Math. Z. 209, 1–16 (1992)

Mathematische Zeitschrift

On interior and boundary regularity of weak solutions to a certain quasilinear elliptic system

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Received March 23, 1990; in final form November 30, 1990

0 Introduction

We consider weak solutions to the quasilinear elliptic system

(0.1) $$\Delta x = 2H(x)x_u \wedge x_v \, ,$$

i.e. mappings $x \in W^{1,2}(\Omega, \mathbb{R}^3)$ satisfying

(0.2) $$\int_\Omega (\nabla\Phi \cdot \nabla x + \Phi \cdot 2H(x)x_u \wedge x_v)\,dudv = 0$$

for all $\Phi \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty$. Here Ω is a domain in $\mathbb{R}^2$, H is a realvalued function in $\mathbb{R}^3$, and $\wedge$ denotes the vector product in $\mathbb{R}^3$. x can also be considered as a critical point of the functional

(0.3) $$E(x) = \int_\Omega (|\nabla x|^2 + A(x) \cdot (x_u \wedge x_v))\,dudv \, ,$$

where $\operatorname{div} A(x) = 4H(x)$. On the coefficient $H(x)$ we impose the following conditions:

(0.4) $$H(x) = H_1(x) + H_2(x) \, ,$$

(0.5) $$\sup_{x \in \mathbb{R}^3} (|H_1(x)| + (1 + |x|)|\nabla H_1(x)|) < \infty \, ,$$

and

(0.6) $$\sup_{x \in \mathbb{R}^3} (|H_2(x)| + |\nabla H_2(x)|) < \infty, \qquad \sup_{|x| \geqq K} |x\, H_2(x)| < 1 \, .$$

As our principal result (Theorem 3.3) we show that $x \in C^{2,\mu}(\Omega, \mathbb{R}^3)$ holds for all $\mu \in (0, 1)$. In the case where $H_2(x) \equiv 0$ this theorem reduces to a result of Heinz [8], whereas for $H_1(x) \equiv 0$ the statement is contained in a more general result of Tomi [16].

Moreover, for $z \in W^{1,2}(\Omega, \mathbb{R}^3)$ we consider weak solutions to the *Dirichlet problem*

$$\text{(0.7)} \qquad \mathrm{DP}\begin{cases} \Delta x = 2H(x)x_u \wedge x_v & \text{in } \Omega \\ x = z & \text{on } \partial\Omega \,, \end{cases}$$

i.e. mappings $x \in W^{1,2}(\Omega, \mathbb{R}^3)$ satisfying (0.2) and $x - z \in W_0^{1,2}(\Omega, \mathbb{R}^3)$. Under the additional assumptions $z \in L^\infty(\Omega)$ and $z \in C^0(\bar{\Omega})$ we prove that x belongs to $L^\infty(\Omega)$ and $C^0(\bar{\Omega})$, respectively. In the case where $H \equiv$ const this follows from a result of Brezis and Coron (see [1, A1]). Moreover, if the boundary function z belongs to a higher regularity class, corresponding results of the boundary behavior of the solution x can be inferred from the papers [5, 11, 18, 19].

The Eq. (0.1) arises in connection with *surfaces of prescribed mean curvature* (see [9] and [10] for a more detailed discussion of this topic and related results).

The essential point in the proofs is to show that the solution x is locally bounded. The main tool in this paper, which is a shortened version of the author's doctoral thesis [12], are estimates of Dirichlet integrals over level sets. This idea has been used previously by Hildebrandt and Widman [11] for treating quasilinear elliptic partial differential equations and systems. Using the special structure of the underlying system (0.1), the *Courant-Lebesgue Lemma*, and a special version of the *isoperimetric inequality*, we first prove a result concerning *partial interior regularity* (Theorem 3.1), namely continuity of a weak solution x to (0.1) in all points $w_0 \in \Omega$ where

$$\text{(0.8)} \qquad \lim_{r \downarrow 0}\left(\left|\frac{1}{2\pi r} \int_{\partial B_r(w_0)} x\, ds\right|^2 \int_{B_r(w_0)} |\nabla x|^2\, dw \right) = 0 \,.$$

Continuity in all of Ω (Theorem 3.2) is then deduced from the facts

$$\text{(0.9)} \qquad \frac{1}{2\pi r} \int_{\partial B_r(w_0)} x\, ds = o(\log^{\frac{1}{2}} r^{-1}), \quad r \downarrow 0 \,,$$

(Lemma 1.3) and

$$\text{(0.10)} \qquad \int_{B_\rho(w_0)} \log|w - w_0|^{-1} |\nabla x(w)|^2\, dw < \infty$$

(Theorem 3.2). We note that the proof of (0.10) makes use of some ideas taken from Heinz [8] (see below).

Boundary regularity (Theorem 4.1) is finally obtained by controlling $(2\pi r)^{-1} \int_{\partial B_r(w_0)} x\, ds$ by the boundary data in a small neighborhood of $\partial\Omega$. We use the interior estimates to establish the stated results.

1 Preliminaries

In this paper we use the following notations: Ω denotes a domain in $\mathbb{R}^n$, $B_r(w_0)$ the open ball in $\mathbb{R}^n$ with center w_0 and radius r. We have $n = 2$ and write $w = (u, v)$ or $w = re^{i\varphi}$, if $w \in \mathbb{R}^2$. Only in Lemmas 2.1 and 2.2 $n \geqq 2$ is admitted.

In usual manner $C^k(\Omega, \mathbb{R}^m)(C^{k,\mu}(\Omega, \mathbb{R}^m))$ denotes the space of functions with (Hölder) continuous partial derivatives of order k. A subscript $_0$ is added to refer to functions with compact support in Ω.

For norms of the Banach spaces $L^p(\Omega, \mathbb{R}^m)$ we use $\|\cdot\|_{p;\Omega}$, and by $W^{1,2}(\Omega, \mathbb{R}^m)$ we mean the well known Sobolev space consisting of the square integrable functions in Ω with square integrable distributional derivatives. Moreover, $W_0^{1,2}(\Omega, \mathbb{R}^m)$ denotes the completion of $C_0^1(\Omega, \mathbb{R}^m)$ in $W^{1,2}(\Omega, \mathbb{R}^m)$. sup often means ess sup.

