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The Conley index over a space*

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1 Introduction

Let $p: X \to \Lambda$ be a locally trivial bundle where X is a locally compact metric space and Λ is compact. Consider a (local) flow φ on X that respects the fibres, i.e., $X_\lambda = p^{-1}(\lambda)$ is flow invariant for any $\lambda \in \Lambda$. Suppose we are given an isolated invariant set $S \subset X$ of φ. In this situation two versions of the Conley index have been defined. One is the usual Conley index of S with respect to φ. It is the homotopy type of the based space N/M, where (N, M) is an index pair of S. The second version, the global Conley index of S, introduced by Salamon [S] consists of the family N_λ/M_λ, $\lambda \in \Lambda$, together with certain maps $N_\lambda/M_\lambda \to N_\mu/M_\mu$. Here $N_\lambda = N \cap X_\lambda$ etc. Of course, N_λ/M_λ is the Conley index of S_λ for the flow φ_λ.

In this paper we shall develop a third (intermediate) point of view which seems to be the most natural. The Conley index of S over Λ, $\mathscr{C}(S, \Lambda)$, is defined as $[N/\sim]$ where $x \sim y$ if $x, y \in M_\lambda$ for some $\lambda \in \Lambda$. $N/\sim$ is a space over Λ with base points $\beta(\lambda)$ in each fibre N_λ/M_λ. $[-]$ denotes based fibre homotopy type. Thus we retain the "over Λ" structure inherent to the situation as well as the topology on N. Under mild assumptions on Λ (connected and locally contractible) $N/\sim$ is locally fibre homotopy trivial (preserving the base points of the fibres). This implies — in fact it is equivalent to — that $N/\sim$ is a weak fibration, i.e., it satisfies the weak covering homotopy property WCHP in the sense of Dold ([D, Definition 5.1]). This notion does not even make sense for the other two versions of the Conley index mentioned in the beginning.

Given a path w in Λ from λ to μ we get a map $f_w: N_\lambda/M_\lambda \to N_\mu/M_\mu$ determined by w up to homotopy. If w' is homotopic to w relative to the end points then $f_w \simeq f_{w'}$. These are the maps constructed by Salamon. Their existence is inherent to the WCHP. This property (in a version with base points) also suffices to yield the long exact homotopy sequence (and the spectral sequences in homology or

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cohomology) usually associated to fibrations. In fact, the sequence

$$N_\lambda/M_\lambda \xrightarrow{i} N/\sim \xrightarrow{\pi} \Lambda$$

of based spaces is exact, where we choose $\lambda \in \Lambda$ and $\beta(\lambda) \in N_\lambda/M_\lambda \subset N/\sim$ as base points. This concept is dual to that of a coexact sequence which one obtains from an attractor-repeller pair as in [S].

Some easy consequences of the above results are the following. If Λ is a contractible then $\mathscr{C}(S, \Lambda)$ is fibre homotopy trivial. This answers a question posed by Salamon. An immediate corollary of this observation is the following. Suppose Λ is contractible and S is an isolated invariant set over a subset K of Λ, $S \subset X_K = p^{-1}(K)$. If $\mathscr{C}(S, K)$ is not fibre homotopy trivial then there cannot exist an isolated invariant set T of X such that $T \cap X_K = S$. In other words, if S_λ is the only compact invariant set of φ_λ for all $\lambda \in K$ then S cannot be "separated from infinity". We illustrate our results by showing how the Conley index over Λ can be used in bifurcation theory.

In the following we assume the reader to be familiar with the basic notions of Conley index theory as developed in [C] or [S].

2 The Conley index over Λ

Consider a locally trivial bundle $p: X \to \Lambda$ of the locally compact, metric space X over the compact, connected space Λ. Let φ be a local flow over Λ as in the introduction and $S \subset X$ an isolated invariant set (which includes compactness). In order to define the Conley index of S over Λ we recall the notion of fibre homotopy equivalence. Two spaces $\pi: E \to \Lambda$, $\pi': E' \to \Lambda$ over Λ are fibre homotopy equivalent if there exist maps $f: E \to E'$, $f': E' \to E$ over Λ, i.e., $\pi' f = \pi$, $\pi f' = \pi'$, and homotopies $H: E \times [0, 1] \to E$, $H': E' \times [0, 1] \to E'$ over Λ such that $H_0 = H(-, 0) = \mathrm{id}_E$, $H_1 = f' \circ f$, $H'_0 = \mathrm{id}_{E'}$, $H'_1 = f \circ f'$. If in addition there exist sections $\beta: \Lambda \to E$, $\beta': \Lambda \to E'$ of base points $\beta(\lambda) \in \pi^{-1}(\lambda) = E_\lambda$ etc., and all maps respect the base points then E and E' are called based fibre homotopy equivalent.

