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On the L^2 cohomology of complex spaces

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Introduction

In the theory of analytic spaces, the topological invariants are often hard to compute because of the presence of the singular points. One of the theoretical obstructions is that we do not have the Poincaré duality on the spaces with singularities in the naive sense. In [G-M-1], Goresky and MacPherson introduced the notion of intersection (co)homology to study the intersections of cycles with prescribed behavior along the singularities of a given space, and established a natural generalization of the Poincaré duality for such a modified notion of (co)homology groups. As in the case of manifolds, analytic expressions for the intersection cohomology have been looked for, since they will naturally lead us to the extension of the theory of elliptic operators on manifolds, the Hodge theory for instance. A candidate was proposed in [C-G-M] after several works of Cheeger on the L^2 cohomology groups (cf. [C-1, 2]). It is the following.

Conjecture. *Let X be a complex analytic subvariety of $\mathbb{CP}^N$. Then the L^2 de Rham cohomology group of X is canonically isomorphic to its intersection cohomology group.*

Since there exists a sheaf theoretic characterization of the intersection cohomology, the conjecture is in fact equivalent to the vanishment of certain L^2 cohomology groups which is a purely local question. The purpose of the present article is to prove the conjecture completely in the affirmative. Since we restrict ourselves to the above mentioned local problem, our result is stated as follows.

Theorem. *Let X be a complex space of pure dimension n. Then the L^2_{loc} de Rham cohomology group of X is canonically isomorphic to its intersection cohomology group.*

In case X admits only isolated singularities, the result has been verified in a series of the author's papers [O-1, 2, 3], where the corresponding extension of the

Hodge theory was also given. The proof of our result here will be based on the methods developed there. They are applicable without any significant changes once we know the local existence of a Lipschitz continuous stratification of X, which is due to Mostowski [M] (see also [P]).

1 Preliminaries

For the reader's convenience we shall present a basic result of Donnelly and Fefferman [D-F] and its technical improvements given in [O-3]. The details will be often omitted. Let N be a complex manifold equipped with a Hermitian metric, say ds_N^2. The (complex) Hilbert space of square integrable r-forms on N will be denoted by $L^r(N)$ and the set of compactly supported C^∞ r-forms will be denoted by $C_0^r(N)$ which is canonically identified with a dense subset of $L^r(N)$ with respect to the L^2 norm. The exterior derivative acting on the set of currents on N will be denoted by d and $d^r_{\max}$ will denote the maximal closed extension of $d|C_0^r(N)$ as a linear operator from $L^r(N)$ to $L^{r+1}(N)$. The r-th L^2 de Rham cohomology group of (N, ds_N^2) is defined as the factor space $\operatorname{Ker} d^r_{\max}/\operatorname{Im} d^{r-1}_{\max}$ which we shall denote by $H^r_{(2)}(N)$ or $H^r_{(2)}(N, ds_N^2)$.

Theorem 1.1 (cf. [D-F]) *Let (N, ds_N^2) be a connected complete Kähler manifold of dimension n. Suppose that N admits a real-valued C^∞ function with bounded gradient such that its complex Hessian coincides with ds_N^2. Then $H^r_{(2)}(N) = 0$ for $r \neq n$.*

Let $\bar\partial$(resp. ∂) be the (0, 1)-component (resp. the (1, 0)-component) of d and ϑ(resp. $\bar\vartheta$) the formal adjoint of $\bar\partial$ (resp. ∂). The formal adjoint of d will be denoted by δ. Then the ingredient of Theorem 1.1 is contained in the following.

Proposition 1.2 (cf. [D-F]) *Let (N, ds_N^2) be as in Theorem* 1.1. *Then there exists a constant C such that*

$$\|u\| \leqq C(\|\bar\partial u\| + \|\vartheta u\|) \leqq C(\|du\| + \|\delta u\|)$$

for all $u \in C_0^r(N)$ with $r \neq n$. Here $\|\ \|$ denotes the L^2 norm.

We are going to state a generalization of Theorem 1.1 for certain modified L^2 norms. Given a real valued C^∞ function φ on N we put

$$\|u\|_\varphi^2 = \int_N e^{-\varphi}|u|^2\, dV \quad \text{for } u \in C_0^r(N)\,,$$

where $|u|$ denotes the length of u and dV the volume form of N. We shall call $\|u\|_\varphi$ the L^2 norm of u with respect to ds_N^2 and φ. Subscripts φ will be omitted if $\varphi = 0$. Letting $L^r_\varphi(N)(= L^r_\varphi(N, ds^2))$ be the set of locally square integrable r-forms u satisfying $\|u\|_\varphi < \infty$, we define the maximal closed extension $d^r_{\max,\varphi}$ of $d|C_0^r(N)$ as an operator from $L^r_\varphi(N)$ to $L^{r+1}_\varphi(N)$, and set $H^r_{(2),\varphi}(N)(= H^r_{(2),\varphi}(N, ds^2))$ $:= \operatorname{Ker} d^r_{\max,\varphi}/\operatorname{Im} d^{r-1}_{\max,\varphi}$. Let us denote by $(\,,\,)_\varphi$ the inner product associated to the (weighted) L^2 norm $\|\ \|_\varphi$. Then we denote by ϑ_φ the formal adjoint of $\bar\partial$ with respect to $(\,,\,)_\varphi$ and put $\Box_\varphi = [\bar\partial, \vartheta_\varphi]_{gr}$, where $[S, T]_{gr}$ will stand for $ST - (-1)^{\deg S \deg T} TS$ in general. Given a differential form η, $e(\eta)$ will denote the exterior multiplication from the left hand side by η. We put $\partial_\varphi = \partial - e(\partial\varphi)$ and $\bar\Box_\varphi = [\partial_\varphi, \bar\vartheta]_{gr}$. Since ∂_φ is

the formal adjoint of $\bar{\vartheta}$ with respect to $(,)_\varphi$, $\bar{\Box}_\varphi$ is a formally self-adjoint positive operator as well as $\Box_\varphi$. Multiplication by the fundamental form of ds_N^2 will be simply denoted by L and the adjoint of L will be denoted by Λ. Note that $[L, \Lambda]u = (r - n)u$ for any r-form u.

