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Filtrations of the modules for Chevalley groups arising from admissible lattices

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Introduction

For a semisimple complex Lie algebra $\mathfrak{g}$, let G be the simply connected Chevalley group of the same type over a field k of characteristic $p>0$. Each admissible lattice M in a finite-dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ gives rise to a rational G-module $\bar{M}$ through reduction modulo p. In this paper we will define a natural filtration for the G-module $\bar{M}$. Using this filtration, we give a description of the module graph of $\bar{M}$ for the groups of type A_1.

For a finite group, it is well known from Brauer's theory that the modular characters of those modular representations arising from different invariant lattices in an irreducible ordinary representation are the same. However, due to a result of Feit in 1967 (cf. [10, p. 70]), the module structure of these modular representations could be quite wild if the integral domain in the modular system has a large ramification index over $\mathbb{Z}_p$. It is not clear whether these modular representations are indecomposable if one considers only lattices over the integral domain $\mathbb{Z}_p$. The same question can be stated for the modular representations of Chevalley groups arising from admissible lattices. There are many interesting modules for a Chevalley group arising in this way such as the cohomology modules of line bundles over the flag varieties in the generic situation.

This paper is organized as follows: In contrast to the result of Feit for finite groups, we will prove, in Sect. 1, that the module $\bar{M}$ is indecomposable if all weight spaces of $V(\lambda)$ are 1-dimensional. The proof is based on the result for the groups of type A_1 proved in [15]. In Sect. 2, we construct a filtration for each module $\bar{M}$ generalizing the Jantzen filtration for a Weyl module [13] and the Andersen filtration for a cohomology module [2] in the generic situation. It turns out the filtration layers are self-dual under the transpose dual. In particular they are semisimple if the corresponding Weyl module is multiplicity free.

We consider only the groups of type A_1 in the last two sections. The submodule structure of Weyl modules has been described by Carter and Cline [4] (also see [8, 6]). By calculating the filtrations explicitly, we study some properties of the module graph of the induced module $H_k^0(\lambda)$ in Sect. 3. The graph of $\bar{M}$ is studied in Sect. 4. It turns out that the nondirected graph associated to the module graph of $\bar{M}$ is independent of the admissible lattice M. An important fact we use in the proof is the structure theorem for Weyl modules in terms of extensions between simple G-modules by Cline in [6]. All the possible graphs for $\bar{M}$ can be characterized from the module graph of $H_k^0(\lambda)$ in a purely combinatorial manner. The last theorem deals with the Loewy length of $\bar{M}$. All the modules $\bar{M}$ with maximal Loewy length, which is the Loewy length of $H_k^0(\lambda)$, are classified.

1 Indecomposability

1.1 Let R be the root system of $\mathfrak{g}$ and $S \subseteq R^+$ the set of simple roots in the set of positive roots. By X we denote the integral weight lattice and X^+ the set of dominant weights. For $\lambda, \mu \in X$, we say $\lambda \geqq \mu$ if $\lambda - \mu \in \mathbb{N}R^+$.

Choose a Chevalley basis $\{x_\alpha, h_i | \alpha \in R, 1 \leqq i \leqq n\}$ for $\mathfrak{g}$. Here n is the rank of $\mathfrak{g}$. The Kostant $\mathbb{Z}$-form $U_{\mathbb{Z}}(\mathfrak{g})$ is the subring of the enveloping algebra $U(\mathfrak{g})$ generated by $\{x_\alpha^{(s)} = x_\alpha^s / s! | \alpha \in R, s \geqq 0\}$. For $\lambda \in X^+$, $V(\lambda)$ is the finite-dimensional irreducible $\mathfrak{g}$-module of highest weight λ. The $\mathbb{Z}$-span M of a basis of $V(\lambda)$ is called a lattice in $V(\lambda)$. M is called admissible if $U_{\mathbb{Z}}(\mathfrak{g}) M \subseteq M$. We denote by M_μ (or $V(\lambda)_\mu$) the μ-weight space of M (or $V(\lambda)$). It is known that $M_\mu = M \cap V(\lambda)_\mu$ and M is a direct sum of its weight spaces (cf. [11, 27.1]).

Let $G_{\mathbb{Z}}$ be the group over $\mathbb{Z}$ defined by Kostant in [14]. Then G is obtained from $G_{\mathbb{Z}}$ through base change. Since $U_{\mathbb{Z}}(\mathfrak{g})$ is the distribution algebra of $G_{\mathbb{Z}}$, $\bar{M} = M \otimes k$ is a rational G-module for each admissible lattice M in $V(\lambda)$ (cf. [12, II 1.12, 1.20]). Throughout this paper, the tensor products will be taken over $\mathbb{Z}$ unless otherwise indicated. Furthermore, we denote by $L(\mu)$, for each $\mu \in X^+$, the irreducible G-module of highest weight μ.

1.2 Theorem. *If $M \subseteq V(\lambda)$ is an admissible lattice and all the weight spaces of $V(\lambda)$ are 1-dimensional, then $\bar{M}$ is an indecomposable G-module.*

Proof. Suppose $\bar{M} = E \oplus F$ is a decomposition of $\bar{M}$ into a direct sum of two nonzero submodules such that λ is a weight of E. Let μ be a maximal weight of F. So $\lambda \neq \mu$ and $\mu + \alpha$ is a weight of $V(\lambda)$ for a simple root α. Let $\mathfrak{l}_\alpha$ be the subalgebra $\mathfrak{g}$ generated by $\{x_\alpha, x_{-\alpha}\}$ and the Cartan subalgebra $\mathfrak{h}$. Let L_α be the Levi factor of the minimal parabolic subgroup P_α of G [12, II 1.7–1.8]. Then L_α is split over $\mathbb{Z}$ and the distribution algebra of $(L_\alpha)_{\mathbb{Z}}$ is the subalgebra $U_{\mathbb{Z}}(\mathfrak{l}_\alpha) = U_{\mathbb{Z}}(\mathfrak{g}) \cap U(\mathfrak{l}_\alpha)$ of $U_{\mathbb{Z}}(\mathfrak{g})$ [12, II 1.12].

$V(\lambda)$ is completely reducible when restricted to $\mathfrak{l}_\alpha$. Thus $V(\lambda) = \oplus V_i$. Here each $V_i = \oplus V_{v + \mathbb{Z}\alpha}$ for a weight v of $V(\lambda)$ is simple as $\mathfrak{l}_\alpha$-module since all the weight spaces of $V(\lambda)$ are 1-dimensional. Then for each i, $M_i = M \cap V_i$ is an admissible lattice in V_i as an $\mathfrak{l}_\alpha$-module with respect to the $\mathbb{Z}$-form $U_{\mathbb{Z}}(\mathfrak{l}_\alpha)$ (cf. the proof of [7, (23.7)]). By considering the restriction of V_i (or M_i) to a subalgebra of $\mathfrak{l}_\alpha$ isomorphic to sl_2, we can apply the result for sl_2 proved in [15] to conclude that $\bar{M}_i$ is indecomposable as L_α-module. Therefore $\bar{M} = \oplus \bar{M}_i$ is a decomposition of $\bar{M}$ into a direct sum of indecomposable L_α-modules. By the definition of each V_i, μ cannot be a maximal weight for any $\bar{M}_i$.

On the other hand, as an L_α-module, each indecomposable component of F is an indecomposable component of $\bar{M}$ and, therefore, must be isomorphic to an $\bar{M}_i$, for which μ is not a maximal weight. However, since μ is a maximal weight of F, there must be an L_α-indecomposable component of F with μ as a maximal weight. This contradiction shows that $\bar{M}$ is indecomposable. □

Remark. If $\mathfrak{g}$ is of type A_n, then the symmetric powers of the natural $\mathfrak{g}$-module and their dual modules are irreducible $\mathfrak{g}$-modules and their weight spaces are 1-dimensional. Thus the above proposition applies to these modules.