2.1 Definition. An *index pair of S over Λ* is a pair (N, M) of compact sets $M \subset N \subset X$ such that $M_\lambda = M \cap p^{-1}(\lambda) \neq \varnothing$ for all $\lambda \in \Lambda$ and (N, M) is an index pair in the usual sense, i.e.,

(i) $N - M$ is an isolating neighborhood of S,
(ii) M is positively invariant with respect to N,
(iii) M is an exit set for N.

2.2 Lemma. *For any isolated invariant set $S \subsetneq X$ there exists an index pair over Λ.*

Proof. Let (N', M') be an index pair for S in the usual sense. Since $S \neq X$ it follows that $S_\lambda \neq X_\lambda$ for any λ. So we may assume that $N'_\lambda \neq X_\lambda$ for all λ.

Now let C be any compact subset of $X - N'$ with $p(C) = \Lambda$. $M := M' \cup C$ and $N := N' \cup C$ will then be an index pair for S over Λ. □

Given an index pair (N, M) for S over Λ we may form the quotient space $E = N/\sim$ of N where we identify each subset M_λ to a point $\beta(\lambda)$. E is a space over

Λ, $\pi: E \to \Lambda$, and we have a canonical section $\beta: \Lambda \to E$. The continuity of β is an easy consequence of the compactness of M. We chose the non-standard definition of exit pair in order to avoid discussions on the definition and the topology of E in the case where some of the exit sets M_λ are empty and others may be not empty.

2.3 Lemma. *Let (N, M) and (N', M') be two index pairs for S over Λ. Then the associated quotient spaces $E := N/\sim$ and $E' = N'/\sim'$ are based fibre homotopy equivalent.*

Proof. Salamon shows that there exists a $T > 0$ such that the map $f^T: N/M \to N'/M'$

$$f^T(x) = \begin{cases} \varphi(x, 3T) & \text{if } \varphi(x, [0, 2T]) \subset N - M, \quad \varphi(x, [T, 3T]) \subset N' - M' \\ \beta'(p(x)) & \text{otherwise} \end{cases}$$

is well defined and continuous (cf. [S], Lemma 4.7]). This map obviously induces a continuous map $g^T: E \to E'$. Similarly, if T is big enough we get a map $g'^T: E' \to E$. It is easy to see that g^T is a based fibre homotopy equivalence with inverse g'^T (cf. [S, Lemma 4.8]). □

2.4. Definition. Let S be an isolated invariant set in X. The *Conley index of S over Λ*, $\mathscr{C}(S, \Lambda)$, is the based fibre homotopy type of the space $\pi: E = N/\sim \to \Lambda$ over Λ, where (N, M) is an index pair for S over Λ.

This is well defined because of Lemma 2.3.

2.5. Remarks

(a) If $\pi: E \to \Lambda$ is the Conley index of S over Λ and $\beta: \Lambda \to E$ is the base point section then $E/\beta(\Lambda)$ is the usual Conley index of S. Here and in the sequel we do not distinguish between E and its based fibre homotopy type $\mathscr{C}(S, \Lambda)$.

(b) Suppose p: $X \to \Lambda$ is a vector bundle over Λ with zero section s: $\Lambda \to X$. Consider a flow φ on X over Λ such that $s(\lambda)$ is a hyperbolic stationary solution for $\varphi_\lambda = \varphi \mid X_\lambda$, $\lambda \in \Lambda$. Then the tangent spaces $V_\lambda \subset X_\lambda$ of the unstable manifolds of $s(\lambda)$ form a vector bundle $V \to \Lambda$ which should be called the Morse index of $S = s(\Lambda)$ over Λ. The Conley index of S over Λ is the fiberwise one point compactification S^V of V. $S^V \to \Lambda$ is a sphere bundle over Λ with the points at infinity as base points. If we identify the points at infinity we obtain the Thom space TV of V. TV is homeomorphic to the one point compactification of V or to the quotient space DV/SV where DV is the disc bundle and SV the sphere bundle of V. Thus the usual Conley index of $S = s(\Lambda)$ is the Thom space TV. In this hyperbolic situation the Morse index of S over Λ induces an element of $\widetilde{KO}(\Lambda)$ and the Conley index of S over Λ induces an element of $\widetilde{JO}(\Lambda)$ which is by definition the set of fibre homotopy classes of sphere bundles over Λ (cf. [H, §§15.4, 15.5]). Thus the passage from the Morse index over Λ to the Conley index over Λ corresponds to the J-homomorphism $J: \widetilde{KO}(\Lambda) \to \widetilde{JO}(\Lambda)$ in algebraic topology.

