omega\setminus A$. In particular there is an index n_0 with $\mu(A)\cdot f_n^2(\omega)\leqq\varepsilon/2$ for all $n>n_0$ and all $\omega\in A$. But then

$$\int_A f_n^2\,d\mu\leqq\varepsilon/2 \quad\text{and}\quad \int_B f_n^2\,d\mu\leqq\mu(B)<\varepsilon/2$$

for all $n>n_0$ produce a contradiction. $\square$

(2.9) *Remark.* Let $I\subset\mathbb{R}$ be an open interval and $E:=\mathscr{C}_0(I)$. Then by 2.8 we have $\mathrm{CONT}_w(E)=E$. Since $\mathrm{Cont}_w(E)$ is a closed characteristic ideal in E, only the following two possibilities can occur: $\mathrm{Cont}_w(E)=E$ or $\mathrm{Cont}_w(E)=0$. We conjecture that the latter is true.

A connection with compact operators on a Hilbert space is given by the following

(2.10) **Proposition.** *Let H be a complex Hilbert space, $E\subset\mathscr{L}(H)$ a* JB*-*subtriple and $\mathscr{K}(H)$ the ideal of all compact operators on H. Then $E\cap\mathscr{K}(H)\subset\mathrm{Cont}_w(E)$.*

Proof. Denote by S the closed unit ball of $\mathscr{K}(H)$ and put $C:=\mathrm{Cont}_w(E)$ for short. Let $(x_\alpha)_{\alpha\in A}$ be a net in E converging weakly to 0 with $\|x_\alpha\|<1$ for all $\alpha\in A$. Since the finite rank operators are dense in $\mathscr{K}(H)$ and C is closed in E, it is enough to show that every rank-one hermitian projection $a\in E$ is contained in C. So fix such an a, i.e. $a(x)=(x|e)e$ for some unit vector $e\in H$. Then $y_\alpha:=\{x_\alpha a x_\alpha\}\in S$ for all α. By [21, p. 69] the weak topology restricted to S is the same as the weak operator topology on S, i.e. we have to show that for every ξ, $\eta\in H$ the net $(y_\alpha\xi|\eta)$ converges in $\mathbb{C}$ to 0. But this is obvious from

$$|(y_\alpha\xi|\eta)|=|(x_\alpha\xi|e)(x_\alpha e|\eta)|\leqq\|\xi\|\cdot|(x_\alpha e|\eta)|. \quad\square$$

The example $E=\mathbb{C}\cdot\mathrm{id}$ shows that in general in 2.10 equality does not hold. But then the action of E on H is highly non-irreducible. For H finite dimensional it is known [5] that $E=\mathscr{L}(H)$ does not have the Dunford-Pettis property, i.e. $\mathrm{CONT}_w(E)$ is a proper ideal in E. But in case H is separable, $\mathscr{K}(E)$ is the only proper ideal $\neq 0$ in E (compare [4, p. 841]), i.e. $\mathrm{CONT}_w(E)=\mathrm{Cont}_w(E)=\mathscr{K}(H)$. More generally we have

(2.11) **Lemma.** *Let H, K be complex Hilbert spaces. Then $E:=\mathscr{L}(H,K)$ is a* JB*-*triple with respect to the triple product $\{xyz\}=(xy^*z+zy^*x)/2$ and $\mathrm{CONT}_w(E)=\mathrm{Cont}_w(E)=\mathscr{K}(H,K)$ is the space of all compact operators $H\to K$.*