The scalar product in $\mathbb{R}^m$ is denoted by $a \cdot b$, the vector product in $\mathbb{R}^3$ by $c \wedge d$, and the triple scalar product by $(a, b, c) = a \cdot (b \wedge c) = a \cdot b \wedge c$. The following type of *isoperimetric inequality* will be essentially needed in our proofs.

Lemma 1.1 *Let $g \in W^{1,2}(B_r(w_0), \mathbb{R}^3) \cap L^\infty$ and $h \in W_0^{1,2}(B_r(w_0), \mathbb{R}^3)$. Then,*

$$\left| \int_{B_r(w_0)} (g, h_u, h_v)\, du dv \right| \leqq c_0 \left(\int_{B_r(w_0)} |\nabla g|^2\, du dv \right)^{1/2} \int_{B_r(w_0)} |\nabla h|^2\, du dv \tag{1.1}$$

holds with an absolute constant c_0.

For proofs see e.g. [1, 6, 17].

We shall use the following version of the well known *Courant-Lebesgue Lemma.*

Lemma 1.2 *Let $x \in W^{1,2}(B_\rho(w_0), \mathbb{R}^m)$. Then, there exists a set $M_x \subset (\frac{1}{2}\rho, \rho)$ with* meas $M_x \geqq \frac{1}{4}\rho$ *satisfying the following two conditions for all $r \in M_x$*

1. $x_{|\partial B_r(w_0)}$ *is absolutely continuous*
2. $\sup_{w', w'' \in \partial B_r(w_0)} |x(w') - x(w'')| \leqq \left(\frac{\pi}{\log \frac{4}{3}} \int_{B_\rho(w_0)} |\nabla x|^2\, du dv \right)^{1/2}.$

The proof (see [12]) is quite similar to those given in [3, 7] and can be omitted.

Some remarks on a decomposition for $W^{1,2}(B_\rho(w_0), \mathbb{R}^m)$ functions will finish our preparations. We set

$$x(w) = \xi(|w - w_0|) + y(w) \tag{1.2}$$

with *circleline meanvalue*

$$\xi(r) = \xi_{w_0}(r) = \frac{1}{2\pi} \int_0^{2\pi} x(w_0 + re^{i\varphi})\, d\varphi \tag{1.3}$$

and *oscillation part*

$$y(w) = y_{w_0}(w) = x(w) - \xi_{w_0}(|w - w_0|)\,. \tag{1.4}$$

Since x has a trace in $L^2(\partial B_r(w_0), \mathbb{R}^m)$ for all $r \in (0, \rho]$ (see [14]), $\xi = \xi(r)$ is well defined in $(0, \rho]$ and $y = y(w_0 + re^{i\varphi})$ a.e. in $(0, 2\pi)$ for all $r \in (0, \rho]$. Moreover, we have

$$\int_0^{2\pi} y(w_0 + re^{i\varphi})\, d\varphi = 0 \tag{1.5}$$

and

(1.6)
$$\int_0^{2\pi} |x(w_0 + re^{i\varphi})|^2 \, d\varphi = 2\pi|\xi(r)|^2 + \int_0^{2\pi} |y(w_0 + re^{i\varphi})|^2 \, d\varphi$$

for all $r \in (0, \rho)$. Then, approximation and (1.5) yield

(1.7)
$$\int_0^{2\pi} |\nabla x(w_0 + re^{i\varphi})|^2 \, d\varphi = 2\pi|\xi'(r)|^2 + \int_0^{2\pi} \left(|y_r(w_0 + re^{i\varphi})|^2 + \frac{|y_\varphi(w_0 + re^{i\varphi})|^2}{r^2} \right) d\varphi$$
$$\geqq \frac{1}{r^2} \int_0^{2\pi} |y(w_0 + re^{i\varphi})|^2 \, d\varphi$$

a.e. in $(0, \rho)$, from which we also obtain that ξ is absolutely continuous in $(0, \rho]$.

The following lemma concerning circleline meanvalues can be proved in an elementary way.

Lemma 1.3 *$x \in W^{1,2}(B_\rho(w_0), \mathbb{R}^m)$ implies*

(1.8)
$$\lim_{r \downarrow 0} \frac{|\xi_{w_0}(r)|^2}{\log r^{-1}} = 0 \, .$$

Proof. It suffices to consider the case $m = 1$. Assume that (1.8) is false. Then there exists a sequence $\{r_k\}_{k \in \mathbb{N}} \subset (0, \rho)$ such that $r_k \downarrow 0$, $k \to \infty$, and

(1.9)
$$\frac{|\xi_{w_0}(r_{k+1}) - \xi_{w_0}(r_k)|^2}{\log r_{k+1}^{-1} - \log r_k^{-1}} \geqq \varepsilon > 0, \quad k \in \mathbb{N} \, .$$

Using (1.7) and the Schwarz inequality, we infer

(1.10)
$$\frac{1}{2\pi} \int_{B_{r_k}(w_0) \setminus B_{r_{k+1}}(w_0)} |\nabla x|^2 \, du dv \geqq \int_{r_{k+1}}^{r_k} \xi'_{w_0}(r)^2 r \, dr \geqq \varepsilon, \quad k \in \mathbb{N} \, ,$$

thus

(1.11)
$$\int_{B_{r_1}(w_0) \setminus B_{r_{n+1}}(w_0)} |\nabla x|^2 \, du dv \geqq 2\pi n \varepsilon, \quad n \in \mathbb{N} \, ,$$

which contradicts $x \in W^{1,2}(B_\rho(w_0))$.

2 Estimates for Dirichlet integrals over level sets

In this section we deduce estimates for (weighted) Dirichlet integrals of mappings $x \in W^{1,2}(\Omega, \mathbb{R}^m)$ satisfying the Poisson equation $\Delta x = h$, where $h \in L^1(\Omega, \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^n$. Instead of x we consider $\Psi_M(|x - a|^2)$ with a cut-off function Ψ_M and $a \in \mathbb{R}^m$, which allows to use a technique well known from proofs of maximum principles (see [11, 4, 13]). The following lemma is fundamental in our argumentation.