(c) We assumed Λ to be compact only for simplicity. If Λ is locally compact the whole theory works without major changes. Instead of S being compact one has to assume that $p \mid S: S \to \Lambda$ is proper. This means that $S \cap p^{-1}(K)$ is compact for any compact subset K of Λ. Probably one can even rewrite the paper for paracompact Λ and locally compact fibres. But in this case Salamon's paper does not apply and has to be rewritten, too. Of course, one can also look at the more general situation where the fibres are not locally compact using the ideas of Rybakowski [R].

(d) One can also introduce connected simple systems over Λ. These consist of a family of spaces over Λ with base point sections β and a family of fibre homotopy classes of maps over Λ between these having properties analogous to those for $\Lambda = \{pt\}$; see [S, Definition 2.6]. We leave this straightforward generalization to the reader.

3 Statement of results

As above we consider a local flow φ on the total space X of the locally trivial bundle $p: X \to \Lambda$ such that the fibres are flow invariant. Λ is compact and connected and X is locally compact and metric. Let S be an isolated invariant set.

3.1 Theorem. *If Λ is locally contractible then the Conley index of S over Λ, $\mathscr{C}(S, \Lambda) = [\pi: E \to \Lambda]$, is locally fibre homotopy trivial. This means that any $\lambda \in \Lambda$ has a neighborhood $K = K(\lambda)$ such that the bundle $\pi \mid K$ is fibre homotopy equivalent to a trivial bundle $K \times F$. Moreover, if $\beta: \Lambda \to E$ denotes the base point section of π then all maps and homotopies can be chosen to respect the base points.*

According to Dold ([D, Theorem 6.4]) a space $\pi: E \to \Lambda$ over Λ is locally fibre homotopy trivial if and only if π satisfies the weak covering homotopy property WCHP. This means that for any map $f: Z \to E$ and any homotopy $\bar{H}: Z \times I \to \Lambda$ with $\pi \circ f(z) = \bar{H}(z, 0)$ there exists a homotopy $H: Z \times I \to E$ covering $\bar{H}$(i.e. $\pi \circ H = \bar{H}$) and such that $H(-, 0): Z \to E$ is fibre homotopic to f(cf. Dold [D, Definition 5.1 and Proposition 5.13]). So we obtain the following corollary.

3.2 Corollary. *If Λ is locally contractible then the Conley index $\mathscr{C}(S, \Lambda) = [\pi: E \to \Lambda]$ of S over Λ has the* WCHP. *Moreover, fixing base points $\lambda_0 \in \Lambda$ and $\beta(\lambda_0) \in \pi^{-1}(\lambda_0)$ then π has the based* WCHP, *i.e., if f and $\bar{H}$ as above fix the base points then so does the covering homotopy H.*

An immediate consequence of the based WCHP is that the sequence

$$F := \pi^{-1}(\lambda_0) \xrightarrow{i} E \xrightarrow{\pi} \Lambda$$

is exact. This means that for every based space Z the sequence

$$[Z, F] \xrightarrow{i_*} [Z, E] \xrightarrow{\pi_*} [Z, \Lambda]$$

of based homotopy classes of maps has the property image $(i_*) = \pi_*^{-1}$ (const) where const is the homotopy class of the constant map $Z \ni z \mapsto \lambda_0 \in \Lambda$. It is an elementary fact of homotopy theory (cf. [W], for example) that an exact sequence

$F \xrightarrow{i} E \xrightarrow{\pi} \Lambda$ induces a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{\pi_*} \pi_n(\Lambda) \xrightarrow{\delta} \pi_{n-1}(F) \xrightarrow{i_*} \cdots$$

Observe that F is the Conley index $\mathscr{C}(S_{\lambda_0})$ of the isolated invariant set $S_{\lambda_0} \subset X_{\lambda_0} = p^{-1}(\lambda_0)$. So the homotopy groups of $\mathscr{C}(S_{\lambda_0})$, $\mathscr{C}(S, \Lambda)$ and Λ are related by a long exact sequence. Also the homology and cohomology groups of $\mathscr{C}(S_{\lambda_0})$, $\mathscr{C}(S, \Lambda)$ and Λ are related. But this involves spectral sequences as in the case of the (co)homology of fibrations.