Proposition 1.3 *Let η be a C^∞ $(0, 1)$-form on N and let c_0 be a positive number satisfying $2c_0 > \sup|\eta|^2$. If ds_N^2 is a Kähler metric,*

$$((\Box_\varphi + \bar{\Box}_\varphi)u, u)_\varphi + c_0 \|u\|_\varphi^2 \geqq ([e(\sqrt{-1}\partial\eta), \Lambda]u, u)_\varphi \tag{1}$$

for any compactly supported C^∞ form u.

Proof. The graded Jacobi identity for $\bar{\partial}$, $e(\eta)$ and Λ yields

$$[\bar{\partial}, e(\eta)^*]_{gr} + [\bar{\vartheta}, e(\bar{\eta})]_{gr} = [\sqrt{-1}e(\partial\eta), \Lambda], \tag{2}$$

where $e(\eta)^*$ denotes the adjoint of $e(\eta)$. By the Cauchy-Schwartz inequality

$$\begin{aligned} |([\bar{\partial}, e(\eta)^*]_{gr}u, u)_\varphi| &\leqq \sup|\eta|\sqrt{(\|\bar{\partial}u\|_\varphi^2 + \|\vartheta_\varphi u\|_\varphi^2)}\,\|u\|_\varphi \\ &\leqq \|\bar{\partial}u\|_\varphi^2 + \|\vartheta_\varphi u\|_\varphi^2 + \tfrac{1}{4}\sup|\eta|^2\|u\|_\varphi^2 \\ &= (\Box_\varphi u, u)_\varphi + \tfrac{1}{4}\sup|\eta|^2\|u\|_\varphi^2 . \end{aligned} \tag{3}$$

Similarly we obtain

$$|([\bar{\vartheta}, e(\bar{\eta})]_{gr}u, u)_\varphi| \leqq (\bar{\Box}_\varphi u, u)_\varphi + \tfrac{1}{4}\sup|\eta|^2\|u\|_\varphi^2 . \tag{4}$$

Combining (2) through (4) we obtain the desired inequality. □

Combining Proposition 1.3 with the identity

$$\Box_\varphi - \bar{\Box}_\varphi = [e(\sqrt{-1}\partial\bar{\partial}\varphi), \Lambda] \tag{5}$$

which is also valid under the Kählerness assumption, we obtain

Proposition 1.4 *Under the situation of Proposition* 1.3, *suppose moreover that* $e(\sqrt{-1}(\partial\bar{\partial}\varphi + \partial\eta)) = 2L$ and $4\sup|\eta|^2 < 1$. Then

$$(\Box_{-\varphi}u, u)_{-\varphi} \geqq \|u\|_{-\varphi}^2 \quad \text{if } u \in C_0^r(N) \text{ and } r < n \tag{6}$$

and

$$(\Box_\varphi u, u)_\varphi \geqq \|u\|_\varphi^2 \quad \text{if } u \in C_0^r(N) \text{ and } r > n . \tag{7}$$

In order to show the vanishing property of $H^n_{(2)}$ the inequality (6) will be crucial. First we mention an immediate consequence of Proposition 1.4.

Proposition 1.5 *Under the situation of Proposition* 1.4, *suppose moreover that ds_N^2 is a complete metric. Then, for any $\bar{\partial}$-closed (p, q)-form v satisfying $\|v\|_{-\varphi} < \infty$ and $p + q < n$ (resp. $\|v\|_\varphi < \infty$ and $p + q > n$), there exists a $(p, q-1)$-form u satisfying $\bar{\partial}u = v$ and $\|u\|_{-\varphi} \leqq \|v\|_{-\varphi}$ (resp. $\|u\|_\varphi \leqq \|v\|_\varphi$). Furthermore, if $\partial\varphi$ and $\partial\bar{\partial}\varphi$ are both bounded, every d-closed r-form g satisfying $\|g\|_{-\varphi} < \infty$ and $r < n$ (resp. $\|g\|_\varphi < \infty$ and $r > n$) can be expressed as $g = df$ for some $(r-1)$-form f satisfying*

$\|f\|_{-\varphi} \leqq C_1 \|g\|_{-\varphi}$ (*resp.* $\|f\|_{\varphi} \leqq C_1 \|g\|_{\varphi}$). *Here* C_1 *is a number depending only on* $\sup(|\partial\varphi| + |\partial\bar{\partial}\varphi|)$.

Proof. Since we have already à priori estimates (6) and (7), the first part of the statement is straightforward. As for the latter, one may decompose g into the (p, q)-components and apply the first part successively to get the desired solution f, since the canonical solutions of the $\bar{\partial}$-equations belong to the domain of ∂ by the restriction we have imposed on φ. □

That the estimates (6) and (7) remain valid under suitable modifications of metrics and weights makes it possible to apply a perturbation technique.