Lemma 2.1 *Let* $\Omega \subset \mathbb{R}^n$, $f \in W^{1,2}(\Omega, \mathbb{R})$, $\gamma \in L^\infty(\Omega, \mathbb{R})$, $\gamma \geqq 0$ *in* Ω, *and* $g \in L^1(\Omega, \mathbb{R})$. *Furthermore, let* f *be a weak solution to* $\nabla \cdot (\gamma \nabla f) = g$ *in* Ω, *i.e.*

$$\int_\Omega (\nabla\varphi \cdot \gamma\nabla f + \varphi g)\, dw = 0 . \tag{2.1}$$

for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}) \cap L^\infty$.

Then,

$$\int_{\{w \in \Omega:\, f(w) > f_0\}} g(w)\, dw \leqq 0 \tag{2.2}$$

for all $f_0 \in \mathbb{R}$ *such that* $\max\{f - f_0, 0\} \in W_0^{1,2}(\Omega, \mathbb{R})$.

If $\operatorname{supp}\gamma \Subset \Omega$, (2.2) *holds for all* $f_0 \in \mathbb{R}$.

Proof. Assume $\varepsilon > 0$ and $\psi_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $\psi_{\varepsilon|(-\infty,\varepsilon)} \equiv 0$, $\psi_{\varepsilon|(2\varepsilon,\infty)} \equiv 1$, $0 \leqq \psi_\varepsilon \leqq 1$, and $0 \leqq \psi'_\varepsilon$. We set $\Phi \equiv 1$, or, if $\Omega \Supset \operatorname{supp}\gamma$ $(\supset \operatorname{supp} g)$, choose $\Phi \in C_0^1(\Omega, \mathbb{R})$ such that $\Phi_{|\operatorname{supp}\gamma} \equiv 1$. Noting $\Phi(w)\psi_\varepsilon(f(w) - f_0) \in W_0^{1,2}(\Omega, \mathbb{R}) \cap L^\infty$ and (2.1), we obtain

$$\begin{aligned} 0 &\geqq -\int_\Omega \Phi(w)\psi'_\varepsilon(f(w) - f_0)\gamma(w)|\nabla f(w)|^2\, dw \\ &= -\int_\Omega \nabla[\Phi(w)\psi_\varepsilon(f(w) - f_0)] \cdot \gamma(w)\nabla f(w)\, dw \\ &= \int_\Omega \Phi(w)\psi_\varepsilon(f(w) - f_0)g(w)\, dw . \end{aligned} \tag{2.3}$$

Letting $\varepsilon \to 0$, the assertion follows.

We define the following cut-off functions $\Psi_M \in C^{1,1}(\mathbb{R}_0^+, \mathbb{R})$, $M > 0$,

$$\Psi_M(t) = \begin{cases} t & \text{if } 0 \leqq t \leqq M \\ t(2 + \log M/t) - M & \text{if } M \leqq t \leqq eM \\ (e-1)M & \text{if } eM \leqq t \ . \end{cases} \tag{2.4}$$

They fulfil the relations $0 \leqq \Psi'_M \leqq 1$, $\Psi'_M(t) = 0$ if $t > eM$, $\Psi''_M(t) = -t^{-1}$ if $M < t < eM$, and $\Psi''_M(t) = 0$ elsewhere.

Lemma 2.2 *Let* $h \in L^1(B_r(w_0), \mathbb{R}^m)$, *and let* $x \in W^{1,2}(B_r(w_0), \mathbb{R}^m)$ *be a weak solution to* $\Delta x = h$ *in* $B_r(w_0) \subset \mathbb{R}^n$, *i.e.*

$$\int_{B_r(w_0)} (\nabla\Phi \cdot \nabla x + \Phi \cdot h)\, dw = 0 \tag{2.5}$$

for all $\Phi \in W_0^{1,2}(B_r(w_0), \mathbb{R}^m) \cap L^\infty$.

Moreover, let $\gamma \in C^{0,1}(\overline{B_r(w_0)})$, $\gamma \geqq 0$ *in* $\overline{B_r(w_0)}$, $a \in \mathbb{R}^m$, *and* $\eta > 0$ *such that*

$$\sup_{w \in \partial B_r(w_0)} |x(w) - a| \leqq \eta < \infty \ . \tag{2.6}$$

Then,

$$(2.7)\qquad \int_{\{w\in B_r(w_0):|x(w)-a|>\eta\}} \gamma|\nabla x|^2\,dw$$

$$\leqq -\limsup_{M\to\infty} \int_{\{w\in B_r(w_0):|x(w)-a|>\eta\}} \gamma\Psi'_M(|x-a|^2)(x-a)\cdot h\,dw$$

$$-\frac{1}{2}\int_{\{w\in B_r(w_0):|x(w)-a|>\eta\}} \nabla\gamma\cdot\nabla[|x-a|^2]\,dw\,.$$

If $\operatorname{supp}\gamma \Subset B_r(w_0)$, (2.7) *holds for all* $\eta\in\mathbb{R}$.

Proof. Using (2.5), we obtain by a simple calculation that

$$(2.8)\qquad \frac{1}{2}\int_{B_r(w_0)} \nabla\varphi\cdot\gamma\nabla[\Psi_M(|x-a|^2)]\,dw$$

$$= -\int_{B_r(w_0)} \varphi\{\gamma\Psi'_M(|x-a|^2)(x-a)\cdot h + \gamma\Psi'_M(|x-a|^2)|\nabla x|^2$$

$$+ 2\gamma\Psi''_M(|x-a|^2)(|(x-a)\cdot x_u|^2 + |(x-a)\cdot x_v|^2)$$

$$+ \Psi'_M(|x-a|^2)(\gamma_u(x-a)\cdot x_u + \gamma_v(x-a)\cdot x_v)\}\,dw$$

holds for all $\varphi\in W_0^{1,2}(B_r(w_0),\mathbb{R})\cap L^\infty$ and $M>0$.

Lemma 2.1 with $f=\Psi_M(|x-a|^2)$, $f_0=\eta^2$, $g=2\{\ldots\}\in L^1(B_r(w_0),\mathbb{R})$ now yields

$$(2.9)\qquad \int_{\{w\in B_r(w_0):|x(w)-a|>\eta\}} \gamma\Psi'_M(|x-a|^2)|\nabla x|^2\,dw$$

$$\leqq \int_{\{w\in B_r(w_0):|x(w)-a|>\eta\}} (-\gamma\Psi'_M(|x-a|^2)(x-a)\cdot h$$

$$- 2\gamma\Psi''_M(|x-a|^2)(|(x-a)\cdot x_u|^2 + |(x-a)\cdot x_v|^2)$$

$$-\tfrac{1}{2}\Psi'_M(|x-a|^2)\nabla\gamma\cdot\nabla[|x-a|^2])\,dw$$

for all $M>\eta^2$. Recalling the properties of Ψ_M, we obtain the assertion after letting $M\to\infty$.