2 Filtrations

2.1 Throughout this section, we will fix a dominant weight $\lambda \in X^+$ and a highest weight vector v^+ in $V(\lambda)$. Let $\langle\ ,\ \rangle$: $V(\lambda) \times V(\lambda) \to C$ be the bilinear contravariant form (see [17, (1B)]) such that $\langle v^+, v^+ \rangle = 1$. For each admissible lattice M in $V(\lambda)$, we define $M' = \{v \in V(\lambda) | \langle v, M \rangle \subseteq \mathbb{Z}\}$. One can easily check that M' is also an admissible lattice and $(M')' = M$. $M_\lambda = \mathbb{Z} v^+$ if and only if $M'_\lambda = \mathbb{Z} v^+$. It is proved in [15] that $(\bar{M})^{tr} \cong \bar{M}'$. Here for a G-modules E, E^{tr} denotes the transpose dual of E, which has the vector space E^* and the G-module structure given by $gf(x) = f(tr(g)x)$ for all $f \in E^*$, $x \in E$ and $g \in G$. tr: $G \to G$ is the composite of the inverse map $g \mapsto g^{-1}$ (for all $g \in G$) and the involutory automorphism ϕ in [17, (2H)].

Let $\mathbb{Z}_p$ be the localization of $\mathbb{Z}$ at the prime p. Since $M \otimes k = M \otimes \mathbb{Z}_p \otimes_{\mathbb{Z}_p} k$, we only consider the admissible lattices over $\mathbb{Z}_p$ in $V(\lambda)$ with respect to the $\mathbb{Z}_p$-form $U_{\mathbb{Z}}(\mathfrak{g}) \otimes \mathbb{Z}_p$ in this paper. Note that all admissible lattices over $\mathbb{Z}_p$ are of the form $M \otimes \mathbb{Z}_p$ (cf. [7, (23.13)]). Without loss of generality, we will consider only admissible lattices M with $M_\lambda = \mathbb{Z}_p v^+$, although other admissible lattices will be occasionally involved in some proofs (this should not contradict the assumption here).

2.2 Let M be an admissible lattice over $\mathbb{Z}_p$ such that $M_\lambda = \mathbb{Z}_p v^+$. There exists a minimal integer e such that $M \subseteq p^{-e} M'$ in $V(\lambda)$. It is easily seen from the λ-weight space that e has to be nonnegative. Set $M^j = M \cap p^{j-e} M'$, then M^j is also an admissible lattice in $V(\lambda)$ for each $j = 0, 1, \ldots$. The inclusion of M^j into M gives a homomorphism ϕ_j: $M^j \otimes k \to \bar{M}$. Set $\bar{M}^j = \operatorname{Im}(\phi_j)$, which is a G-submodule of $\bar{M}$. Following the inclusion $M^{j+1} \subseteq M^j$, we get a filtration of G-modules $\bar{M} = \bar{M}^0 \supseteq \bar{M}^1 \supseteq \ldots \supseteq \bar{M}^l \supseteq 0$. It is immediate from the definition that $\bar{M}^0 \neq \bar{M}^1$.

Lemma. *$M^j = \{m \in M | \langle m, M \rangle \subseteq p^{j-e} \mathbb{Z}_p\}$.*

Proof. In fact M' is the $\mathbb{Z}_p$-dual of M by the nondegeneracy of the contravariant form $\langle\ ,\ \rangle$. Then the lemma follows from a standard argument. □

Remark. Using the set-up in the lemma, one can define the filtration via the bilinear form $\langle\ ,\ \rangle_M = p^e \langle\ ,\ \rangle$ in a similar way as Jantzen defines the filtration for a Weyl module [13]. In particular, if we take M to be the minimal admissible lattice $U_{\mathbb{Z}}(\mathfrak{g}) v^+$, the filtration we defined here is the Jantzen filtration for the Weyl module, denoted by $\overline{V(\lambda)}$, since $e = 0$ in this case.

2.3 Let W be the Weyl group. For each root $\alpha \in R^+$, s_α denotes the associated reflection in W. $\ell(w)$ is the length of w for $w \in W$ and w_0 is the unique element of W with maximal length. Let $M = H_f^{\ell(w)}(w \cdot \lambda)$ be the $\mathbb{Z}_p$-free quotient of $H^{\ell(w)}(w \cdot \lambda)$, which is defined via the derived functor of the induction functor for the group schemes $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ (see Andersen [2, (4.2)] for detailed information). Then M can be regarded as an admissible lattice in $V(\lambda)$ with $M_\lambda = \mathbb{Z}_p v^+$ via a natural embedding $H_f^{\ell(w)}(w \cdot \lambda) \to H^0(\lambda)$ (cf. [15]). $H^0(\lambda)$ is always $\mathbb{Z}_p$-free. Using the Serre duality derived by Andersen in [2, 2.10], one gets $M' \cong H_f^{\ell(w_0 w)}(w_0 w \cdot \lambda)$.

For each $\alpha \in S$ such that $\ell(s_\alpha w) = \ell(w)+1$, there exist maps T_{s_α}: $H_f^{\ell(s_\alpha w)}(s_\alpha w \cdot \lambda) \to H_f^{\ell(w)}(w \cdot \lambda)$ and T'_{s_α}: $H_f^{\ell(w)}(w \cdot \lambda) \to H_f^{\ell(s_\alpha w)}(s_\alpha w \cdot \lambda)$. $T'_{s_\alpha} T_{s_\alpha}$ and $T_{s_\alpha} T'_{s_\alpha}$ are the scalar multiplications by p^{m_α} (up to units in $\mathbb{Z}_p$).

By induction on $\ell(w)$, one can show that there exist maps

$$H_f^{\ell(w)}(w \cdot \lambda) \xrightarrow{T_w} H^0(\lambda) \xrightarrow{T'_w} H_f^{\ell(w)}(w \cdot \lambda)$$

such that $T'_w T_w$ is the multiplication by $p^{m(\lambda, w)}$. Here T_w is the natural embedding of $H_f^{\ell(w)}(w \cdot \lambda)$ into $H^0(\lambda)$. Thus $p^{m(\lambda, w)} H^0(\lambda) \subseteq H_f^{\ell(w)}(w \cdot \lambda)$. In particular, for $M = H_f^{\ell(w)}(w \cdot \lambda)$ we have $M \subseteq H^0(\lambda) \subseteq p^{-m(\lambda, w_0 w)} M'$. If we denote by ϕ the embedding $M \subseteq p^{-m(\lambda, w_0 w)} M'$, then $T_{w_0} = p^{m(\lambda, w w_0)} \phi$: $H_f^{\ell(w)}(w \cdot \lambda) \to H_f^{\ell(w_0 w)}(w_0 w \cdot \lambda)$ is the map defined by Andersen in [2, (2.10)]. The integer $m(\lambda, w)$ can be calculated through the p-adic expressions of $\langle \lambda, \alpha^\vee \rangle$ for $\alpha \in R^+$ with $w(\alpha) \in R^-$. By the definition of e, we have $e \leqq m(\lambda, w_0 w)$. If $e < m(\lambda, w_0 w)$, then $T_{w_0}(H_f^{\ell(w)}(w \cdot \lambda)) \subseteq p H_f^{\ell(w_0 w)}(w_0 w \cdot \lambda)$ and the induced map $T_{w_0} \otimes 1$: $H_f^{\ell(w)}(w \cdot \lambda) \otimes k \to H_f^{\ell(w_0 w)}(w_0 w \cdot \lambda) \otimes k$ is 0. In the generic situation (see [3, 2.1] for precise conditions), $T_{w_0} \otimes 1 \neq 0$ and its image is the simple socle of $H_f^{\ell(w_0 w)}(w_0 w \cdot \lambda) \otimes k$. Thus we have proved

Proposition. *For $M = H_f^{\ell(w)}(w \cdot \lambda)$, the filtration defined in 2.2 is a shift of the filtration defined by Andersen in [2, 4.5] such that $\bar{M}/\bar{M}^1 \neq 0$. In particular, for generic λ, the filtration defined here for $H_k^{\ell(w)}(w \cdot \lambda)$ is the Andersen filtration.*

2.4 Let l be the minimal number such that $\bar{M}^{l+1} = 0$. Then the length of the filtration for $\bar{M}$ is $l+1$. We denote by e' the minimal number such that $M' \subseteq p^{-e'} M$.