Proposition 1.6 (cf. [O-3, § 2, Theorem 8]) *Let the situation be as in Proposition* 1.5. *Suppose moreover that* $\eta = \bar{\partial}(-\log(-\omega) + \psi)$ *for some* C^{∞} *plurisubharmonic functions* ω (< 0) *and* ψ. *Let* $r \leqq n$ *and let* v *be a* d*-closed square integrable* r*-form with respect to* $\partial\eta$. *Then there exists a square integrable* $(r-1)$*-form* u *with respect to* $\partial\eta$ *satisfying* $du = v$ *if and only if there exist* $(r-1)$*-forms* u_ε *satisfying* $du_\varepsilon = v$ *for all* $\varepsilon > 0$, *such that the* L^2 *norms of* u_ε *with respect to* $\partial(\varepsilon\bar{\partial}\omega + \eta)$ *and* $-\varepsilon\omega$ *are finite.*

Proof. If such u_ε exist, similarly as in the proof of Proposition 1.5 one can find the solutions u'_ε, possibly different from u_ε, whose L^2 norms with respect to $\partial(\varepsilon\bar{\partial}\omega + \eta)$ and $-2^n\varepsilon\omega$ are bounded by those of v. Taking the perturbation terms $\varepsilon\partial\bar{\partial}\omega$ and $-2^n\varepsilon\omega$ into account, it turns out that these L^2 norms of v depending on ε are uniformly bounded by const $\|v\|$. By taking a weakly convergent subsequence of u'_ε we obtain an L^2 solution to the equation $du = v$ as a limit. The converse is obvious.

Finally we recall Zucker's L^2 Künneth formula in a simplified situation that we need later. Let (M, ds^2_M) be another Hermitian manifold and let $(M \times N, ds^2_{M\times N})$ be the product of (M, ds^2_M) and (N, ds^2_N). Then we have

Proposition 1.7 (cf. [Z-1]) *Suppose that* $\dim H^s_{(2)}(M) < \infty$ *and* $\dim H^t_{(2)}(N) < \infty$ *for all* s *and* t. *Then*

$$H^r_{(2)}(M \times N) \cong \bigoplus_{s+t=r} H^s_{(2)}(M) \otimes H^t_{(2)}(N) \quad \text{for all } r\,.$$

2 Whitney stratification and intersection cohomology

Let X be a (reduced) complex space of dimension n. A stratification of X is by definition a family $\mathscr{S} = \{X_i\}_{i=-1}^{n}$ of closed complex analytic subsets of X such that

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \phi$$

and $X_i^\circ := X_i - X_{i-1}$ is, for each i, a complex manifold of dimension i unless it is empty. The connected components of X_i° are called the strata of $\mathscr{S}$. $\mathscr{S} = \{X_i\}$ is called a Whitney stratification if the following is satisfied.

(W) Let $X_i \supset X_j$ and let $x_\mu \in X_i^\circ$ (resp. $y_\mu \in X_j^\circ$) be a sequence of points converging to some $x \in X_j^\circ$ in such a way that, with respect to any local holomorphic

embedding of X into some $\mathbb{C}^N$ around x, the lines passing through x_μ and y_μ converge to some line ℓ and that the tangent spaces $T_{x_\mu}X_i^\circ$ converge to some affine space τ. Then $\ell \subset \tau$.

It is well known that a Whitney stratification exists for any complex space X (cf. [G-M-2]).

Let $\mathcal{S} = \{X_i\}_{i=-1}^n$ be a Whitney stratification of X and let $\mathcal{L}^\cdot = \{\mathcal{L}^k, d^k\}$ be a complex of fine sheaves over X. $\mathcal{L}^\cdot$ will be called an intersection complex if the following properties are satisfied.

(i) $\mathcal{L}^k = 0$ if $k < 0$.

(ii) There is a quasi-isomorphism $\mathbb{C}^\cdot \to \mathcal{L}^\cdot|_{X_n^\circ}$, where $\mathbb{C}^\cdot$ denotes the trivial extension of the constant sheaf $\mathbb{C}$ as a sheaf complex over X_n°.

(iii) The cohomology sheaves of $\mathcal{L}^\cdot$ are locally constant on each stratum of $\mathcal{S}$.

(iv) For each i and $x \in X_i^\circ$, there exists an arbitrarily small neighbourhood $U \ni x$ in X such that

$$H^r(U, \mathcal{L}^\cdot|_U) = \begin{cases} 0 & \text{if } r \geqq n - i \\ H^r(U - X_i, \mathcal{L}^\cdot | U - X_i) & \text{if } r < n - i \,. \end{cases}$$

Since any Whitney stratification of X defines a topological pseudomanifold structure of X in a canonical way (cf. [G-M-3, 1.1]), we have

Theorem 2.1 (cf. [G-M-3]) *Let X be a complex space of pure dimension. Then any Whitney stratification of X admits an intersection complex over X. Moreover, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Whitney stratifications of X and let $\mathcal{L}_i^\cdot$ be intersection complexes associated to $\mathcal{S}_i$ $(i = 1, 2)$. Then $H^r(X, \mathcal{L}_1^\cdot)$ and $H^r(X, \mathcal{L}_2^\cdot)$ are canonically isomorphic to each other.*

Thus the (r-th) intersection cohomology group $IH^r(X)$ is well defined as the (hyper-)cohomology group $H^r(X, \mathcal{L}^\cdot)$ for any intersection complex $\mathcal{L}^\cdot$ associated to any Whitney stratification of X.