3 Interior regularity

Our main result in this section concerns interior continuity of weak solutions to

$$(3.1)\qquad \Delta x = 2H(x)x_u\wedge x_v$$

in $\Omega\subset\mathbb{R}^2$, where H satisfies, for some $K>0$,

$$(3.2)\qquad H(x)=H_1(x)+H_2(x),\ \sup_{x\in\mathbb{R}^3}(|H_1(x)|+(1+|x|)|\nabla H_1(x)|)<\infty\ ,$$

$$(3.3)\qquad \sup_{x\in\mathbb{R}^3}(|H_2(x)|+|\nabla H_2(x)|)<\infty\,,\qquad \sup_{|x|\geqq K}|x\,H_2(x)|<1\,.$$

This result will be obtained in two steps by first proving *partial interior regularity*, i.e. continuity a.e. in Ω, and then excluding points where continuity could be false. This procedure was used by Heinz [7, 8], who proved interior regularity if $H(x) = H_1(x)$. After locally re-constructing a solution to (3.1) and thus demonstrating its regularity a.e. he showed that the condition which allowed this construction in fact holds throughout Ω. We note that Grüter [3] also gave a proof for partial interior regularity, if $H(x) = H_1(x)$, avoiding existence methods.

Our result of partial interior regularity reads

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^2$, $H_1, H_2 \in C^{0,1}(\mathbb{R}^3, \mathbb{R})$, and let $x \in W^{1,2}(\Omega, \mathbb{R}^3)$ be a weak solution to* (3.1) *in Ω where H satisfies* (3.2), (3.3) *for some $K > 0$.*

Then, x is continuous in all points

$$w_0 \in \Omega_0 := \left\{ w \in \Omega : \lim_{r \downarrow 0} \left(|\xi_w(r)|^2 \int_{B_r(w)} |\nabla x|^2 \, dw \right) = 0 \right\} \tag{3.4}$$

where $\xi_w(r)$ denotes the circleline meanvalue defined in (1.3).

Proof. Let $w_0 \in \Omega_0$ and assume w. l. o. g. that $\sup_{x \in \mathbb{R}^3} |x\, H_2(x)| \leqq 1 - 4\delta$ for some $\delta \in (0, \frac{1}{4})$. We define

$$\alpha_i := \sup_{x \in \mathbb{R}^3} (1 + |H_i(x)| + |\nabla H_i(x)|) < \infty \quad (i = 1, 2)\,, \tag{3.5}$$

$$\beta_1 := \sup_{x \in \mathbb{R}^3} |x|\,|\nabla H_1(x)| < \infty \tag{3.6}$$

and choose some $s \in (0, 1)$ with $(1 - 4\delta)(1 + 3s)^2 \leqq 1 - 3\delta$. We pick a $\mu \in (0, (\alpha_1 + \alpha_2)^{-1}\delta)$. Using Lemma 1.2 and $w_0 \in \Omega_0$, we find some $\rho > 0$ satisfying $B_\rho := B_\rho(w_0) \subset \Omega$,

$$\omega_\rho := \sup_{w \in \partial B_\rho} |x(w) - \xi_{w_0}(\rho)| \leqq 2^{-(s^{-1})}\mu\,, \tag{3.7}$$

and

$$32 c_0 (\alpha_1 + \beta_1 + (\alpha_1 + \alpha_2)|\xi_{w_0}(\rho)|) \left(\int_{B_\rho} |\nabla x|^2 \, dw \right)^{1/2} \leqq \delta\,, \tag{3.8}$$

where c_0 denotes the constant of Lemma 1.1.

The main tool is Lemma 2.2. Setting $r = \rho$, $h = 2H(x) x_u \wedge x_v$, $\gamma \equiv 1$, $a = \xi_{w_0}(\rho) =: \xi_\rho$, $\eta = \omega_\rho$, we obtain

$$\begin{aligned} &\int_{\{w \in B_\rho : |x(w) - \xi_\rho| > \omega_\rho\}} |\nabla x|^2 \, dw \\ &\leqq - \limsup_{M \to \infty} \int_{\{w \in B_\rho : |x(w) - \xi_\rho| > \omega_\rho\}} \Psi'_M(|x - \xi_\rho|^2)(x - \xi_\rho) \cdot (2H(x) x_u \wedge x_v) \, dw\,. \end{aligned} \tag{3.9}$$

Let $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $\Phi_{|(-\infty, \omega_\rho^s)} \equiv 0$, $\Phi_{|(\mu^s, \infty)} \equiv 1$, $0 \leqq \Phi \leqq 1$, and $|\Phi'| \leqq 3\mu^{-s}$, then we have $1 - \Phi(|x(w) - \xi_\rho|^s)^2 = 0$ for all $w \in B_\rho$ with $|x(w) - \xi_\rho| > \mu$. Making use of $1 = (1 - \Phi(|x(w) - \xi_\rho|^s)^2) + \Phi(|x(w) - \xi_\rho|^s)^2$ and the special structure of the triple scalar product, we estimate the r.h.s. of (3.9) for $M > \mu^2$:

$$(3.10) \qquad - \int\limits_{\{w \in B_\rho : |x(w) - \xi_\rho| > \omega_\rho\}} 2H(x)\Psi'_M(|x - \xi_\rho|^2)(x - \xi_\rho, x_u, x_v)\, dw$$

$$\leqq \int\limits_{\{w \in B_\rho :\, \omega_\rho < |x(w) - \xi_\rho| < \mu\}} \mu \, \|H\|_\infty |\nabla x|^2 \, dw$$

$$- \int\limits_{B_\rho} 2H(x)\Psi'_M(|x - \xi_\rho|^2)(x - \xi_\rho) \cdot$$

$$\cdot \frac{\partial}{\partial u}[\Phi(|x - \xi_\rho|^s)(x - \xi_\rho)] \wedge \frac{\partial}{\partial v}[\Phi(|x - \xi_\rho|^s)(x - \xi_\rho)]\, dw\,.$$