Lemma. $l = e + e'$.

Proof. Since $\bar{M}^{l+1} = 0$, we have $M^{l+1} = M \cap p^{l+1-e} M' \subseteq pM$. For each $m' \in p^{l+1-e} M'$, there exists $t \geqq 0$ such that $p^t m' \in M$. Since $p^t m' \in M \cap p^{l+1-e} M' \subseteq pM$, we have $p^{t-1} m' \in M$ and, therefore, $m' \in M$ by repeating the above argument. Hence $p^{l+1-e} M' \subseteq M$ and $p^{l+1-e} M' = M \cap p^{l+1-e} M' \subseteq pM$. Thus $M' \subseteq p^{-l+e} M$ and $e' \leqq l - e$. However, if $e' < l - e$, $M' \subseteq p^{-e'} M \subseteq p^{e-l+1} M$ and $p^{l-e} M' \subseteq pM$. This shows that $M^l \subseteq pM$, which contradicts $\bar{M}^l \neq 0$. □

Corollary. *The filtrations for $\bar{M}$ and $\bar{M}'$ have the same length.*

2.5 Let $\mathcal{M}$ be an Abelian category and D: $\mathcal{M} \to \mathcal{M}$ an exact contravariant functor. If M is an object in $\mathcal{M}$ and $M = M^0 \supseteq M^1 \supseteq \ldots \supseteq M^l \supseteq 0$ is a filtration for M in $\mathcal{M}$. Then $D(M)^j = D(M/M^{l+1-j})$ defines a filtration for $D(M)$. We call the filtration $\{D(M)^j\}$ the D-dual filtration of the filtration $\{M^j\}$. In particular, $D(M^j/M^{j+1}) \cong D(M)^{l-j}/D(M)^{l+1-j}$.

We will take $D = tr$, the transpose dual functor on the category of rational G-modules. For an admissible lattice M, there are two filtrations for $\bar{M}'$. The first one is the filtration we defined in 2.2 for $\bar{M}'$ and the other one is the dual filtration of $\{\bar{M}^j\}$ since $(\bar{M})^{tr} \cong \bar{M}'$ (cf. [15]).

Proposition. *The two filtrations for $\bar{M}'$ are the same.*

Proof. Following Corollary 2.4, the two filtrations on $\bar{M}'$ have the same length, say, $l+1$. It suffices to prove $(\bar{M}/\bar{M}^j)^{tr} \cong \bar{M}'^{l+1-j}$ for each j.

Let ϕ: $M^j \to M$ be the embedding and ϕ^*: $M' \to (M^j)'$ the dual map over $\mathbb{Z}_p$. Set $\phi \otimes 1$: $M^j \otimes k \to M \otimes k$, and $\phi^* \otimes 1$: $M' \otimes k \to (M^j)' \otimes k$. Since both M and M^j are free $\mathbb{Z}_p$-modules, a dual basis argument will show that $(\phi \otimes 1)^* = \phi^* \otimes 1$ under the identifications $M' \otimes_{\mathbb{Z}_p} k \cong (M \otimes_{\mathbb{Z}_p} k)^*$ and $(M^j)' \otimes_{\mathbb{Z}_p} k \cong (M^j \otimes_{\mathbb{Z}_p} k)^*$ via the bilinear form $\langle\, , \rangle$. By definition, $\bar{M}^j = \mathrm{Im}(\phi \otimes 1)$. Thus we have $(\bar{M}/\bar{M}^j)^{tr} = \ker(\phi^* \otimes 1)$. We will show $\ker(\phi^* \otimes 1) = \bar{M}'^{l+1-j}$.

Using the nondegeneracy of the bilinear form $\langle\, , \rangle$ on $V(\lambda)$, one can verify that $(M+N)' = M' \cap N'$ and $(M \cap N)' = M' + N'$ for any two lattices M and N in $V(\lambda)$. Consider the map ϕ^*: $M' \to (M^j)' = M' + (p^{j-e} M')' = M' + p^{e-j} M$, which is the natural embedding.

Lemma. *If M_1 and M_2 are two $\mathbb{Z}_p$-lattices in a finite dimensional $\mathbb{Q}$-vector space V, then the natural embeddings $M_1 \cap pM_2 \xrightarrow{\theta} M_1 \xrightarrow{\psi} M_1 + M_2$ induce an exact sequence*

$$(M_1 \cap pM_2) \otimes \mathbb{F}_p \xrightarrow{\theta \otimes 1} M_1 \otimes \mathbb{F}_p \xrightarrow{\psi \otimes 1} (M_1 + M_2) \otimes \mathbb{F}_p.$$

Proof. Let $m \in M_1$ such that $\psi(m) \in p(M_1 + M_2)$. Then $m = pm_1 + pm_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Thus $m - pm_1 = pm_2 \in M_1 \cap pM_2$ and $m - pm_1 \equiv m \pmod p$. This shows $\mathrm{Im}(\theta \otimes 1) \supseteq \mathrm{Ker}(\psi \otimes 1)$. However, $\mathrm{Im}(\theta \otimes 1) \subseteq \mathrm{Ker}(\psi \otimes 1)$ is obvious. □

To continue the proof of the proposition, we take $M_2 = p^{e-j} M = p^{l-j-e'} M$ and $M_1 = M'$ in the lemma. We have $M_1 \cap pM_2 = (M')^{l+1-j}$. Thus $\mathrm{Im}(\theta \otimes 1) = \bar{M}'^{l+1-j}$ and the proposition follows from the lemma above since $\psi = \phi^*$ in this case. □

Set $\bar{M}_j = \bar{M}^j / \bar{M}^{j+1}$, which is called the jth filtration layer of $\bar{M}$.

Corollary. *For each admissible lattice M and $j \geq 0$, $(\bar{M}^j)^{tr} \cong \bar{M}'/\bar{M}'^{l+1-j}$ and $(\bar{M}_j)^{tr} \cong \bar{M}'_{l-j}$.*

2.6 Proposition. *For each admissible lattice M in $V(\lambda)$, $\bar{M}_j$ is selfdual, i.e., $\bar{M}_j^{tr} \cong \bar{M}_j$.*

Proof. Using Corollary 2.5, we only need to show $\bar{M}_j \cong \bar{M}'_{l-j}$. Note that

$$\bar{M}_j \cong ((M \cap p^{j-e} M')/(pM \cap p^{j-e} M' + M \cap p^{j-e+1} M')) \otimes_{\mathbb{F}_p} k.$$

There is also a similar form for $\bar{M}_{l-j}$. The multiplication by p^{e-j} gives an isomorphism of admissible lattices $M^j \to M' \cap p^{e-j} M = (M')^{l-j}$, which sends $pM \cap p^{j-e} M' + M \cap p^{j-e+1} M'$ onto $p^{e-j+1} M \cap M' + p^{e-j} M \cap pM'$. The proposition follows by tensoring k to the induced isomorphism over $\mathbb{F}_p$. □

Corollary. *If all composition factors of the Weyl module $\overline{V(\lambda)}$ have multiplicities at most* 1, *then $\bar{M}_j$ is semisimple for all j.*

3 Structure graphs of Weyl modules for $\mathrm{SL}_2(k)$

3.1 In the rest of this paper we will consider only $\mathfrak{g} = sl_2$ and $G = \mathrm{SL}_2(k)$. Let $\{X, H, Y\}$ be the standard Chevalley basis for $\mathfrak{g}$. We fix a dominant integral weight λ, which is a nonnegative integer. Write $\lambda+1=a_r p^r+\ldots+a_{r_1} p^{r_1}$ with $0\leqq a_t\leqq p-1$ for $r\geqq t\geqq r_1\geqq 0$ and $a_r>0$, $a_{r_1}>0$. For $e\geqq 0$, let ρ_e: $\mathbb{N}\to\mathbb{Z}$ be the reflection such that $\rho_e(n)=n-2k$ whenever $n=mp^e+k$ with $0\leqq k\leqq p^e-1$. ρ_e is called n-admissible if $m\not\equiv 0 \pmod p$, which is equivalent to $\rho_e(n)\neq\rho_{e+1}(n)$. Carter and Cline [4] proved that $L(\mu)$ is a composition factor of $V(\lambda)$ if and only if $\mu+1=\rho_{e_s}\rho_{e_{s-1}}\ldots\rho_{e_1}(\lambda+1)$ for a sequence $e_1>e_2>\ldots>e_s\geqq r_1$ such that ρ_{e_t} *is* $\rho_{e_{t-1}}\ldots\rho_{e_1}(\lambda+1)$-admissible for $t=1,\ldots,s$. Denote by $\mathscr{V}(\lambda)$ the set of all such μ.