We say that a vector field ξ defined on a subset of X is $\mathcal{S}$-compatible if ξ is tangent to the strata of $\mathcal{S}$. We shall call $\mathcal{S}$ an ℓ-stratification if it admits a Lipschitz continuous $\mathcal{S}$-compatible vector field. Since any ℓ-stratification defines also a pseudomanifold structure, Theorem 2.1. holds for the ℓ-stratifications, too, although we do not know their existence in general. Because of the purely local nature of the axioms of the intersection complex, the following is important for our purpose.

Theorem 2.2 (cf. [M] or [P]) *For any point $x \in X$ there exist a neighbourhood U and an ℓ-stratification of U.*

3 Application of the L^2 method

In this section we shall show the existence of a complete Kähler metric on certain part of a given complex space for which the L^2 estimates are true and give certain cohomology vanishing results.

Let X be a complex space as before and let W be any relatively compact open subset of X. By Hironaka's theorem there exists a complex manifold $\tilde{W}$ together

with a proper holomorphic map $\pi: \tilde{W} \to W$ satisfying the following properties:

(i) $\pi|\tilde{W} - \pi^{-1}(\mathrm{Sing}\, W)$ is a biholomorphic map onto $W - \mathrm{Sing}\, W$.

(ii) $\pi^{-1}(\mathrm{Sing}\, W)$ is supported on a divisor of simple normal crossings.

(iii) π is a succession of blowing-ups along nonsingular centers.

Here Sing W denotes the singular points of W. In what follows we shall often identify $\tilde{W} - \pi^{-1}(\mathrm{Sing}\, W)$ with $W - \mathrm{Sing}\, W$ via π.

Let $x \in W$ be any point and let $U \ni x$ be a Stein neighbourhood contained in W, and let $E_i (i = 1, \ldots, I)$ be the irreducible components of $\pi^{-1}(\mathrm{Sing}\, U)$. Since π is a succession of blowing-ups, there exist positive integers p_i such that the line bundle $\sum_{i=1}^{I} p_i[E_i]$ is negative on $\pi^{-1}(U)$. Hence one can find a positive integer J and an $I \times J$ matrix (p_{ij}) so that

(I) Given $1 \leqq i_1 < \cdots < i_k \leqq I$ so that $\bigcap_{\alpha=1}^{k} E_{i_\alpha} \neq \phi$, there exist $j_\beta (\beta = 1, \ldots, k)$ such that $\det(p_{i_\alpha j_\beta}) \neq 0$.

(II) The bundles $\sum_{i=1}^{I} p_{ij}[E_i]$ are negative for all j.

Let s_j be canonical sections of $\sum_{i=1}^{I} p_{ij}[E_i]$ and let $|s_j|$ be the length of s_j with respect to some fiber metric whose curvature form is negative. Then we put

$$\psi_0 = -\sum_{j=1}^{J} \log(-\log|s_j|)$$

by shrinking U if necessary so that $|s_j| < 1$ on $\pi^{-1}(U)$. Then ψ_0 is a strictly plurisubharmonic function on $U - \mathrm{Sing}\, U$.

On the other hand there exists a proper holomorphic embedding $\iota: U \to \mathbb{C}^N$ for some N such that $\iota(x)$ is the origin. Let ds_U^2 be the restriction of the euclidean metric of $\mathbb{C}^N$ to $U - \mathrm{Sing}\, U$. Here we naturally identify U with $\iota(U)$. Using the coordinate $z = (z_1, \ldots, z_N)$ of $\mathbb{C}^N$ one may express ds_U^2 as $\partial\bar{\partial}\|z\|^2$, where we put $\|z\|^2 = \sum_{i=1}^{N} |z_i|^2$ and identify $\partial\bar{\partial}\|z\|^2$ with the complex Hessian of $\|z\|^2$ by an abuse of notation. These conventions will be used throughout unless there is fear of confusion. Then we put

$$U_\delta = \{z \in U; \|z\| < \delta\}$$

and define a family of metrics on $U_\delta - \mathrm{Sing}\, U_\delta$ by

$$d\sigma_\lambda^2 = \partial\bar{\partial}(\lambda(-\log(\delta^2 - \|z\|^2)) + \psi_0)\,.$$

Here $\lambda: \mathbb{R} \to \mathbb{R}$ is a C^∞ function satisfying $\lambda' \geqq 1$ and $\lambda'' \geqq 0$. We note that $d\sigma_\lambda^2$ are complete Kähler metrics on U_δ-Sing U_δ.

Let us denote by $\|u\|_{\pm\lambda}$ the L^2 norms of u with respect to $d\sigma_\lambda^2$ and $\pm\lambda(-\log(\delta^2 - \|z\|^2))$. Then, as a special case of Proposition 1.4 we obtain

Proposition 3.1 *Under the above situation, assume that U_δ is of pure dimension n. Then there exists a constant C such that for any choice of λ,*

$$\|u\|_{-\lambda} \leqq C(\|\bar{\partial}u\|_{-\lambda} + \|\vartheta_{-\lambda}u\|_{-\lambda}) \tag{8}$$

for all $u \in C^r_0(U_\delta - \operatorname{Sing} U_\delta)$ *with* $r < n$ *and*

$$\|u\|_\lambda \leqq C(\|\bar\partial u\|_\lambda + \|\vartheta_\lambda u\|_\lambda) \tag{9}$$

for all $u \in C^r_0(U_\delta - \operatorname{Sing} U_\delta)$ *with* $r > n$. *Here* $\vartheta_{\pm\lambda}$ *denote the adjoints of* $\bar\partial$ *with respect to* $\| \ \|_{\pm\lambda}$.