In this way we obtain from (3.9), (3.10), (3.5)

$$(3.11) \qquad \int\limits_{\{w \in B_\rho : |x(w) - \xi_\rho| > \omega_\rho\}} (1 - \mu(\alpha_1 + \alpha_2))|\nabla x|^2 \, dw \leqq F_1 + F_2$$

where $(i = 1, 2)$

$$(3.12) \qquad F_i = \limsup_{M \to \infty} \left| \int\limits_{B_\rho} 2H_i(x)\Psi'_M(|x - \xi_\rho|^2)(x - \xi_\rho) \cdot \right.$$

$$\left. \cdot \frac{\partial}{\partial u}[\Phi(|x - \xi_\rho|^s)(x - \xi_\rho)] \wedge \frac{\partial}{\partial v}[\Phi(|x - \xi_\rho|^s)(x - \xi_\rho)]\, dw \right| .$$

F_1 and F_2 are now estimated by Lemma 1.1 and in an elementary way:
Recalling the properties of H_1, H_2, and Ψ_M, we obtain

(3.13)

$$F_1 \leqq 2c_0 \limsup_{M \to \infty} \left(\int\limits_{B_\rho} |\nabla[H_1(x)\Psi'_M(|x - \xi_\rho|^2)(x - \xi_\rho)]|^2 \, dw \right)^{1/2}$$

$$\int\limits_{\{w \in B_\rho : |x(w) - \xi_\rho| > \omega_\rho\}} |\nabla[\Phi(|x - \xi_\rho|^s)(x - \xi_\rho)]|^2 \, dw$$

$$\leqq 2c_0(\alpha_1 + \beta_1 + \alpha_1|\xi_\rho|) \left(\int\limits_{B_\rho} |\nabla x|^2 \, dw \right)^{1/2}$$

$$\int\limits_{\{w \in B_\rho : |x(w) - \xi_\rho| > \omega_\rho\}} \left(1 + \sup_{\mathbb{R}} |\Phi'|\, s\mu^s \right)^2 |\nabla x|^2 \, dw$$

$$\leqq 32c_0(\alpha_1 + \beta_1 + \alpha_1|\xi_\rho|) \left(\int\limits_{B_\rho} |\nabla x|^2 \, dw \right)^{1/2} \int\limits_{\{w \in B_\rho : |x(w) - \xi_\rho| > \omega_\rho\}} |\nabla x|^2 \, dw$$

and

(3.14)

$$F_2 \leqq \limsup_{M\to\infty} \left| \int_{B_\rho} 2H_2(x)\Psi'_M(|x-\xi_\rho|^2)x \cdot \right.$$
$$\left. \cdot \frac{\partial}{\partial u}[\Phi(|x-\xi_\rho|^s)(x-\xi_\rho)] \wedge \frac{\partial}{\partial v}[\Phi(|x-\xi_\rho|^s)(x-\xi_\rho)]\,dw \right|$$
$$+ \limsup_{M\to\infty} \left| \int_{B_\rho} 2H_2(x)\Psi'_M(|x-\xi_\rho|^2)\xi_\rho \cdot \right.$$
$$\left. \cdot \frac{\partial}{\partial u}[\Phi(|x-\xi_\rho|^s)(x-\xi_\rho)] \wedge \frac{\partial}{\partial v}[\Phi(|x-\xi_\rho|^s)(x-\xi_\rho)]\,dw \right|$$
$$\leqq \left((1-4\delta) + 2c_0 \limsup_{M\to\infty} \left(\int_{B_\rho} |\nabla[H_2(x)\Psi'_M(|x-\xi_\rho|^2)\xi_\rho]|^2\,dw \right)^{1/2} \right)$$
$$\int_{\{w\in B_\rho: |x(w)-\xi_\rho|>\omega_\rho\}} (1 + \sup_{\mathbb{R}} |\Phi'|\, s\mu^s)^2 |\nabla x|^2\,dw$$
$$\leqq \left((1-3\delta) + 32 c_0 \alpha_2 |\xi_\rho| \left(\int_{B_\rho} |\nabla x|^2\,dw \right)^{1/2} \right) \int_{\{w\in B_\rho: |x(w)-\xi_\rho|>\omega_\rho\}} |\nabla x|^2\,dw\,.$$

Combining (3.11), (3.13), (3.14), (3.8), we deduce

(3.15)
$$\int_{\{w\in B_\rho: |x(w)-\xi_\rho|>\omega_\rho\}} |\nabla x|^2\,dw \leqq 0\,.$$

Therefore, we finally have

(3.16)
$$\sup_{w\in B_\rho} |x(w) - \xi_{w_0}(\rho)| \leqq \omega_\rho \leqq 2^{-(s^{-1})}\mu \leqq \frac{\mu}{2}$$

and

(3.17)
$$\operatorname{osc}_{B_\rho} x \leqq \mu\,.$$

In fact, the condition which guarantees continuity (see (3.4)) holds at least a.e. in Ω. This can be seen from $|\xi_{w_0}(r)| \leqq c_1(r^{-1}\|x\|_{2;B_r(w_0)} + \|\nabla x\|_{2;B_r(w_0)})$ (see [7, 12]), by applying a classical theorem of Lebesgue. Therefore it seems appropriate to speak of *partial interior regularity*.

To prove that condition (3.4) holds for all $w_0 \in \Omega$, we take the result of Lemma 1.3 into account. Since $\xi_w(r) = o(\log^{\frac{1}{2}} r^{-1})$, $r \downarrow 0$, it can be seen easily that the boundedness of the weighted Dirichlet integral which will be established in the following theorem ensures continuity of weak solutions to (3.1), provided H satisfies the stated assumptions.

Theorem 3.2 *Under the assumptions of Theorem* 3.1

(3.18)
$$\int_{B_\rho(w_0)} \log|w-w_0|^{-1} |\nabla x(w)|^2\,dw < \infty$$

holds for all $B_\rho(w_0) \subset \Omega$.

Hence, $\Omega_0 = \Omega$ *and* $x \in C^0(\Omega, \mathbb{R}^3)$ *(see* Theorem 3.1).