Consider the set $(r, r_1]=\{r-1,\ldots,r_1\}\subseteq\mathbb{N}$. For e, $f\in\mathbb{N}$ with $r\geqq e>f\geqq r_1$, we call the subset $(e,f]=\{e-1,\ldots,f\}$ of $(r,r_1]$ an *interval* for λ if $a_e>0$ and either $f=r_1$ or $a_f<p-1$. For each interval $(e,f]$, we define a measure

$$i((e,f])=\begin{cases} a_{e-1}p^{e-1}+\ldots+(a_f+1)p^f & \text{if } f>r_1,\\ a_{e-1}p^{e-1}+\ldots+a_f p^f & \text{if } f=r_1.\end{cases}$$

Let $\mathscr{S}(\lambda)$ be the subset of the power set of $(r, r_1]$ consisting of unions of intervals. Then each $I\in\mathscr{S}(\lambda)$ is a disjoint union of maximal intervals, which will be called *components* of I. The measure i can be extended naturally to a measure defined on $\mathscr{S}(\lambda)$. For a given finite set E, by $|E|$ we denote the cardinality of E.

Lemma. *If I, $I_1\in\mathscr{S}(\lambda)$, then*
(i) $i(I)=i(I_1)$ *if and only if* $I=I_1$;
(ii) $\mu\in\mathscr{V}(\lambda)$ *if and only if* $\lambda-\mu=2i(I)$ *for some* $I\in\mathscr{S}(\lambda)$;
(iii) $v_p\left(\binom{\lambda}{i(I)}\right)=|I|$.

Proof. (i) follows from the uniqueness of the p-adic expression of $i(I)$. To show (ii), one sees that $\mu\in\mathscr{V}(\lambda)$ if and only if $\mu+1=\rho_{e_s}\ldots\rho_{e_1}(\lambda+1)$ with $r\geqq e_1>\ldots>e_s\geqq r_1$ such that $a_{e_t}>0$ for odd t and $a_{e_t}<p-1$ for even t with $e_t>r_1$. Corresponding to $\mu\in\mathscr{V}(\lambda)$, we set $I(\mu)=(e_1,e_2]\cup\ldots\cup(e_{s-1},e_s]$ if s is even, or $I(\mu)=(e_1,e_2]\cup\ldots\cup(e_s,r_1]$ if s is odd. Thus $I(\mu)\in\mathscr{S}(\lambda)$ and this defines a bijection $\mathscr{V}(\lambda)\to\mathscr{S}(\lambda)$. A direct calculation will show $\rho_f\rho_e(\lambda+1)=\lambda+1-2i((e,f])$ for each interval $(e,f]$. Now (ii) follows from the induction on the number of components of $I(\mu)$.

To show (iii), we recall some facts. For a pair of nonnegative integers $m\leqq n$, write $n=\sum_t b_t p^t$ and $m=\sum_t c_t p^t$ in p-adic forms. For $e>f\geqq 0$, we call the set $(e,f]=\{e-1,\ldots,f\}$ a *component* for the pair (m,n) if it is a maximal set with the property: $b_t\leqq c_t$ for $t\in(e,f]$ and $b_f<c_f$. It is proved in [15, (3.4)] that $v_p\left(\binom{n}{m}\right)$ is the sum of cardinalities of all components for the pair (m,n). In our case we can write I as a disjoint union of intervals and each component of I corresponds to a component for the pair $(i(I),\lambda)$. (Compare the p-adic expressions of $\lambda+1$ and λ.) Thus (iii) follows. □

3.2 In [1], Alperin defines a module space to be a finite topological space such that every point is relatively closed. For each finite poset X, there is a natural way to make X into a module space by defining the open sets to be the ideals of X. In our case, $\mathscr{S}(\lambda)$ is a poset under the inclusion order. Therefore $\mathscr{S}(\lambda)$ is a module space in a natural way. We will denote by $\Gamma_0(\lambda)$ the module graph associated to the module space $\mathscr{S}(\lambda)$, which has vertex set $\mathscr{S}(\lambda)$. $I \to J$ is a directed edge in $\Gamma_0(\lambda)$ if $I \supsetneqq J$ and there is no $I_1 \in \mathscr{S}(\lambda)$ with $I \supsetneqq I_1 \supsetneqq J$. For our convenience, we denote by $S(I)$ the irreducible G-module $L(\lambda - i(I)\alpha)$ for each $I \in \mathscr{S}(\lambda)$.

Theorem. *$\Gamma_0(\lambda)$ is the structure graph of the G-module $H_k^0(\lambda)$ under the correspondence $I \mapsto S(I)$.*

Proof. The proof can be found in [8] by Deriziotis keeping in mind that $i(I) = (\lambda - \mu)/2$. □

3.3 Lemma. *For $I_1, I_2 \in \mathscr{S}(\lambda)$, $I_1 \to I_2$ is an edge in $\Gamma_0(\lambda)$ if and only if $I_1 \supseteq I_2$ and $|I_1| = |I_2| + 1$.*

Proof. It is clear that the condition given here is sufficient for $I_1 \to I_2$ to be an edge. Suppose $I_1 \supseteq I_2$ and $|I_1| > |I_2| + 1$. We will show that there exists $I \in \mathscr{S}(\lambda)$ with $I_1 \supseteq I \supseteq I_2$ such that either $|I| = |I_1| - 1$ or $|I| = |I_2| + 1$. Since $I_1 \supsetneqq I_2$, each component of I_2 is contained in a component of I_1. Let $(e, f]$ be a component of I_1 but not a component of I_1. At least one of the following three cases will occur.

Case 1 $(e, f]$ contains a component $(e', f']$ of I_2 such that $e > e'$. Then take $I = I_2 \cup (e' + 1, f]$ if $a_{e'+1} \neq 0$ or $I = I_1 \setminus \{e'\}$ if $a_{e'+1} = 0$.

Case 2 $(e, f]$ contains a component $(e', f']$ of I_2 such that $f' > f$. Then take $I = I_2 \cup (e', f' - 1]$ if $a_{f'-1} < p - 1$ or $I = I_1 \setminus \{f' - 1\}$ if $a_{f'-1} = p - 1$.

Case 3 $(e, f]$ contains no component of I_2. In this case we will find I such that $I \subseteq (e, f]$ and $|I| = e - f - 1$. We can take $I = (e, f] \setminus \{f\}$ if either $e = f + 1$ or $a_{f+1} < p - 1$, or $I = (e, f] \setminus \{e - 1\}$ if $a_{e-1} > 0$. Otherwise, there exists f_1 such that $e > f_1 > f$ with $a_{f_1} < p - 1$ and $a_{f_1 - 1} > 0$. Take $I = (e, f_1] \cup (f_1 - 1, f]$ in this case. □

Corollary. *For $I_1, I_2 \in \mathscr{S}(\lambda)$, if $\operatorname{Ext}_G^1(S(I_1), S(I_2)) \neq 0$, then $|I_1| = |I_2| \pm 1$ and either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$.*

Proof. This follows from the above lemma and the structure theorem of Cline [6]. □

3.4 Let Γ be a directed graph. We denote by $\bar{\Gamma}$ the nondirected graph associated to Γ. With fixed λ, the maximal value $v_p\left(\binom{\lambda}{i(I)}\right)$ for $I \in \mathscr{S}(\lambda)$ is $r - r_1$ by Lemma 3.1. Since $H_k^0(\lambda)$ (the induced G-module) is indecomposable, the graph $\overline{\Gamma_0(\lambda)}$ is path connected. For $I, J \in \mathscr{S}(\lambda)$, we denote by $d(I, J)$ the distance between I and J, which is the minimal number of edges in paths connecting I and J in $\overline{\Gamma_0(\lambda)}$.