We note that Proposition 1.5 can be applied if $\lambda(t) = t$ so that one has the corresponding L^2 cohomology vanishing in this situation. To state it in a neat way we shall assume that U is of pure dimension n from now on.

Proposition 3.2 *If* $\lambda(t) = t$,

$$H^r_{(2),\log(\delta^2 - \|z\|^2)}(U_\delta - \operatorname{Sing} U_\delta, d\sigma^2_\lambda) = 0 \quad \text{for } r < n \tag{10}$$

and

$$H^r_{(2),-\log(\delta^2 - \|z\|^2)}(U_\delta - \operatorname{Sing} U_\delta, d\sigma^2_\lambda) = 0 \quad \text{for } r > n\,. \tag{11}$$

From now on we put $V' = V - \operatorname{Sing} V$ for any open subset $V \subset X$, and λ_δ will stand for $\lambda(-\log(\delta^2 - \|z\|^2))$ for simplicity.

Proposition 3.3 *Let* $r > n$. *Then, for any choice of* λ *there exists a* C^∞ *convex increasing function* τ *such that*

$$L^r_{\lambda_\delta}(U'_\delta, d\sigma^2_\lambda) \subset L^r_{\tau_\delta}(U'_\delta, d\sigma^2_\tau) \tag{12}$$

and the image of the homomorphism

$$\iota_{\lambda,\tau}\colon H^r_{(2),\lambda_\delta}(U'_\delta, d\sigma^2_\lambda) \to H^r_{(2),\tau_\delta}(U'_\delta, d\sigma^2_\tau) \tag{13}$$

induced by the inclusion is zero.

Proof. It is clear that (12) is always attained without any restriction on r by choosing τ to be of sufficiently rapid growth compared to λ. Taking into account the L^2 cohomology vanishing for the $\bar\partial$-operator and successive use of it to solve the equation $du = v$, it is also easy to see that $\iota_{\tau,\lambda} = 0$ is attained by choosing a rapidly growing τ. □

Let us introduce a few conventional notations. Given an open subset $V \subset X$, a Hermitian metric ds^2 on V' and a real valued C^∞ function φ on V', we denote by $L^r_{\mathrm{loc}}(V, ds^2, \varphi)$ the set of locally square integrable r-forms f on V' satisfying $f|U' \in L^r_\varphi(U', ds^2)$ for any relatively compact open subset $U \subset V$. Furthermore we put $L^r_0(V, ds^2, \varphi) = \{f \in L^r_{\mathrm{loc}}(V, ds^2, \varphi);\ \operatorname{supp} f \Subset V\}$. With respect to the maximal closed extensions of $d|C^r_0(V')$ to $L^r_{\mathrm{loc}}(V, ds^2, \varphi)$ and $L^r_0(V, ds^2, \varphi)$, the cohomology groups $H^r_{\mathrm{loc}}(V, ds^2, \varphi)$ and $H^r_0(V, ds^2, \varphi)$ are defined in the same manner as in Sect. 1. Then Propositions 3.2 and 3.3 imply in particular the following.

Theorem 3.4 *Let* U_δ *and* ψ_0 *be as above, and let* $ds^2 = \partial\bar\partial(\|z\|^2 + \psi_0)$. *Then* $H^r_0(U_\delta, ds^2, \|z\|^2) = 0$ *if* $r < n$ *and* $H^r_{\mathrm{loc}}(U_\delta, ds^2, \|z\|^2) = 0$ *if* $r > n$.

Lemma. *Let* ψ_0 *be as above. Then* $-\exp(-t\psi_0)$ *is plurisubharmonic if* $t \in [0, \frac{1}{2}]$.

Proof. By definition $-\exp(-t\psi_0) = -\prod_{j=1}^{J}(-\log|s_j|)^t$. Hence

$$\partial\bar\partial(-\exp(-t\psi_0)) = \exp(-t\psi_0)\left\{\sum_{j=1}^{J}(-\log|s_j|)^{-t}\,\partial\bar\partial(-(-\log|s_j|)^t) - t^2\sum_{i,j}\frac{\partial\log|s_i|\bar\partial\log|s_j|}{\log|s_i|\log|s_j|}\right\}.$$

Since $\partial\bar\partial(-\log|s_i|)^t \geqq t(1-t)(-\log|s_i|)^{t-2}\partial\log|s_i|\bar\partial\log|s_i|$, we obtain the pluri-subharmonicity of $-\exp(-t\psi_0)$ for $t\in[0,\frac{1}{2}]$.

We put $\Omega = -\exp(-\frac{1}{2}\psi_0)$. We are going to show that $H_0^n(U_\delta, ds^2, \|z\|^2) = 0$. Similarly as in [O-3, §4] one can find an operator $K_\delta: L_0^n(U_\delta, ds^2, \|z\|^2) \to L_{\mathrm{loc}}^{n-1}(U'_\delta, ds^2, \|z\|^2)$, i.e. integration along a vector field that retracts U_δ to $\pi^{-1}(\mathrm{Sing}\, U_\delta)$, such that for any $u\in \mathrm{Ker}\, d\cap L_0^n(U_\delta, ds^2, \|z\|^2)$ one has $d(K_\delta u) = u$ and $K_\delta u\in L^{n-1}(U'_\delta, ds^2 + \varepsilon\partial\bar\partial\Omega, -\varepsilon\Omega)$ for all $\varepsilon > 0$, hence by Proposition 1.6 there exists a $v\in L^{n-1}(U'_\delta, ds^2)$ satisfying $dv = u$.