To formulate the next result we need the notion of pull back. Given a space $\pi: E \to \Lambda$ over Λ and a map $f: K \to \Lambda$ then the pull back $f^*\pi: f^*E \to K$ is defined as follows.

$$f^*E := \{(\kappa, e) \in K \times E : \pi(e) = f(\kappa)\}, \quad f^*\pi(\kappa, e) := \kappa .$$

If π has the WCHP then so does $f^*\pi$. And if $\pi': E' \to \Lambda$ is fibre homotopy equivalent to π then $f^*\pi'$ and $f^*\pi$ are fibre homotopy equivalent. If K is a subset of Λ, $i: K \hookrightarrow \Lambda$, then $i^*\pi$ is simply the restriction of π to the part over K. We write $\mathscr{C}(S, K)$ for $i^*\mathscr{C}(S, \Lambda)$.

3.3 Theorem. *If K is a deformation retract of Λ, $r: \Lambda \to K$ the retraction, then $\mathscr{C}(S, \Lambda) = r^*\mathscr{C}(S, K)$. In particular, if Λ is contractible then $\mathscr{C}(S, \Lambda)$ is fibre homotopy trivial: $\mathscr{C}(S, \Lambda) = \Lambda \times \mathscr{C}(S_{\lambda_0})$.*

As a consequence, the original Conley index of S, which is given by

$$\mathscr{C}(S, \Lambda)/\beta(\Lambda) \cong (\Lambda \times \mathscr{C}(S_{\lambda_0}))/(\Lambda \times \beta(\lambda_0))$$

is homotopy equivalent to $\mathscr{C}(S_{\lambda_0})$ is Λ is contractible. This answers a question posed by Salamon (cf. [S, p. 36]).

3.4 Theorem. *Let K be a subset of Λ that can be deformed inside Λ to a point. Consider an isolated invariant set S of $X_K = p^{-1}(K)$. Suppose $\mathscr{C}(S, K)$ is not fibre homotopy trivial. Then there cannot exist an isolated invariant set T of X such that $T \cap X_K = S$.*

We conclude this section with a result about attractor-repeller pairs. Suppose we are given two isolated invariant sets S and A of X such that $A \subset S$ is an attractor in S. Let $A^* \subset S$ be the complementary repeller.

3.5 Theorem. *There exists a coexact sequence*

$$\mathscr{C}(A, \Lambda) \to \mathscr{C}(S, \Lambda) \to \mathscr{C}(A^*, \Lambda)$$

of spaces over Λ.

The proof is similar to the corresponding one in the case $\Lambda = \{pt\}$; cf. [S, §5.2]. It requires no new ideas and will therefore be omitted.

4 An application to bifurcation theory

Let $F: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be C^2 and such that $0 \in \mathbb{R}^n$ is a critical point of $F_\lambda = F(\lambda, \cdot)$ for every $\lambda \in \mathbb{R}^k$. We are interested in other critical points bifurcating from $\mathbb{R}^k \times \{0\}$. Assume that there exists $R > 0$ such that 0 is a non-degenerate critical point of F_λ if $|\lambda| = R$. Let $m(\lambda)$ be the Morse index of 0 for $|\lambda| = R$. In the case $k = 1$ it is well known that critical points of F_λ bifurcate from $(-R, R) \times \{0\}$ if $m(+R) \neq m(-R)$. This is due to the homotopy invariance of the Conley index. We shall generalize this result to the case $k \geqq 2$. Then $m = m(\lambda)$ is independent of λ since $S^{k-1} = \{\lambda \in \mathbb{R}^k : |\lambda| = R\}$ is connected and $m(\lambda) \in \mathbb{Z}$ depends continuously on λ.

Instead of looking at the Morse indices $m(\lambda)$ we consider the map

$$S^{k-1} \ni \lambda \mapsto A_\lambda := D^2 F_\lambda(0) \in \mathrm{GL}(n) \cap \mathrm{Sym}(n) = \mathrm{GLS}(n)$$

where $\mathrm{Sym}(n)$ denotes the set of all symmetric $n \times n$-matrices. Fixing a base point λ_0 in S^{k-1} this map induces an element β_F of π_{k-1} $(\mathrm{GLS}(n); A_{\lambda_0})$. Let $\mathrm{GLS}(n, m)$ denote the component of A_{λ_0} in $\mathrm{GLS}(n)$. This depends only on the number of negative eigenvalues of A_{λ_0} which is just the Morse index of 0 for F_{λ_0}. We assume for simplicity that the Morse index satisfies the inequality $k < m < n - k$. This is no restriction since we may suspend F, i.e. we replace $\mathbb{R}^n$ by $\mathbb{R}^n \times \mathbb{R}^{k+1} \times \mathbb{R}^{k+1}$ and F by the map

$$\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} \ni (\lambda, x, y, z) \mapsto F(\lambda, x) + \tfrac{1}{2}\|y\|^2 - \tfrac{1}{2}\|z\|^2 \in \mathbb{R} .$$