Lemma. *Let $I_1, I_2 \in \mathscr{S}(\lambda)$ such that $|I_1| \geqq |I_2|$. Then $d(I_1, I_2) \geqq |I_1| - |I_2|$. Furthermore, $d(I_1, I_2) = |I_1| - |I_2|$ if and only if $I_1 \supseteq I_2$.*

Proof. Suppose $t = d(I_1, I_2)$. Let $I_1 = J_0, J_1, \ldots, J_t = I_2$ be a path in $\overline{\Gamma_0(\lambda)}$, i.e., (J_{s-1}, J_s) are edges in $\overline{\Gamma_0(\lambda)}$ for $s = 1, \ldots, t$. By Lemma 3.3, $||J_{s-1}| - |J_s|| = 1$ and

$$t = \sum_{s=1}^{t} ||J_{s-1}| - |J_s|| \geqq \sum_{s=1}^{t} (|J_{s-1}| - |J_s|) = |I_1| - |I_2|.$$

This proves the first statement. If $d(I_1, I_2)=|I_1|-|I_2|$, we must have $|J_{s-1}|-|J_s|=1$ for all $s=1, \ldots, t$ and $J_{s-1}\supseteq J_s$ by Lemma 3.3. Thus $I_1\subseteq I_2$.

On the other hand, if $I_1\supseteq I_2$, we use induction on $|I_1|-|I_2|$ to show $d(I_1, I_2)=|I_1|-|I_2|$. This is clear if $|I_1|-|I_2|\leqq 1$. If $|I_1|-|I_2|\geqq 2$ and $I_1\supseteq I_2$, the proof of Lemma 3.3 has shown that there exists $I\in\mathscr{S}(\lambda)$ such that $I_1\supseteq I\supseteq I_2$ and $|I|$ is either $|I_1|-1$ or $|I_2|+1$. Now the assertion follows from induction hypothesis. □

3.5 Recall that $\mathscr{S}(\lambda)$ is closed under union. One can verify that it is also closed under intersection. For $I_1, I_2\in\mathscr{S}(\lambda)$, $I_1\cup I_2$ is the unique least upper bound of I_1 and I_2, and $I_1\cap I_2$ the unique largest lower bound of I_1 and I_2 in the poset $\mathscr{S}(\lambda)$. Thus $\mathscr{S}(\lambda)$ is a lattice. The simple formula, $|I_1\cap I_2|+|I_1\cup I_2|=|I_1|+|I_2|$, will be used many times in the sequel.

Proposition. *For $I_1, I_2\in\mathscr{S}(\lambda)$, $d(I_1, I_2)=|I_1\cup I_2|-|I_1\cap I_2|$.*

Proof. Using Lemma 3.4, it is easy to see

$$\begin{aligned} d(I_1, I_2)&\leqq d(I_1, I_1\cup I_2)+d(I_2, I_1\cup I_2)\\ &=2|I_1\cup I_2|-(|I_1|+|I_2|)=|I_1\cup I_2|-|I_1\cap I_2|. \end{aligned}$$

To show $d(I_1, I_2)\geqq|I_1\cup I_2|-|I_1\cap I_2|$, we use induction on $d(I_1, I_2)$. If $d(I_1, I_2)=1$, then, say, $I_1\to I_2$ is an edge in $\Gamma_0(\lambda)$. Thus $I_1\cup I_2=I_1$ and $I_1\cap I_2=I_2$. The assertion follows from Lemma 3.3. Assume $d(I_1, I_2)=t\geqq 2$. Let $I_1=J_0$, $J_1, \ldots, J_t=I_2$ be a shortest path in $\Gamma_0(\lambda)$. Then

$$d(I_1, I_2)=d(I_1, J_1)+d(J_1, I_2)=|I_1\cup J_1|-|J_1\cap I_1|+|J_1|+|I_2|-2|J_1\cap I_2|.$$

If $J_1\subseteq I_1$, we have $d(I_1, I_2)=|I_1|+|I_2|-2|J_1\cap I_2|\geqq|I_1\cup I_2|-|I_1\cap I_2|$. Otherwise we have $I_1\subseteq J_1$ and therefore

$$\begin{aligned} d(I_1, I_2)&=|J_1|-|I_1|+|I_2|+|J_1|-2|I_2\cap J_1|\\ &=|I_1\cup I_2|-|I_1\cap I_2|+2(|J_1|-|I_1|+|I_1\cap I_2|-|J_1\cap I_2|)\\ &=|I_1\cup I_2|-|I_1\cap I_2|+2(|J_1\backslash I_1|-|(J_1\backslash I_1)\cap I_2|)\\ &\geqq|I_1\cup I_2|-|I_1\cap I_2|. \quad \square \end{aligned}$$

3.6 The following lemma will be used in the next section.

Lemma. *Let $I, J\in\mathscr{S}(\lambda)$ such that $i(I)>i(J)$. Then $|J|-|I\cap J|\leqq v_p\left(\binom{i(I)}{i(J)}\right)$.*

Proof. Let $K=I\cap J$. Using Proposition 3.4 of [15], we only need to show that each $g\in J\backslash K$ is contained in a component for the pair $(i(J), i(I))$ (cf. the proof of 3.1). There exists a component $(j_1, j_1']$ of J such that $g\in(j_1, j_1']\backslash K$. There exists $s\geqq 0$ maximal such that $g-t\in(j_1, j_1']\backslash K$ for $t=0, \ldots, s$. Thus $g-t\notin I$ and the coefficient of p^{g-t} in the p-adic expansion of $i(I)$ is 0 for $t=0, 1, \ldots, s$.

(A) If $g-s=j_1'$, the coefficient of p^{g-s} in the p-adic expansion of $j(J)$ is $a_{j_1'}+1>0$ or $a_{r_1}>0$ if $j_1'=r_1$ (cf. 3.1).

(B) If $g-s>j_1'$, then $g-s-1\in(k_1, k_1']$ for a component $(k_1, k_1']$ of K. Thus $k_1=g-s$ and the coefficient of p^{g-s} in the p-adic expansion of $i(J)$ is $a_{k_1}>0$.

In the both cases we discussed above, g is in a component of the pair $(i(J), i(I))$. □

4 Modules for $\mathrm{SL}_2(k)$ arising from admissible lattices

4.1 We still assume that $\mathfrak{g}=sl_2$ and $G=\mathrm{SL}_2(k)$ in this section. Let $\lambda\geqq 0$ be a fixed dominant integral weight for $\mathfrak{g}$ and $\{v_0, v_1, \ldots, v_\lambda\}$ a basis for $H^0(\lambda)$ with $v_i\in V(\lambda)_{\lambda-i\alpha}$ in as in [15] (Sect. 3). $H^0(\lambda)$ is the *maximal* admissible lattice in $V(\lambda)$ (with respect to the fixed weight vector $v^+=v_0$) (cf. [15]). Let M be a lattice in $V(\lambda)$ with a $\mathbb{Z}$-basis $\{z_0, z_1, \ldots, z_\lambda\}$ such that $z_i=x_i v_i$ for some $x_i\in\mathbb{Z}$. Then

$$X^{(s)} z_i=\frac{x_i}{x_{i-s}}\binom{i}{i-s} z_{i-s} \qquad Y^{(s)} z_i=\frac{x_i}{x_{i+s}}\binom{\lambda-i}{\lambda-i-s} z_{i+s}.$$

M is admissible if and only if $\frac{x_i}{x_j}\binom{i}{j}$ and $\frac{x_j}{x_i}\binom{\lambda-j}{\lambda-i}$ are in $\mathbb{Z}$ whenever $\lambda\geqq i>j\geqq 0$. We can further assume that $x_0=1$ and $x_i>0$ for $i=1, \ldots, \lambda$. Though we are considering the admissible lattices over $\mathbb{Z}_p$, each admissible lattice M can be represented by a sequence $\{x_0, \ldots, x_\lambda\}$ satisfying the above conditions. For $i>j$, $z_i\otimes 1$ is in the submodule of $\bar{M}$ generated by $z_j\otimes 1$ if $v_p\left(\frac{x_j}{x_i}\binom{\lambda-j}{\lambda-i}\right)=0$ and $z_j\otimes 1$ is in the submodule of $\bar{M}$ generated by $z_i\otimes 1$ if $v_p\left(\frac{x_i}{x_j}\binom{i}{j}\right)=0$. In particular, $\bar{M}=\overline{V(\lambda)}$ if $x_i=\binom{\lambda}{i}$ for $i=0, \ldots, \lambda$.