Proof. Put $C:=\mathrm{CONT}_w(E)$ and $\mathscr{K}:=\mathscr{K}(H,K)$ for short. Since the JB*-triples $\mathscr{L}(H,K)$ and $\mathscr{L}(K,H)$ are isomorphic, we may assume $\dim H\leqq\dim K$. Since $\mathscr{L}(H,K)$ is isomorphic to a subtriple of $\mathscr{L}(H\oplus K)$ we get $\mathscr{K}\subset C$ by 2.10. For the proof of the opposite inclusion suppose there exists a non-compact operator $a\in C$. There is a closed linear subspace $L\subset K$ with $a(H)\subset L$ and $\dim L=\dim H$. Replacing $\mathscr{L}(H,K)$ by the subtriple $\mathscr{L}(H,L)$ we may assume without loss of generality that $K=L$ and hence even that $K=H$, i.e. $E=\mathscr{L}(H)$. We may assume furthermore that a is self-adjoint and – after applying the functional calculus for Borel functions in a – that a is a projection on H with infinite dimensional range $R\subset H$. Replacing $\mathscr{L}(H)$ by the subtriple $\mathscr{L}(R)$ we may finally assume that $R=H$ and that a is the identity on H. For every pair of vectors η, $\xi\in H$ let $t_{\eta,\xi}\in\mathscr{L}(H)$ be the rank-one operator defined by $t_{\eta,\xi}(\zeta)=(\zeta|\eta)\xi$ for all $\zeta\in H$. Fix a unit vector $\eta\in H$. Then $\xi\mapsto t_{\eta,\xi}$ defines a linear isometry $H\to E$. For an orthonormal sequence (ξ_n) in H, the corresponding sequence (x_n) in E with $x_n := t_{\eta,\xi_n}$ converges weakly to 0 and violates condition (3) in 2.4, i.e. $a \notin C$ in contradiction to our assumption. $\square$

The JB*-triples $\mathscr{L}(H, K)$ form the first out of 6 types of the so-called *Cartan factors* (compare [14, p. 473]). The types II, III can be handled in the same way as above whereas the types V, VI are finite dimensional and hence of no interest in this context. Cartan factors of type IV are the *spin factors* defined as follows: Let H be a complex Hilbert space of dimension >2 with conjugation $x \mapsto \bar{x}$. Then there is an equivalent norm $\| \;\|_\infty$ on H such that $E := (H, \| \;\|_\infty)$ becomes a JBW*-triple in the triple product

$$\{x y z\} := (x \mid y) z + (z \mid y) x - (x \mid \bar{z}) \bar{y}.$$

(2.12) **Lemma.** $\mathrm{CONT}_w(E) = \mathrm{Cont}_w(E) = 0$ *for every spin factor* E *of infinite dimension.*

Proof. Fix an arbitrary element $a \in \mathrm{Cont}_w(E)$. Then $\dim(E) = \infty$ implies the existence of an orthonormal sequence (x_n) in H with $x_n = -\bar{x}_n$ and $x_n \perp a$, i.e. $q_a(x_n) = \bar{a}$ for all n. But then $0 = w - \lim q_a(x_n) = \bar{a}$ and hence $a = 0$. $\square$

3 Holomorphic automorphisms

Let again E be a JB*-triple. In the section before, we have studied for every $a \in E$ the corresponding squaring operator q_a: $E \to E$. Of importance is also the conjugate linear operator Q_a: $E \to E$ defined by $Q_a(z) := \{a z a\} = q_z(a)$. This operator is called the *quadratic representation*, and it satisfies the fundamental formula

$$Q_{Q_a(b)} = Q_a \, Q_b \, Q_a$$

for all a, $b \in E$ (compare [17]). For every x, $y \in E$ the *Bergman operator* $B(x, y) \in \mathscr{L}(E)$ is defined by

$$B(x, y) = \mathrm{id} - 2 x \,\square\, y + Q_x \, Q_y.$$

In case $\|x \,\square\, y\| < 1$ the spectrum of $B(x, y)$ lies in $\{z \in \mathbb{C}\colon |z-1| < 1\}$; in particular the fractional power $B(x, y)^r \in \mathrm{GL}(E)$ exists for every $r \in \mathbb{R}$ in a natural way (compare [13, p. 517]).