Combining this with a general theorem of Tomi [15], we obtain the final statement of *interior regularity* of solutions to (3.1), namely

Theorem 3.3 *Under the assumptions of Theorem* 3.1 $x \in C^{2,\mu}(\Omega, \mathbb{R}^3)$ *holds for all* $\mu \in (0, 1)$.

Proof of Theorem 3.2 By Lemma 1.3 and Theorem 3.1, we only have to establish (3.18). W. l. o. g. we set $w_0 = 0$ and $\rho \leqq \frac{1}{2}$. Let $r = |w|$ and $B_\rho = B_\rho(0)$. From the proof of Theorem 3.1 we take over the quantities $\delta, \alpha_1, \alpha_2, \beta_1$. We define

$$\log_t s = \begin{cases} \log s & \text{if } 0 < s < t \\ \log t & \text{if } t \leqq s \end{cases} \tag{3.19}$$

for $t > 4$ and choose some $\Phi_\rho \in C_0^1((-\rho, \rho))$ satisfying $\Phi_{\rho|(0,\frac{\rho}{2})} \equiv 1$. Finally we set

$$p = p_t = p_t(r) = p_t(|w|) = \Phi_\rho(|w|)^2 \log_t |w|^{-1} \tag{3.20}$$

$$q = q_t = q_t(r) = q_t(|w|) = \Phi_\rho(|w|) \log_t^{\frac{1}{2}} |w|^{-1} .$$

Lemma 2.2 with $h = 2H(x) x_u \wedge x_v$, $\gamma = p(|\cdot|) \in C_0^{0,1}(B_\rho)$, $a = 0$, $\eta = -1$ yields

$$\int_{B_\rho} p |\nabla x|^2 \, dw \leqq - \limsup_{M \to \infty} \int_{B_\rho} 2p(H_1(x) + H_2(x)) \Psi_M'(|x|^2)(x, x_u, x_v) \, dw \tag{3.21}$$

$$- \frac{1}{2} \int_{B_\rho} \nabla p \cdot \nabla [|x|^2] \, dw .$$

For brevity's sake we use

(3.22)

$$R_\rho := \int_{B_\rho \setminus B_{\rho/2}} (|p'(r)| + |q'(r)|^2)(|\nabla x|^2 + 2|x|^2) \, dw + \int_{B_{\rho/2}} |\nabla x|^2 \, dw + \frac{1}{\rho} \int_{\partial B_{\rho/2}} |x|^2 \, ds .$$

Combining (3.21), $\sup_{x \in \mathbb{R}^3} |x \, H_2(x)| \leqq 1 - 4\delta$,

$$- \frac{1}{2} \int_{B_{\rho/2}} \nabla p \cdot \nabla [|x|^2] \, dw \leqq \frac{1}{\rho} \int_{\partial B_{\rho/2}} |x|^2 \, ds , \tag{3.23}$$

and (3.22), we deduce

$$4\delta \int_{B_\rho} p |\nabla x|^2 \, dw \leqq - \limsup_{M \to \infty} \int_{B_\rho} 2pH_1(x) \Psi_M'(|x|^2)(x, x_u, x_v) \, dw + R_\rho . \tag{3.24}$$

We have to estimate the first term on the r.h.s. of (3.24), which can be done using the following definitions and transformations taken from Heinz [8]:

$$\eta = \eta(|w|) = - \int_{|w|}^{\rho} q(\tau) \xi'(\tau) \, d\tau \tag{3.25}$$

$$h = h(w) = \eta + qy$$

where $\xi = \xi_{w_0}$, $y = y_{w_0}$, and $q = q_t$ denote the quantities already defined in (1.3), (1.4), and (3.20). A.e. in B_ρ we have

$$(3.26)\qquad p(r)(x, x_u, x_v) = \frac{p(r)}{r}(x, x_r, x_\varphi) = \frac{1}{r}(x, qx_r, qx_\varphi)$$

$$= \frac{1}{r}(x, q\xi_r + qy_r, qy_\varphi) = \frac{1}{r}(x, \eta_r + qy_r, qy_\varphi)$$

$$= \frac{1}{r}(x, h_r - q_r y, h_\varphi)$$

$$= \frac{1}{r}(x, h_r, h_\varphi) - \frac{p'(r)}{2r}(x, y, y_\varphi)$$

$$= (x, h_u, h_v) - \frac{p'(r)}{2r}(\xi, y, y_\varphi)\,,$$

and thus

$$(3.27)\qquad -\limsup_{M\to\infty}\int_{B_\rho} 2p\,H_1(x)\Psi'_M(|x|^2)(x, x_u, x_v)\,dw \leqq G_1 + G_2$$

where

$$(3.28)\qquad G_1 = \limsup_{M\to\infty}\left|\int_{B_\rho} 2H_1(x)\Psi'_M(|x|^2)(x, h_u, h_v)\,dw\right|$$

and

$$(3.29)\qquad G_2 = \limsup_{M\to\infty}\left|\int_{B_\rho}\frac{p'(r)}{r}H_1(x)\Psi'_M(|x|^2)(\xi, y, y_\varphi)\,dw\right|.$$

The estimate of G_1 is the same as in [8] but will be recalled for the reader's convenience. First, we consider $|\nabla h|$ and find, noting $h_r = qx_r + q_r y$ and $h_\varphi = qy_\varphi$,

$$(3.30)\qquad |\nabla h|^2 = |qx_r + q_r y|^2 + \frac{q(r)^2}{r^2}|y_\varphi|^2 = q^2|\nabla x|^2 + p'x_r\cdot y + q'^2|y|^2\,.$$