This does not change the critical points of F but only their index. We now compute $\pi_{k-1}(\mathrm{GLS}(n); A_{\lambda_0}) = \pi_{k-1}(\mathrm{GLS}(n, m))$.

4.1 Proposition. *If $k > 1$ and $k < m < n - k$ then*

$$\pi_{k-1}(\mathrm{GLS}(n, m)) = \begin{cases} \mathbb{Z} & \text{if } k \equiv 1 \text{ or } 5 \bmod 8 ; \\ \mathbb{Z}/2 & \text{if } k \equiv 2 \text{ or } 3 \bmod 8 ; \\ 0 & \text{otherwise} . \end{cases}$$

In order to state our bifurcation result we need to introduce certain integers. For each integer n and prime p let $v_p(n)$ be the largest integer v such that p^v divides n. Then we define natural numbers b_l for $l \geqq 1$ as follows:

$$v_2(b_l) := 3 + v_2(l)$$

and for $p > 2$

$$v_p(b_l) := \begin{cases} 1 + v_p(2l) & \text{if } p - 1 \text{ does not divide } 2l ; \\ 0 & \text{otherwise} . \end{cases}$$

Observe that b_l is always divisible by 8. b_l is closely related to the Bernoulli numbers (cf. [MK]).

4.2 Theorem. *Nontrivial critical points of F bifurcate from $\mathbb{R}^k \times \{0\}$ if either*

(i) $\beta_F \neq 0$ *for* $k \equiv 2$ *or* 3 mod 8 *or*

(ii) $\beta_F \not\equiv 0 \bmod b_l$ *for* $k = 4l + 1$.

Theorem 4.2 is a consequence of Theorem 3.4. Set $\Lambda := \{\lambda \in \mathbb{R}^k : |\lambda| \leqq R\}$ and $K := \partial\Lambda$. We consider the flow φ on $\Lambda \times \mathbb{R}^n$ induced by the ordinary differential equation $\dot{x} = -\nabla F_\lambda(x)$. $S = K \times \{0\}$ is an isolated invariant set of $K \times \mathbb{R}^n$ and we shall show that $\mathscr{C}(S, K)$ is not fibre homotopy trivial. Then Theorem 3.4 tells us that $\Lambda \times \{0\}$ cannot be an isolated invariant set. Since φ is a gradient flow there must exist stationary orbits of φ bifurcating from $\Lambda \times \{0\}$ which proves Theorem 4.2.

In order to compute $\mathscr{C}(S, K)$ we first compute the Morse index of S over K. Let G_m^n denote the Grassmannian manifold of m-dimensional linear subspaces of $\mathbb{R}^n$.

4.3 Lemma. *The map* $\mathrm{GLS}(n, m) \to G_m^n$, $L \mapsto V^-(L)$, *which associates to each* $L \in \mathrm{GLS}(n, m)$ *the generalized eigenspace* $V^-(L)$ *belonging to the negative part of the spectrum of* L, *is a strong deformation retraction.*

Proof. We consider G_m^n as a subspace of $\mathrm{GLS}(n, m)$ via the inclusion $V \mapsto L_V$ defined by $L_V | V = -\mathrm{id}$ and $L_V | V^\perp = +\mathrm{id}$. Obviously the composition $V \mapsto L_V \mapsto V^-(L_V)$ is the identity. Given any $L \in \mathrm{GLS}(n, m)$ consider the path $I \ni t \mapsto L_t \in \mathrm{GLS}(n, m)$ defined by $L_t | V^-(L) = -t \cdot \mathrm{id} + (1-t) \cdot L$ and $L_t | V^-(L)^\perp = +t \cdot \mathrm{id} + (1-t) \cdot L$. These maps define a deformation $\mathrm{GLS}(n, m) \times I \to \mathrm{GLS}(n, m)$, $(L, t) \mapsto L_t$, of $\mathrm{GLS}(n, m)$ to G_m^n. □