Thus we obtain

Theorem 3.5 $H_0^n(U_\delta, ds^2, \|z\|^2) = 0$.

Remark. In case X is the Satake compactification of an arithmetic (torsion free) quotient of a bounded symmetric domain Ξ, it is well known that the Bergman metric of Ξ induces a metric $d\sigma^2$ on X' which is locally (on X) quasi-isometric to ds^2 with respect to the toroidal resolution (cf. [A-M-R-T]). Since X allows an equisingular stratification with respect to the toroidal resolution, Theorems 3.4 and 3.5 (with its dual) imply the following.

Theorem (cf. [S-S]) *Let X and $d\sigma^2$ be as above. Then*

$$H^r_{(2)}(X', d\sigma^2) \cong IH^r(X).$$

4 Proof of theorem

In what follows let X be a complex space of pure dimension n. Recall that a Hermitian metric of X is by definition a Hermitian metric on X' which extends to a C^∞ Hermitian metric of the ambient space for any local holomorphic embedding of X to complex manifolds. We shall fix an arbitrary Hermitian metric ds_X^2 of X and call $H^r_{\mathrm{loc}}(X, ds_X^2)$ the (r-th) L^2_{loc} de Rham cohomology group of X, where we put $H^r_{\mathrm{loc}}(X, ds_X^2) := H^r_{\mathrm{loc}}(X, ds_X^2, 0)$. Let us define a sheaf $\mathscr{L}^r$ over X by putting

$$\mathscr{L}^r(W) = \{f\in L^r_{\mathrm{loc}}(W, ds_X^2);\, df\in L^{r+1}_{\mathrm{loc}}(W, ds_X^2)\}$$

for any open subset $W\subset X$, where $L^r_{\mathrm{loc}}(W, ds_X^2) := L^r_{\mathrm{loc}}(W, ds_X^2, 0)$. Clearly $\mathscr{L}^r$ are fine sheaves, and the maximal closed extensions of $d|C_0^r(W')$ define coboundary maps $d^r: \mathscr{L}^r \to \mathscr{L}^{r+1}$. We shall call the complex $\mathscr{L}^\cdot = \{\mathscr{L}^r, d^r\}$ the L^2 complex over X.

Theorem 4.1 *The L^2 complex over X is an intersection complex with respect to any Whitney stratification.*

Corollary. *The L^2_{loc} de Rham cohomology group of X is canonically isomorphic to the intersection cohomology group of X.*

The proof of Theorem 4.1 will be carried out in the following manner. We denote the assertion of Theorem 4.1 as P_1, and mean by $\mathrm{P}_1(n)$ that P_1 holds for all X satisfying $\dim X < n$. Let $d^r_{\max}$(resp. $\delta^r_{\max}$) be the maximal closed extension of $d|C^r_0(X')$ (resp. $\delta|C^r_0(X')$) to $L^r(X, ds^2_X)$. Then we denote by P_2 the assertion of the following.

Sublemma. $\mathrm{Im}(d^{r-1}_{\max}|L^{r-1}_0(U, ds^2_X)) \perp \mathrm{Im}(\delta^{r+1}_{\max}|L^{r+1}_0(U, ds^2_X))$ *for all r and any open subset $U \subset X$.*

By $\mathrm{P}_2(n)$ we shall mean that P_2 holds for all X satisfying $\dim X < n$. We are going to prove $\mathrm{P}_1 \wedge \mathrm{P}_2$ by induction on n. $\mathrm{P}_1(1) \wedge \mathrm{P}_2(1)$ is trivial. We assume $\mathrm{P}_1(n) \wedge \mathrm{P}_2(n)$. Since the questions are local, we may assume that X admits an ℓ-stratification by Theorem 2.2. Let $\mathscr{S} = \{X_i\}$ be any ℓ-stratification of X. Combining $\mathrm{P}_1(n)$ and Proposition 1.7, one sees immediately that the axioms of intersection complex are satisfied by $\mathscr{L}^\bullet$ on $X - X_0$. Moreover P_2 also holds for $X - X_0$ because of the local triviality of X along the strata and the induction hypothesis $\mathrm{P}_2(n)$. Let us prove that $H^r_0(U_\delta, ds^2_X) = 0$ for $r < n$ by modifying the argument in [O-1, Supplement] with certain simplification. The proof of the following is easy and left to the reader.

Lemma. *There exists a C^∞ function $\chi: \mathbb{R} \to \mathbb{R}$ satisfying $\chi(t) = t$ on $(-\infty, -100)$, $\chi(t) = -t^{-1}$ on $(100, \infty)$ and $\chi'(t) \geqq 2\chi''(t)$ everywhere.*

Choosing δ smaller if necessary so that $\psi_0 < 0$ and $\|z\| < 1$ on U'_δ, we put

$$F_\tau = 2(n-r)\log\|z\| + \chi(\log\log(\delta/\|z\|)) + \tau\psi_0 - 1 .$$

Clearly $ds^2_\tau := \partial\bar\partial(-\log(-F_\tau))$ is a complete Kähler metric on U'_δ if $\tau > 0$. Let $\lambda_1 \leqq \cdots \leqq \lambda_n$ be the eigenvalues of $\partial\bar\partial F_\tau$ with respect to ds^2_τ. Then we have

$$\inf \lambda_1 > 0 \qquad \text{and} \qquad \lambda_i = -F_\tau \quad \text{for } 2 \leqq i \leqq n .$$