Then, using $|p'(r)| \leqq r^{-1}$ and $|q'(r)| = (2r\log_t^{\frac{1}{2}} r^{-1})^{-1} \leqq (2r)^{-1}$ if $0 < r \leqq \frac{1}{2}\rho \leqq \frac{1}{4}$, and (1.5), (1.7), we obtain

$$(3.31)\qquad \int_{B_\rho}|\nabla h|^2\,dw = \int_0^\rho\int_0^{2\pi}(p(r)|\nabla x|^2 + p'(r)y_r\cdot y + q'(r)^2|y|^2)\,d\varphi\,r\,dr$$

$$\leqq \int_0^\rho\int_0^{2\pi} p(r)|\nabla x|^2\,d\varphi\,r\,dr + \int_0^{\rho/2}\int_0^{2\pi}\left(|y_r|^2 + \frac{|y|^2}{2r^2}\right)d\varphi\,r\,dr$$

$$+ \int_{\rho/2}^{\rho}(|p'(r)| + |q'(r)|^2)\int_0^{2\pi}(|y_r||y| + |y|^2)\,d\varphi\,r\,dr$$

$$\leqq \int_{B_\rho} p(r)|\nabla x|^2\,dw + R_\rho\,.$$

Hence, by Lemma 1.1 and the properties of H_1 and Ψ_M, we have

$$(3.32)\qquad G_1 \leqq \limsup_{M\to\infty} 2c_0\left(\int_{B_\rho} |\nabla[H_1(x)\Psi_M'(|x|^2)x]|^2\,dw\right)^{1/2} \int_{B_\rho} |\nabla h|^2\,dw$$
$$\leqq 2c_0(\alpha_1+\beta_1)\left(\int_{B_\rho}|\nabla x|^2\,dw\right)^{1/2}\left(\int_{B_\rho} p(r)|\nabla x|^2\,dw + R_\rho\right).$$

Next, treating G_2 we use Lemma 1.3 and obtain

$$(3.33)\qquad \alpha_1|\xi(r)| \leqq \delta \log^{\frac{1}{2}} r^{-1} \leqq \frac{1}{r}$$

for $0 < r \leqq r_0$ with some $r_0 \leqq \frac{1}{2}$. Therefore, if $0 < r < \frac{1}{2}\rho \leqq \frac{1}{2}r_0$, we have

$$(3.34)\qquad \alpha_1|p'(r)\xi(r)| \leqq \delta\,\frac{p(r)^{1/2}}{r},$$

and from (3.29), (3.33), (3.34), and (1.7) we deduce

$$(3.35)\qquad G_2 \leqq \delta \int_0^{\rho/2}\int_0^{2\pi} \frac{|y|}{r}\,\frac{p^{1/2}(r)|y_\varphi|}{r}\,d\varphi\, r\,dr + \int_{\rho/2}^{\rho}\int_0^{2\pi} |p'(r)|\,\frac{|y|}{r}\,\frac{|y_\varphi|}{r}\,d\varphi\, r\,dr$$
$$\leqq \delta \int_{B_{\rho/2}} p(r)|\nabla x|^2\,dw + R_\rho\,.$$

Now, we choose $\rho_0 \in (0, r_0)$ such that

$$(3.36)\qquad c_0(\alpha_1+\beta_1)\left(\int_{B_{\rho_0}} |\nabla x|^2\,dw\right)^{1/2} \leqq \delta$$

where again c_0 denotes the constant of Lemma 1.1. We combine (3.24), (3.27), (3.32), (3.35), (3.36) and deduce

$$(3.37)\qquad \int_{B_{\rho_0}} p(|w|)|\nabla x(w)|^2\,dw \leqq 3R_{\rho_0}\delta^{-1}$$

where

$$(3.38)\qquad p(|w|) = p_t(|w|) = \Phi_{\rho_0}(|w|)^2 \log_t |w|^{-1}\,.$$

Finally, letting $t\to\infty$ Fatou's Lemma yields the desired assertion.

4 Boundary regularity

In the last section of this paper we consider weak solutions to the *Dirichlet problem*

$$(4.1)\qquad \mathrm{DP}\begin{cases} \Delta x = 2H(x)x_u \wedge x_v & \text{in } \Omega \\ x = z & \text{on } \partial\Omega \end{cases}$$

where $z \in W^{1,2}(\Omega, \mathbb{R}^3)$.

For H satisfying (0.5), Heinz's result [7] concerning partial interior regularity can easily be transferred to the boundary problem, if $\operatorname{osc} z < \|H\|_\infty^{-1}$ locally holds near $\partial\Omega$ (cf. [17, Theorem 5.3]). In case of higher regularity for z, i.e. $z \in C^{0,\mu}$, $C^{1,\mu}$, and C^2, $C^{2,\mu}$, results of Widman [18, 19] and Heinz [5] (in connection with Schauder estimates) then ensure corresponding regularity for x.

Here, we shall give a proof for continuity or boundedness of solutions x to (4.1) under the corresponding assumption on z, and those on H formulated in Theorem 3.1. According to an idea of Courant [2, Sec. I.5.4], we shall control the circleline meanvalue $(2\pi r)^{-1}\int_{\partial B_r(w_0)} x\,ds$ using an arc crossing $\partial B_r(w_0)$ and $\partial\Omega$, if w_0 is near $\partial\Omega$. In this way we do not need any information about the oscillation of z. Our result on boundary regularity is formulated as follows:

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^2$, $z \in W^{1,2}(\Omega, \mathbb{R}^3)$, and let $x \in W^{1,2}(\Omega, \mathbb{R}^3)$ be a weak solution to the Dirichlet problem (4.1)/(0.7) where H satisfies*

$$(4.2) \qquad H(x) = H_1(x) + H_2(x), \ \sup_{x\in\mathbb{R}^3} (|H_1(x)| + (1 + |x|)|\nabla H_1(x)|) < \infty\ ,$$

$$(4.3) \qquad \sup_{x\in\mathbb{R}^3} |H_2(x)| + |\nabla H_2(x)| < \infty\,, \qquad \sup_{|x|\geqq K} |x\, H_2(x)| < 1$$

for some H_1, $H_2 \in C^{0,1}(\mathbb{R}^3, \mathbb{R})$, $K > 0$.

Moreover, let $w_a \in \partial\Omega$, $z_a \in \mathbb{R}^3$, and $r_a, \sigma_a > 0$ such that

$$(4.4) \qquad \|z - z_a\|_{\infty;\, B_{r_a}(w_a)\cap\Omega} \leqq \sigma_a$$

and $\partial\Omega \cap \partial B_{r_b}(w_b) \neq \emptyset$ holds for all $w_b \in \partial\Omega \cap B_{r_a}(w_a)$ and all $r_b \in (0, r_a)$. Then, for all $\varepsilon > 0$ there exists some $r_\varepsilon > 0$ such that

$$(4.5) \qquad \|x - z_a\|_{\infty;\, B_{r_\varepsilon}(w_a)\cap\Omega} \leqq \sigma_a + \varepsilon\ .$$

In particular, if Ω is bounded by a finite number of Jordan curves, $x \in L^\infty(\Omega, \mathbb{R}^3)$ follows from $z \in W^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty$, and $x \in C^0(\bar\Omega, \mathbb{R}^3)$ with $x_{|\partial\Omega} = z_{|\partial\Omega}$ is guaranteed by $z \in W^{1,2}(\Omega, \mathbb{R}^3) \cap C^0(\bar\Omega)$.