4.2 The following facts can be verified easily by using the filtrations for Weyl modules and the above basis.

(A) If $L(\lambda)$ is a simple G-module and v_λ, v_μ are non-zero weight vectors in $L(\lambda)$ of weights λ and μ respectively such that $\lambda-\mu=s\alpha$, then $Y^{(s)} v_\lambda\neq 0$, $X^{(s)} v_\mu\neq 0$, and $X^{(s)} Y^{(s)} v_\lambda=\binom{\lambda}{s} v_\lambda$.

(B) If $0\to L(\lambda)\to E\to L(\mu)\to 0$ is a nonsplit extension of two different simple G-modules, then all weight spaces of E are 1-dimensional. If v_μ is a non-zero weight vector in E of weight μ, then $Y^{(s)} v_\mu\neq 0$ if $\mu-\lambda=s\alpha$ with $s>0$ and $X^{(s)} v_\mu\neq 0$ if $\lambda-\mu=s\alpha$ with $s>0$.

Lemma. *Let $I, J\in\mathscr{S}(\lambda)$ and $i=i(I)>j=i(J)$. If $\mathrm{Ext}_G^1(S(I), S(J))\neq 0$, then*

(i) *either* $v_p\left(\binom{i}{j}\right)=0$ *or* $v_p\left(\binom{\lambda-j}{\lambda-i}\right)=0$;

(ii) *either* $v_p\left(\frac{x_i}{x_j}\binom{i}{j}\right)=0$ *or* $v_p\left(\frac{x_j}{x_i}\binom{\lambda-j}{\lambda-i}\right)=0$.

Proof. To prove (i), we consider the module $H_k^0(\lambda)$. By the structure theorem of Cline [6], there is a subquotient of $H_k^0(\lambda)$ which is an extension of $S(I)$ and $S(J)$. (i) follows from the actions of $X^{(s)}$ and $Y^{(s)}$ on the basis elements of $H_k^0(\lambda)$ and (B). (ii) is a consequence of (i) and the facts: $||I|-|J||=1$ and $\binom{\lambda}{i}\binom{i}{j}=\binom{\lambda}{j}\binom{\lambda-j}{\lambda-i}$, which imply $v_p\left(\binom{i}{j}\right)-v_p\left(\binom{\lambda-j}{\lambda-i}\right)=\pm 1$ by 3.1 (iii). □

4.3 Let M' be the dual lattice of M. If M' is represented by $\{y_0, \ldots, y_\lambda\}$ as in 4.1, then one can easily verify from the contravariant form that $x_i y_i=\binom{\lambda}{i}$ for $i=0, 1, \ldots, \lambda$. Let e be the minimal integer such that $M\subseteq p^{-e}M'$ as in 2.2. Then $e=\max\{v_p(y_i)-v_p(x_i)\}=\max\left\{v_p\left(\binom{\lambda}{i}\right)-2\,v_p(x_i)\right\}$. Let $\{z_0, \ldots, z_\lambda\}$ be a basis for M as in 4.1. Then $z_i\in M^j=M\cap p^{j-e}M'$ if and only if $v_p(x_i)\geqq v_p(p^{j-e}y_i)$ $=j-e+v_p(y_i)$. Equivalently, $\bar{z}_i\in \bar{M}^j$ if and only if $2\,v_p(x_i)-v_p\left(\binom{\lambda}{i}\right)+e\geqq j$. For any module E of finite length and any simple module L, by $[E:L]$ we denote the multiplicity of L in a composition series of E. Thus we have proved

Lemma. *$[\bar{M}_j: S(I)]\neq 0$ for $I\in\mathscr{S}(\lambda)$ if and only if $2\,v_p(x_i)-|I|+e=j$.*

4.4 Now we are ready to prove the structure theorem of $\bar{M}$ for $\mathfrak{g}=sl_2$.

Theorem. *Let I, $J\in\mathscr{S}(\lambda)$ and $\mathrm{Ext}^1_G(S(I), S(J))\neq 0$. Then, for any admissible lattice M,*

(i) *$S(I)$ and $S(J)$ appear in the adjacent layers of the filtration for $\bar{M}$;*
(ii) *there is a subquotient of $\bar{M}$ which is an extension of $S(I)$ and $S(J)$;*
(iii) *if $\Gamma(M)$ denotes the module graph of $\bar{M}$, then $\overline{\Gamma(M)}=\overline{\Gamma_0(\lambda)}$.*

Proof. (i) Set $i=i(I)$ and $j=i(J)$. We may assume $i>j$. By Lemma 4.2, there are four possible situations:

(A) $v_p\left(\binom{i}{j}\right)=0$ and $v_p\left(\frac{x_i}{x_j}\binom{i}{j}\right)=0$;

(B) $v_p\left(\binom{\lambda-j}{\lambda-i}\right)=0$ and $v_p\left(\frac{x_j}{x_i}\binom{\lambda-j}{\lambda-i}\right)=0$;

(C) $v_p\left(\binom{i}{j}\right)=0$ and $v_p\left(\frac{x_j}{x_i}\binom{\lambda-j}{\lambda-i}\right)=0$;

(D) $v_p\left(\binom{\lambda-j}{\lambda-i}\right)=0$ and $v_p\left(\frac{x_i}{x_j}\binom{i}{j}\right)=0$.

We have $v_p(x_i)=v_p(x_j)$ in cases (A) and (B). In (C) and (D),

$$v_p(x_i)-v_p(x_j)=v_p\left(\binom{\lambda-j}{\lambda-i}\right)-v_p\left(\binom{i}{j}\right)=v_p\left(\binom{\lambda}{i}\right)-v_p\left(\binom{\lambda}{j}\right)=\pm 1$$

by Corollary 3.3. If $[\bar{M}_t: S(I)]\neq 0$ and $[\bar{M}_s: S(J)]\neq 0$, by Lemma 4.3,

$$t-s=2(v_p(x_i)-v_p(x_j))-\left(v_p\left(\binom{\lambda}{i}\right)-v_p\left(\binom{\lambda}{j}\right)\right)=\pm\left(v_p\left(\binom{\lambda}{i}\right)-v_p\left(\binom{\lambda}{j}\right)\right)=\pm 1.$$

(ii) We may assume $s=t-1$ in the proof of (i). Consider the submodule U of $\bar{M}$ generated by $\bar{z}_i$. Then U has simple head $S(I)$. By Lemma 4.2, $S(J)$ has to be a composition factor of U. Since all the filtration layers are semisimple, $\operatorname{Rad} U \subseteq \bar{M}^s$. Then the assertion follows.
(iii) It follows from (ii) that either $I \to J$ or $J \to I$ is an edge in $\Gamma(M)$. Conversely, if $I \to J$ is an edge of $\overline{\Gamma(M)}$, then $\operatorname{Ext}^1_G(S(I), S(J)) \neq 0$. Thus all edges of $\overline{\Gamma(M)}$ have to be edges of $\overline{\Gamma_0(\lambda)}$. □

Corollary. *If* $\bar{M}/\operatorname{Rad}\bar{M}$ (*or* $\operatorname{Soc}\bar{M}$) *is simple, then the filtration for* $\bar{M}$ *is the radical* (*or socle*) *filtration.*