Therefore, with respect to ds^2_τ and the weight $-F_\tau$, one has precise L^2 estimates for the $\bar\partial$-operator in degrees less than n. Namely

$$\|\rho_{\tau,r} v\|_{(\tau)} \leqq \mathrm{const}(\|\bar\partial v\|_{(\tau)} + \|\vartheta_{(\tau)} v\|_{(\tau)}) - \|\bar\vartheta v\|_{(\tau)} - \|\partial_{(\tau)} v\|_{(\tau)} \tag{14}$$

for any $v \in C^r_0(U')$ with $r < n$. Here $\rho_{\tau,r} := \sum_{i=1}^{n-r} \lambda_i$, and $\|\ \|_{(\tau)}$ and $\vartheta_{(\tau)}$ (resp. $\partial_{(\tau)}$) denote the L^2 norm and the formal adjoint of $\bar\partial$ (resp. $\bar\vartheta$) with respect to ds^2_τ and $-F_\tau$, respectively. For simplicity we put $L^r_\tau = L^r_{-F_\tau}(U'_\delta, ds^2_\tau)$ for $\tau \geqq 0$. Given any $u \in \mathrm{Ker}\, d \cap L^r_0(U_\delta, ds^2_X)$ with $r < n$, we decompose u into the (p,q)-components $u^{p,q}$ as $u = u^{0,r} + \cdots + u^{r,0}$. Then we have $\bar\partial u^{0,r} = 0$. Let $\{K_\mu\}_{\mu=1}^\infty$ be an increasing sequence of compact subsets of U'_δ such that $\bigcup_{\mu=1}^\infty K_\mu = U'_\delta$. Let u_{K_μ} be the trivial extension of $u|K_\mu$ to U'_δ. Taking the orthogonal decomposition

$$L^r_\tau = \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^* \quad \text{for } \tau > 0 \text{ and } r < n$$

into account, where $\bar{\partial}^*$ denote the Hilbert space adjoint of $\bar{\partial}$, we obtain correspondingly the canonical solutions to

$$u^{0,r}_{K_\mu} = \bar{\partial} f + \bar{\partial}^* g \,.$$

Let us choose a sufficiently rapidly decreasing sequence $\tau_\mu \searrow 0$ ($\mu \to \infty$) so that we have

$$\| u^{0,r}_{K_\mu} \|_{(\tau_\mu)} \leqq 2 \| u \| \,. \tag{15}$$

Then we decompose $u^{0,r}_{K_\mu}$ in $L^r_{\tau_\mu}$ as $u^{0,r}_{K_\mu} = \bar{\partial} f_\mu + \bar{\partial}^* g_\mu$, where $\bar{\partial}^* f = 0$ and $\bar{\partial} g = 0$.

Now passing to the limit we obtain a decomposition

$$u^{0,r} = \bar{\partial} f + \vartheta_{(0)} g \quad \text{in } L^r_0 \tag{16}$$

such that $\| \partial_{(0)} f \|_{(0)} \leq \infty$, $\| F_0 f \|_{(0)} < \infty$ and $\delta g \in L^r_{\mathrm{loc}}(U_\delta - \{x\}, ds^2_X)$. Note that $\| df \|_{(0)} < \infty$ since $\| \bar{\partial} f \|_{(0)} < \infty$, $\| \partial_{(0)} f \|_{(0)} < \infty$ and $\| F_0 f \|_{(0)} < \infty$.

Let $\{\kappa_\nu\}_{\nu=1}$ be a C^∞ partition of unity of $U_\delta - \{x\}$ such that $\operatorname{supp} \kappa_\nu \Subset U_\delta - \{x\}$ and $\sup_{U_\delta - \{x\}} |\partial \kappa_\nu|_{(0)} < \text{const}$, where $|\ |_{(0)}$ denotes the pointwise norm with respect to ds^2_0. Note that u is locally contained in $\operatorname{Im} d^{r-1}_{\max}$ on $U_\delta - \{x\}$ because $u \in \operatorname{Ker} d \cap L^r_0(U_\delta, ds^2_X)$. Hence the validity of P_2 for $X - X_0$ implies that $(u, \delta_{(0)}(\kappa_\nu g))_{(0)} = 0$ for all ν, where $\delta_{(0)}$ denotes the formal adjoint of d with respect to ds^2_0.

Therefore

$$(u^{0,r}, \vartheta_{(0)} g)_{(0)} = \left(u, \vartheta_{(0)} \left(\sum_\nu \kappa_\nu g \right) \right)_{(0)} = \sum_\nu (u, \vartheta_{(0)} (\kappa_\nu g))_{(0)}$$

$$= \sum_\nu (u, \delta_{(0)} (\kappa_\nu g))_{(0)} = 0 \,.$$

Since $\| df \|_{(0)} < \infty$ we have

$$(\bar{\partial} f, \vartheta_{(0)} g)_{(0)} = 0$$

similarly as above.

Thus we obtain $\vartheta_{(0)} g = 0$ so that

$$\bar{\partial} u^{0,r} = f \,.$$

Repeating the above procedure we obtain a $w \in L^{r-1}_0$ satisfying $dw = u$ and $\| F_0 w \|_{(0)} < \infty$. By choosing δ smaller if necessary in advance, we may conclude that $H^r_0(U_\delta, ds^2_X) = 0$ for $r < n$. The estimate for the solution w proves at the same time that $P_2(n+1)$ holds for $r < n$.