In the following, denote by $D := \{z \in E\colon \|z\| < 1\}$ the open unit ball of E. Then a function f: $D \to E$ is called holomorphic if the (Fréchet) derivate $f'(a) \in \mathscr{L}(E)$ exists for every $a \in D$. A holomorphic bijection g: $D \to D$ is called an automorphism of D if also g^{-1} is holomorphic. Denote by $G := \mathrm{Aut}(D)$ the group of all automorphisms of D. It is well known that the isotropy subgroup

$$K := \{g \in G : g(0) = 0\}$$

consists of all linear transformations in $\mathrm{Aut}(D)$ (restricted to D), i.e. $K = \mathrm{Aut}(E)$. As a consequence of 1.1 we have $\|z \,\square\, a\| < 1$ for every z, $a \in D$. By [13, p. 515],

$$g_a(z) := a + B(a, a)^{1/2} (1 + z \,\square\, a)^{-1} z \tag{3.1}$$

defines an automorphism $g_a \in G$ with $g_a(0)=a$, $g_a(-a)=0$ and $g_{-a}=g_a^{-1}$. This means in particular that G acts transitively on D and that every $g \in G$ has a unique representation $g=g_a\lambda$, where $a=g(0)$ and $\lambda \in \mathrm{Aut}(E)$. In [15, p. 132] it has been shown that every $g \in G$ has a holomorphic continuation to an open neighbourhood of $\bar{D} \subset E$. With 3.1 we get a quantitative improvement:

(3.2) **Proposition.** *Every $g \in G$ has a holomorphic continuation to the open ball with radius $r=\|a\|^{-1}$, where $a=g(0)$.*

Proof. $\|z\|<r$ implies $\|z\square a\|<1$ by 1.1. Therefore 3.1 defines g_a and hence also $g=g_a\lambda$ holomorphically on the ball with radius r. □

3.1 also gives a way to recover the triple product on E from the group $G=\mathrm{Aut}(D)$ – more precisely:

(3.3) **Lemma.** *For every $a \in D$, choose an automorphism $g \in \mathrm{Aut}(D)$ with $g(-a)=0$. Let $L := g'(0) \in \mathrm{GL}(E)$ be the first and let $Q := g''(0)$: $E \times E \to E$ be the second derivative of g at the origin. Then for all $x, y \in E$ the triple product is given by*

$$\{x a y\} = -L^{-1} Q(x, y).$$

Proof. Let

$$g(z)=\sum_{k=0}^{\infty} p_k(z)$$

be the expansion of g as sum of k-homogeneous polynomials p_k around $0 \in D$. Because of $g(-a)=0$ we have a representation of the form $g=\lambda g_a$ for some $\lambda \in K=\mathrm{Aut}(E)$. This implies $p_1=\lambda B(a,a)^{1/2}$ and $p_2=-\lambda B(a,a)^{1/2} q_a$. □

Let us now fix an admissible topology $\mathfrak{t}$ on E and let $\mathrm{Cont}_{\mathfrak{t}}(E)$ be as before.

(3.4) **Lemma.** *For every $a \in \mathrm{Cont}_{\mathfrak{t}}(E)$ the mapping*

(3.5) $\quad E \times E \to E \quad$ *defined by* $(x, y) \mapsto B(x, a)\, y$

is $\mathfrak{t}^2-\mathfrak{t}$-continuous on bounded subsets.

Proof. Obvious from the definition of $\mathrm{Cont}_{\mathfrak{t}}(E)$ and the Jordan triple identity

$$Q_x Q_a y = 2\{\{y a x\} a x\} - \{y a \{x a x\}\}. \quad \square$$

Denote by $\mathrm{GL}_{\mathfrak{t}}(E)$ the group of all linear transformations g of E such that g and g^{-1} are $\mathfrak{t}-\mathfrak{t}$-continuous on bounded subsets of E, and put $\mathrm{Aut}_{\mathfrak{t}}(E) := \mathrm{Aut}(E) \cap \mathrm{GL}_{\mathfrak{t}}(E)$. In the same way, let $\mathrm{Aut}_{\mathfrak{t}}(D)$ be the group of all $g \in \mathrm{Aut}(D)$ that are $\mathfrak{t}-\mathfrak{t}$-homeomorphisms of D.