Proof. We prove the corollary only for the radical filtration. By Corollary 2.6, we have $\operatorname{Rad}^i \bar{M} \subseteq \bar{M}^i$ for all $i \geqq 0$. To show the equality, we use induction on i. For $i=1$, this follows from the assumption. Suppose that $\operatorname{Rad}^i \bar{M} = \bar{M}^i$. If $[\bar{M}^i/\operatorname{Rad}^{i+1}\bar{M} : S(J)] \neq 0$ for $J \in \mathscr{S}(\lambda)$, there exists $J_1 \in \mathscr{S}(\lambda)$ such that $[\bar{M}_{i-1} : S(J_1)] \neq 0$ and $\operatorname{Ext}^1_G(S(J_1), S(J)) \neq 0$. By (i) of the theorem, $[\bar{M}_i : S(J)] \neq 0$. Therefore $\bar{M}^i/\operatorname{Rad}^{i+1}\bar{M}$ and $\bar{M}^i/\bar{M}^{i+1}$ have the same composition factors and $\operatorname{Rad}^{i+1}\bar{M} = \bar{M}^{i+1}$. □

4.5 We still use $\{x_0, \ldots, x_\lambda\}$ to represent the admissible lattice M as in 4.1. Let $\lambda - i\alpha$ and $\lambda - j\alpha$ be weights of the same composition factor of $H^0_k(\lambda)$. Using the filtrations for $\overline{V(\lambda)}$ and $\bar{M}$ defined in 2.2, we can show $v_p\left(\left(\binom{\lambda}{j}\right)\right) = v_p\left(\left(\binom{\lambda}{i}\right)\right)$ and $v_p(x_i) = v_p(x_j)$.

Define a function $e\colon \mathscr{S}(\lambda) \to \mathbb{N}$ by $e(I) = v_p(x_{i(I)})$ for all $I \in \mathscr{S}(\lambda)$. One can see that M is an admissible lattice over $\mathbb{Z}_p$ if and only if $e(I) + v_p\left(\left(\binom{i(I)}{i(J)}\right)\right) \geqq e(J)$ and $e(J) + v_p\left(\left(\binom{\lambda - i(J)}{\lambda - i(I)}\right)\right) \geqq e(I)$ whenever $I, J \in \mathscr{S}(\lambda)$ with $i(I) > i(J)$. Conversely, for any given function $e\colon \mathscr{S}(\lambda) \to \mathbb{Z}$ satisfying $e(J) - e(I) \leqq v_p\left(\left(\binom{i(I)}{i(J)}\right)\right)$ and $e(I) - e(J) \leqq v_p\left(\left(\binom{\lambda - i(J)}{\lambda - i(I)}\right)\right)$ for all $I, J \in \mathscr{S}(\lambda)$ with $i(I) > i(J)$, there is a unique admissible lattice M over $\mathbb{Z}_p$ such that $v_p(x_{i(I)}) = e(I) - e(\emptyset)$ for all $I \in \mathscr{S}(\lambda)$.

Theorem. *Let* Γ *be a graph with* $\bar{\Gamma} = \overline{\Gamma_0(\lambda)}$. *Then* $\Gamma = \Gamma(M)$ *for some admissible lattice* M *if and only if there exists a function* $e\colon \mathscr{S}(\lambda) \to \mathbb{Q}$ *such that for any edge* $I \to J$ *in* Γ *one has*

$$(1) \qquad e(J) = \begin{cases} e(I) & \text{if } I \to J \text{ is an edge in } \Gamma_0(\lambda), \\ e(I) + 1 & \text{if } J \to I \text{ is an edge in } \Gamma_0(\lambda). \end{cases}$$

Proof. If $\Gamma = \Gamma(M)$, we set $e(I) = v_p(x_{i(I)})$. Let $I \to J$ be an edge in $\Gamma(M)$. Following Theorem 4.4(ii) and Lemma 4.3, we have $(2e(J) - |J|) - (2e(I) - |I|) = 1$, or equivalently $e(J) = e(I) + \frac{1}{2}(|J| - |I| + 1)$. Now Eq. (1) follows from Lemma 3.3.

Suppose e is a function satisfying the Eq. (1). By considering the function $I \mapsto e(I) - e(\emptyset)$ we may assume that $e(\emptyset) = 0$ and $e(I) \in \mathbb{Z}$. One can see that Eq. (1) is equivalent to the following: For any edge $I \to J$ in $\Gamma_0(\lambda)$,

$$(2) \qquad e(I) = \begin{cases} e(J) & \text{if } I \to J \text{ is an edge in } \Gamma, \\ e(J) + 1 & \text{if } J \to I \text{ is an edge in } \Gamma. \end{cases}$$

By induction, we have $0 \leqq e(I)-e(J) \leqq |I|-|J|$ whenever $I \supseteq J$. In particular, $0 \leqq e(I) \leqq |I|$ and $e(I) \geqq e(J)$ for all $I, J \in \mathscr{S}(\lambda)$ with $I \supseteq J$. For any pair $I, J \in \mathscr{S}(\lambda)$ with $i=i(I)>i(J)=j$, set $K=I \cap J$. Following Lemma 3.6 we have $|J|-|K| \leqq v_p\left(\binom{i}{j}\right)$. Thus $e(J)-e(I) \leqq e(J)-e(K) \leqq v_p\left(\binom{i}{j}\right)$. This shows that $e(J)-e(I) \leqq v_p\left(\binom{i(I)}{i(J)}\right)$ when $i(I)>i(J)$ for any graph Γ as in the theorem with a function e satisfying Eq. (1).

Let Γ' be the dual graph of Γ (i.e., the graph obtained by reversing the directions on all edges of Γ). Define $e'\colon \mathscr{S}(\lambda) \to \mathbb{Z}$ by $e'(I)=|I|-e(I)$. One can verify that e' satisfies Eq. (1) for the graph Γ' (use Lemma 3.3). By what we have proved above, we have $e'(J)-e'(I) \leqq v_p\left(\binom{i(I)}{i(J)}\right)$ whenever $i(I)>i(J)$. Therefore

$$e(I)-e(J) \leqq |I|-|J|+v_p\left(\binom{i(I)}{i(J)}\right)=v_p\left(\binom{\lambda-i(J)}{\lambda-i(I)}\right).$$

Thus we have shown that the function e (with $e(\emptyset)=0$) satisfies the conditions for an admissible lattice as we discussed before the theorem. □

Remark. (1) Let $\mathscr{E}$ be the set of edges of $\Gamma_0(\lambda)$. Each graph Γ with $\bar{\Gamma}=\overline{\Gamma_0(\lambda)}$ is uniquely determined by a function $f\colon \mathscr{E} \to \{0, 1\}$ such that for each edge $I \to J$ in $\Gamma_0(\lambda)$, $f(I \to J)=0$ or 1 according to $I \to J$ or $J \to I$ is an edge in Γ. Equation (1) is equivalent to the fact that the sum of f along any directed path in $\Gamma_0(\lambda)$ depends only on the starting and ending vertices. This provides an easy criterion to test if $\Gamma=\Gamma(M)$ for some admissible lattice M.

(2) The filtration for $H_k^0(\lambda)$ divides the set $\mathscr{E}$ of edges into many layers. The jth layer of edges are those edges connecting the composition factors in the $(j-1)$th filtration layer to those in the jth filtration layer for $H_k^0(\lambda)$. As an easy consequence of the above criterion, one can show that for each n with $2 \leqq n \leqq r-r_1+1$ there is an admissible lattice M such that $\bar{M}$ has Loewy length n. The graph of such an $\bar{M}$ can be constructed by setting f to be constant on each layer of $\mathscr{E}$. Any function f which takes a constant value on each layer satisfies the above criterion since any two directed paths in $\Gamma_0(\lambda)$ connecting I and J have the same length and each of such paths has one and only one edge in each layer between I and J (cf. 3.3 and 4.3).

4.6 Let $I_0=(r, r_1]$, which is the unique maximal element in $\mathscr{S}(\lambda)$. For each $I \in \mathscr{S}(\lambda)$, let $I'=I_0 \backslash I$, which may not be in $\mathscr{S}(\lambda)$. To state the last theorem we need one more notation. For a module E, we denote by $\ell\ell(E)$ the Loewy length of E.