Proof. From the proof of Theorem 3.1 we take over the quantities δ, α_1, α_2, β_1, s and assume $\varepsilon \in (0, (\alpha_1 + \alpha_2)^{-1}\delta)$. We choose some $r_\varepsilon \in (0, \frac{1}{2} r_a)$ such that

$$(4.6) \qquad \frac{\pi}{\log\frac{4}{3}} \int_{B_{2r_\varepsilon}(w_a)\cap\Omega} (|\nabla x|^2 + |\nabla(x - z)|^2)\,dw \leqq \left(\frac{\varepsilon}{4}\right)^2$$

and

$$(4.7) \quad 32 c_0(\alpha_1 + \beta_1 + (\alpha_1 + \alpha_2)(|z_a| + \sigma_a + \varepsilon)) \left(\int_{B_{2r_\varepsilon}(w_a)\cap\Omega} |\nabla x|^2\,dw\right)^{1/2} \leqq \delta\ .$$

Let $\{(x - z)_n\}_{n\in\mathbb{N}} \subset C_0^1(\Omega, \mathbb{R}^3)$ be a sequence of functions converging to $x - z$ in $W_0^{1,2}(\Omega, \mathbb{R}^3)$. We define $(x - z)_n$, $n \in \mathbb{N}$, and $x - z$ as vanishing in $\mathbb{R}^2 \backslash \Omega$.

For any $w_2 \in B_{r_\varepsilon}(w_a) \cap \Omega$ we choose $w_1 \in B_{2r_\varepsilon}(w_a) \cap \partial\Omega$ such that $\operatorname{dist}(w_2, \partial\Omega) = |w_2 - w_1|$. Using Lemma 1.2 and (4.6), we find some $n_0 \in \mathbb{N}$ and, for all $n \in \mathbb{N}$, $n > n_0$, some set $M^n_{x-z} \subset (\frac{1}{2}|w_2 - w_1|, |w_2 - w_1|)$ such that $\operatorname{meas} M^n_{x-z} \geqq \frac{1}{4}|w_2 - w_1|$ and

$$(4.8) \qquad \sup_{w', w'' \in \partial B_r(w_1)} |(x - z)_n(w') - (x - z)_n(w'')| \leqq \frac{3\varepsilon}{8}$$

for all $r \in M^n_{x-z}$. In $(\frac{1}{2}|w_2 - w_1|, |w_2 - w_1|)$, we again find some set M'_x such that meas $M'_x \geqq \frac{1}{4}|w_2 - w_1|$ and

$$\sup_{w', w'' \in \partial B_{r'}(w_2)} |x(w') - x(w'')| \leqq \frac{\varepsilon}{4} \tag{4.9}$$

for all $r' \in M'_x$.

Since we have convergence a.e. in Ω for some subsequence of $\{(x-z)_n\}_{n \in \mathbb{N}}$, there exist $n' \in \mathbb{N}$, $n' > n_0$, $\rho_1 \in M^{n'}_{x-z}$, $\rho_2 \in M'_x$ such that $|(x-z)_{n'}(w_3) - (x-z)(w_3)| \leqq \frac{1}{8}\varepsilon$ holds for some $w_3 \in \partial B_{\rho_1}(w_1) \cap \partial B_{\rho_2}(w_2)$ and x, z are absolutely continuous on $\partial B_{\rho_2}(w_2)$. Hence, we obtain

(4.10)

$$\begin{aligned} |\xi_{w_2}(\rho_2) - z_a| \leqq & \sup_{w \in \partial B_{\rho_2}(w_2)} |x(w) - x(w_3)| + |(x-z)(w_3) - (x-z)_{n'}(w_3)| \\ & + |(x-z)_{n'}(w_3) - (x-z)_{n'}(w_4)| + |(x-z)_{n'}(w_4)| + |z(w_3) - z_a| \\ & \leqq \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{3\varepsilon}{8} + 0 + \sigma_a \end{aligned}$$

with some $w_4 \in \partial\Omega \cap \partial B_{\rho_1}(w_1)$, especially,

$$|\xi_{w_2}(\rho_2)| < |z_a| + \sigma_a + \varepsilon\,. \tag{4.11}$$

In contrast to the proofs of interior regularity, we obviously succeed in deriving an estimate of the circleline meanvalue $\xi_{w_2}(\rho_2)$.

What follows now is routine. In the proof of Theorem 3.1 we substitute w_0, ρ, μ by w_2, ρ_2, $\frac{1}{2}\varepsilon$, resp., moreover (3.7) by

$$\omega_{\rho_2} := \sup_{w \in \partial B_{\rho_2}(w_2)} |x(w) - \xi_{w_2}(\rho_2)| \leqq 2^{-(s^{-1})} \frac{\varepsilon}{2} \tag{4.12}$$

and (3.8) by (4.7). The result is taken from (3.16), namely

$$\sup_{w \in B_{\rho_2}(w_2)} |x(w) - \xi_{w_2}(\rho_2)| \leqq \frac{\varepsilon}{4}\,. \tag{4.13}$$

Finally, combining (4.10), (4.13), we obtain

$$\sup_{w \in B_{\rho_2}(w_2)} |x(w) - z_a| \leqq \sigma_a + \varepsilon\,, \tag{4.14}$$

which completes the proof, since $w_2 \in B_{r_\varepsilon}(w_a) \cap \Omega$ has been chosen arbitrarilly.

Remarks. It remains an open question whether weaker assumptions on H guarantee the same regularity proved in this paper, of course without more informations about a given solution. Moreover, under the stated assumptions on H it is not known whether there is a maximum estimate for solutions to (0.7) depending on their $W^{1,2}$ norms and boundary data, similar to that given by Brezis and Coron [1.A1] in the case of $H \equiv$ const.

Acknowledgment. I wish to thank my teacher Professor Dr. E. Heinz for proposing these problems, moreover that I could learn so much from him. Thanks go also to Professor Dr. G. Hellwig for his support.

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