(3.6) **Theorem.** *For every admissible topology $\mathfrak{t}$ on the JB*-triple E the group of all biholomorphic $\mathfrak{t}-\mathfrak{t}$-homeomorphisms of the open unit ball $D \subset E$ is given by*

$$\mathrm{Aut}_{\mathfrak{t}}(D) = \{g_a\lambda : a \in D \cap \mathrm{Cont}_{\mathfrak{t}}(E),\ \lambda \in \mathrm{Aut}_{\mathfrak{t}}(E)\}.$$

Proof. Fix an element $a \in D \cap \mathrm{Cont}_{\mathfrak{t}}(E)$. Then $f(z) := (1+z\square a)^{-1} z = \sum(-z\square a)^k z$ defines a $\mathfrak{t}-\mathfrak{t}$-continuous function on D, since the convergence of the sum is uniform on D and every summand is $\mathfrak{t}-\mathfrak{t}$-continuous. By (3.5) the operator

$B(a,a)$ also is $\mathfrak{t}-\mathfrak{t}$-continuous. The spectral radius of $B(a,a)-\mathrm{id}$ is bounded by $\|a\|^2<1$. Therefore also $B(a,a)^{1/2}$ is $\mathfrak{t}-\mathfrak{t}$-continuous as a norm-convergent power series in $B(a,a)-\mathrm{id}$. This shows that g_a is $\mathfrak{t}-\mathfrak{t}$-continuous for every $a\in D\cap \mathrm{Cont}_{\mathfrak{t}}(E)$. Finally $g_a^{-1}=g_{-a}$ gives $\lambda g_a\in \mathrm{Aut}_{\mathfrak{t}}(D)$ for every $a\in D\cap \mathrm{Cont}_{\mathfrak{t}}(E)$ and $\lambda\in \mathrm{Aut}_{\mathfrak{t}}(E)$.

For the proof of the opposite inclusion, choose an arbitrary $g\in \mathrm{Aut}_{\mathfrak{t}}(D)$ and put $a:=g(0)$. Then $h:=g^{-1}$ satisfies $h(-b)=0$ for $b:=-a$. First and second derivatives of h at the origin are $\mathfrak{t}-\mathfrak{t}$-continuous as a locally uniform limit of $\mathfrak{t}-\mathfrak{t}$-continuous mappings. This implies by 3.3 that q_b is $\mathfrak{t}-\mathfrak{t}$-continuous and hence that $a\in \mathrm{Cont}_{\mathfrak{t}}(E)$. By the first part of this proof, this implies that $\lambda:=g_a^{-1}g\in \mathrm{Aut}_{\mathfrak{t}}(E)$. □

(3.7) **Corollary.** *Suppose* $\mathfrak{t}$ *is an admissible topology on* E *with* $\mathrm{Aut}_{\mathfrak{t}}(E)=\mathrm{Aut}(E)$ *– for instance, if* $\mathfrak{t}=w$ *is the weak topology or – in case of a* JBW*-*triple – if* $\mathfrak{t}$ *is one of the topologies* s^*, τ^*, w^*. *Then for every biholomorphic automorphism* g *of* D, *the following conditions are equivalent*:

(1) g *is* $\mathfrak{t}-\mathfrak{t}$-*continuous on* D,
(2) $g(0)\in \mathrm{Cont}_{\mathfrak{t}}(E)$.

Proof. Suppose (1) holds. Then $g=g_a\lambda$ with $a\in D\cap \mathrm{Cont}_{\mathfrak{t}}(E)$ and λ linear. Then $g(0)=a$ shows (2). The other implication also follows easily. □

As an application of 2.12 and 2.10 we get finally

(3.8) **Corollary.** *Let* E *be a spin factor of infinite dimension. Then the only* $w-w$-*continuous biholomorphic automorphisms of the open unit ball of* E *are the surjective linear isometries.*

(3.9) **Corollary.** *Let* H *be a complex Hilbert space and* $E:=\mathcal{K}(H)$ *the space of all compact operators on* H. *Then every biholomorphic automorphism of the open unit ball of* E *is* $w-w$-*continuous.*

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