Let $r > n$. Instead of providing directly that $H^r_{\mathrm{loc}}(U_\delta, ds^2_X) = 0$ by only modifying the metrics and weights, we observe first that there exist canonical isomorphisms $H^r_{(2)}(U'_\delta, ds^2_X) \cong H^r_{\mathrm{loc}}(U_\delta, ds^2_X)$ for all r. This is because $H^r_{\mathrm{loc}}(U_\delta - \{x\}, ds^2_X)$ is already a topological invariant in virtue of the validity of P_1 for $X - X_0$. Given any $u \in \operatorname{Ker} d \cap L^r(U'_\delta, ds^2_x)$ we proceed as follows.

We put

$$G_\tau = \log \| z \|^2 + \chi(- \log \log (\delta / \| z \|)) + \tau \psi_0 - 1 \,,$$

where $\tau \in [0, 1]$ and χ is as before, and

$$ds^2_{\langle \tau \rangle} = \partial \bar{\partial} (e^{G_\tau} + \tau(- \log(- G_\tau))) \,.$$

We denote by $\| \ \|_{\langle\tau\rangle}$ (resp. $\delta_{\langle\tau\rangle}$) the L^2 norm in $L^r(U'_\delta, ds^2_{\langle\tau\rangle})$ (resp. the formal adjoint of d with respect to $\| \ \|_{\langle\tau\rangle}$).

Lemma. *Let $\tau > 0$ and $r \neq n$. Then.*

$$\| |\partial(-\log(-G_\tau))|_{\langle\tau\rangle} v\|_{\langle\tau\rangle} \leqq \text{const}(\|dv\|_{\langle\tau\rangle} + \|\delta_{\langle\tau\rangle} v\|_{\langle\tau\rangle}) \tag{17}$$

for all $v \in C^r_0(U')$. Here $| \ |_{\langle\tau\rangle}$ denotes the pointwise norm.

Proof. Use (2) for $\eta = \partial(-\log(-G_\tau))$, and apply Cauchy-Schwartz inequality and the Kähler identity.

Hence we can decompose u_{K_μ} canonically in $L^r(U'_\delta, ds^2_{\langle\tau\rangle})$ for all $\tau > 0$ as

$$u_{K_\mu} = df_\mu + d^* g_\mu \ .$$

Similarly as in the case of $r < n$, we pass to the limit and obtain

$$u = df + \delta_{\langle 0\rangle} g$$

where $f \in L^{r-1}(U'_\delta, ds^2_{\langle 0\rangle})$, $df, \delta_{\langle 0\rangle} g \in L^r(U'_\delta, ds^2_{\langle 0\rangle})$ and

$$\| |\partial(-\log(-G_0))|_{\langle 0\rangle} g\|_{\langle 0\rangle} < \infty \ . \tag{18}$$

Since $r > n$, combining (18) with the validity of $P_1 \wedge P_2$ for $X - X_0$ we obtain that $\delta_{\langle 0\rangle} g \perp \text{Ker } d^{r-1}_{\max}$ in $L^r(U'_\delta, ds^2_0)$.

Hence $\delta_{\langle 0\rangle} g$ must be zero, so that the images of u in $H^r_{\text{loc}}(U_{\delta'}, ds^2_X)$ are zero if $\delta' < \delta$, which proves that $H^r_{\text{loc}}(U_\delta, ds^2_X) = 0$ for $r > n$.

$P_2(n+1)$ for $r > n$ is simply the dual of $P_2(n+1)$ for $r < n$.

In order to prove that $H^n_0(U_\delta, ds^2_X) = 0$, we choose our ψ_0 in advance so that the desingularization of U_δ dominates the blowing-up of U_δ centered at x. Then we have

$$\partial\bar\partial(-\log(-G_\tau)) \leqq \text{const}\, \partial\bar\partial(\|z\|^2 + \psi_0) \quad \text{on } U'_{\delta/2} \quad \text{for } \tau \in [0, 1] \ .$$

Let $ds^2_{\tau,\varepsilon} = \partial\bar\partial(-\log(-G_\tau) + \varepsilon(\|z\|^2 + \psi_0))$. Then $ds^2_{\tau,\varepsilon}$ are quasi-isometric to $\partial\bar\partial(\|z\|^2 + \psi_0)$ on $U'_{\delta/2}$. Hence, given any $u \in \text{Ker } d \cap L^n_0(U_{\delta/2}, ds^2_{\langle\tau\rangle})$ one has a $w \in L^{n-1}_0(U_\delta, ds^2_{\tau,\varepsilon})$ satisfying $dw = u$ by Theorem 3.5.

Hence we can decompose u_{K_μ} as

$$u_{K_\mu} = df_\mu + d^* g_\mu \quad \text{in } L^n(U'_\delta, ds^2_{\tau,\varepsilon}) \quad \text{for all } \varepsilon > 0 \ .$$

Similarly as before we conclude that $u \in \text{Im}(d | L^{n-1}(U'_\delta, ds^2_{\langle\tau\rangle}))$ from the above decompositions. Then we apply the same argument with respect to $ds^2_{\langle\tau\rangle}$ and obtain $H^n_0(U_\delta, ds^2_X) = 0$ and that $P_2(n+1)$ is true for $r = n$. In fact, (17) together with P_1 imply

$$\text{Ker } d^n_{\max} \perp \text{Im}(\delta^{n+1}_{\max} | L^{n+1}_0(U_\delta, ds^2_X)) \ . \tag{19}$$

The axioms of the intersection complex for X will follow if we add $H^n_{\text{loc}}(U_\delta, ds^2_X) = 0$ to all these. But this follows from $H^n_0(U_\delta, ds^2_X) = 0$ by the duality in virtue of (19).

Q.E.D.

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