Theorem. (i) *For each admissible lattice M in $V(\lambda)$, $\ell\ell(\bar{M}) \leqq \ell\ell(H_k^0(\lambda))=r-r_1+1$.*
(ii) *For each $I \in \mathscr{S}(\lambda)$, there is an admissible lattice M such that the filtration defined in* 2.2 *for $\bar{M}$ is the socle filtration and $S(J)$ is a composition factor of* $\mathrm{Soc}^n \bar{M}$ *if and only if $d(I, J)<n$. In particular, $S(I)=\mathrm{Soc}\,\bar{M}$. Furthermore, every $\bar{M}$ with a simple socle is of this form.*
(iii) *The following are equivalent for an admissible lattice M. In each case, $\bar{M}$ is rigid.*

(a) *M is constructed as in* (ii) *for some $I\in\mathcal{S}(\lambda)$ with $I'\in\mathcal{S}(\lambda)$;*
(b) $\ell\ell(M)=r-r_1+1$;
(c) *Both* $\operatorname{Soc}\bar{M}$ *and* $\bar{M}/\operatorname{Rad}\bar{M}$ *are simple.*

Proof. (i) Consider the filtration for $\bar{M}$. Each filtration layer is semisimple by Corollary 2.6. Let $I, J\in\mathcal{S}(\lambda)$ such that $[\bar{M}_t: S(I)]\neq 0$ and $[\bar{M}_s: S(J)]\neq 0$. We may assume $t\geqq s$. Using induction on $d(I, J)$ and Theorem 4.4, one can show that $t-s\leqq d(I, J)$. The conclusion of (i) follows from Proposition 3.5.
(ii) For a fixed $I\in\mathcal{S}(\lambda)$, we define a graph Γ with $\bar{\Gamma}=\overline{\Gamma_0(\lambda)}$ as follows: If (I_1, I_2) is an edge in $\overline{\Gamma_0(\lambda)}$, then $I_1\to I_2$ is an edge in Γ if $d(I, I_1)>d(I, I_2)$. Note that $\overline{\Gamma_0(\lambda)}$ is a bipartite graph with the partition $\mathcal{S}(\lambda)=A\cup B$ given by $A=\{J\in\mathcal{S}(\lambda)\,|\,|J|$ is even$\}$ and $B=\{J\in\mathcal{S}(\lambda)\,|\,|J|$ is odd$\}$. One can see that $d(I, I_1)\neq d(I, I_2)$ for each edge $\underline{(I_1, I_2)}$ in $\overline{\Gamma_0(\lambda)}$. Otherwise there will be a circuit with odd number of edges in $\overline{\Gamma_0(\lambda)}$ (see [5, Theorem 1, p. 10]). Thus the graph Γ is well defined with $\bar{\Gamma}=\overline{\Gamma_0(\lambda)}$.

Define the function e: $\mathcal{S}(\lambda)\to\mathbb{Q}$ by $e(J)=\frac{1}{2}(|J|-|I|-d(I, J))$ for each $J\in\mathcal{S}(\lambda)$. For each edge $I_1\to I_2$ in Γ, one must have $d(I, I_1)=d(I, I_2)+1$. Thus $e(I_2)-e(I_1)=\frac{1}{2}(|I_2|-|I_1|+1)$, which is 0 if $I_1\to I_2$ is an edge in $\Gamma_0(\lambda)$ and 1 if $I_2\to I_1$ is an edge in $\Gamma_0(\lambda)$. By Theorem 4.5, $\Gamma=\Gamma(M)$ for some admissible lattice M.

By Lemma 4.3, two composition factors $S(I_1)$ and $S(I_2)$ occur in the same filtration layer of $\bar{M}$ if and only if $d(I, I_1)=d(I, I_2)$. The filtration is the socle filtration follows from the fact that for each $I_1\in\mathcal{S}(\lambda)$ with $d(I_1, I)\geqq 1$, there exists $I_2\in\mathcal{S}(\lambda)$ such that (I_1, I_2) is an edge in $\overline{\Gamma_0(\lambda)}$ and $d(I, I_2)=d(I, I_1)-1$.

Conversely, let M be an admissible lattice such that $\operatorname{Soc}\bar{M}=S(I)$ is simple. By Corollary 4.4, the filtration for $\bar{M}$ is the socle filtration. Using Theorem 4.4(i) and an induction on i one can show that, for each $J\in\mathcal{S}(\lambda)$, $[\operatorname{Soc}^i\bar{M}: S(J)]\neq 0$ if and only if $d(I, J)<i$. Thus the graph $\Gamma(M)$ is the one constructed above.
(iii) (a) $\Rightarrow$ (b) and (c). Suppose $I'=I_0\setminus I\in\mathcal{S}(\lambda)$. Then we have $d(I, I')=|I_0|$ by Proposition 3.5. Following the construction in (ii), $\bar{M}$ has the maximal Loewy length $|I_0|+1$. Using Proposition 3.5, a simple calculation shows

$$d(J, I')+d(J, I)=|I_0| \quad \text{for all } J\in\mathcal{S}(\lambda). \tag{3}$$

By the definition of the graph $\Gamma(M)$ in the proof of (ii), Eq. (3) shows that there is a directed path from I' to each $J\in\mathcal{S}(\lambda)$. Therefore, $\bar{M}/\operatorname{Rad}\bar{M}\cong S(I')$ and (c) follows.
(b) $\Rightarrow$ (a). Since $\ell\ell(\bar{M})=|I_0|+1$, then $\bar{M}_{|I_0|}\neq 0$. Let $S(I)$ be a composition factor of $\bar{M}_{|I_0|}$. If $S(J)$ is a composition factor of $\bar{M}/\bar{M}^1$, by the proof in (i), we have $|I_0|\leqq d(I, J)\leqq|I_0|$ and $J=I'$. Thus $I'\in\mathcal{S}(\lambda)$ since $\bar{M}/\bar{M}^1$ is not zero by the definition of the filtration. Let $S(J)$ be a composition factor of $\bar{M}_s$. By the proof of (i) once again, we have $|I_0|-s\leqq d(J, I)$ and $s\leqq d(J, I')$. Following Eq. (3), we have $d(I, J)=|I_0|-s$. Thus the graph $\Gamma(M)$ is the one constructed in (ii) for I and (a) follows.
(c) $\Rightarrow$ (b). Suppose $S(I_1)=\operatorname{Soc}\bar{M}$ and $S(I_2)=\bar{M}/\operatorname{Rad}\bar{M}$. Let l be the smallest number such that $\operatorname{Rad}^{l+1}\bar{M}=0$. Similar to the last paragraph in the proof of (ii), we have

$$[\operatorname{Soc}^i\bar{M}: S(J)]\neq 0 \quad \text{if and only if } d(I_1, J)\leqq i-1;$$

$$[\operatorname{Rad}^i\bar{M}: S(J)]\neq 0 \quad \text{if and only if } d(I_2, J)\leqq i.$$

Since $\mathrm{Rad}^i \bar{M} \subseteq \mathrm{Soc}^{l+1-i} \bar{M}$, we have $d(I_1, J) + d(I_2, J) \leqq l \leqq |I_0|$ by (i) for all $J \in \mathscr{S}(\lambda)$. Taking $J = \varnothing, I_0$, we have

$$d(\emptyset, I_1) + d(\emptyset, I_2) \leqq l \leqq |I_0|;$$
$$d(I_0, I_1) + d(I_0, I_2) \leqq l \leqq |I_0|.$$

However, by Eq. (3), $d(\emptyset, I_i) + d(I_0, I_i) = |I_0|$ for $i = 1, 2$. Adding the above two inequalities together we get $l = |I_0|$. Thus $\ell\ell(\bar{M}) = |I_0| + 1$ and (b) follows.

Finally, the rigidity of $\bar{M}$ follows from the conditions of (c) and Corollary 4.4. □

Remark. There exists λ and an admissible lattice M in $V(\lambda)$ such that the filtration we constructed for $\bar{M}$ is neither the radical nor the socle filtration.

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