We can now prove Proposition 4.1. The inclusion $\mathbb{R}^N \subsetneq \mathbb{R}^{N+1}$ induces an inclusion $G_m^N \subsetneq G_m^{N+1}$ and we write G_m^∞ for the union of all G_m^N with the direct limit topology. If $k < n - m$ then $\pi_{k-1}(G_m^n) \cong \pi_{k-1}(G_m^\infty)$. This can be proved inductively using the long exact sequence of homotopy groups associated to a fibration. (Remember that $G_m^N \cong O(N)/(O(m) \times O(N-m))$.) Similarly there is an inclusion $G_M^\infty \subsetneq G_{M+1}^\infty$. As usual we write BO for the union of the G_M^∞. If $k < m$ then as above it follows that $\pi_{k-1}(G_m^\infty) \cong \pi_{k-1}(BO)$. Finally, $\pi_{k-1}(BO) \cong \pi_{k-2}(O)$, where O denotes the union of all orthogonal groups $O(N)$. These groups have been computed by Bott, they are periodic in k with period 8. We refer the reader to [H, 15.2.3],for a list of these groups which gives precisely Proposition 4.1. □

Next remember that $\pi_{k-1}(BO) \cong \widetilde{KO}(S^{k-1})$. The element $\mu_F \in \widetilde{KO}(S^{k-1})$ which corresponds to $\beta_F \in \pi_{k-1}(\mathrm{GLS}(n, m))$ is given by the m-dimensional vector bundle which we called the Morse index of $S = K \times \{0\}$ over $K = S^{k-1}$. And the Conley index $\mathscr{C}(S, K)$ is given by the associated sphere bundle with fibre S^m over K; see Remark 2.5(b). Therefore $\mathscr{C}(S, K)$ is fibre homotopy trivial iff μ_F lies in the kernel of the J-homomorphism $J: \widetilde{KO}(K) \to \widetilde{JO}(K)$. This kernel has been computed by Adams et al.; cf. [H, §15.14]. If $K = S^{k-1}$ with $k \equiv 2$ or 3 mod 8 then $\widetilde{KO}(K) \cong \mathbb{Z}/2$ and J is injective. And if $k \equiv 1$ or 5 mod 8 then $\widetilde{KO}(K) \cong \mathbb{Z}$ and the kernel of J consists of the multiples of b_l where $k = 4l + 1$. This proves Theorem 4.2. □

It is interesting to compare this approach with the multiparameter bifurcation results of [A] or [B]. There one was interested in the zeroes of a map $f: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ which was not assumed to be a gradient map. Therefore the map $\lambda \mapsto Df_\lambda(0)$ induces only an element γ_f of $\pi_{k-1}(\mathrm{GL}(n))$ and not of $\pi_{k-1}(\mathrm{GLS}(n))$. If $f(\lambda, x) = -\nabla F_\lambda(x)$ then our bifurcation invariant β_F is mapped to γ_f via the inclusion $\mathrm{GLS}(n) \subsetneq \mathrm{GL}(n)$. Now for $k < n$ the group $\pi_{k-1}(\mathrm{GL}(n)) \cong \pi_{k-1}(O(n))$ is isomorphic to $\pi_{k-1}(O) \cong \pi_k(BO) \cong \widetilde{KO}(S^k)$. Bifurcation of zeroes of f occurs if $J(\gamma_f)$ is not zero in $\widetilde{JO}(S^k)$. This has been proved in [A] in the situation where $K = S^{k-1}$ is the boundary of a small ball around 0 in $\mathbb{R}^k$ and $Df_\lambda(0) \in GL(n)$ for all $\lambda \neq 0$ small. In [B] one can find a proof in a more general situation. The bifurcation index $\mathrm{BI}(f)$ constructed in [B] is precisely $J(\gamma_f)$. There are two important differences between $\mathscr{C}(S, K) = J(\beta_F)$ and $\mathrm{BI}(f) = J(\gamma_f)$. First of all they lie in different groups: $\mathscr{C}(S, K) \in \widetilde{JO}(K) = \widetilde{JO}(S^{k-1})$ and $\mathrm{BI}(f) \in \widetilde{JO}(S^k)$. Since $\beta_F = 0$ implies $\gamma_f = 0$ but not vice versa the Conley index can detect more bifurcation than the bifurcation index. In the trivial case $k = 1$ one has $\widetilde{KO}(S^0) \cong \mathbb{Z}$, $\widetilde{KO}(S^1) \cong \mathbb{Z}/2$ and J is injective in both cases. Here γ_f is the mod 2 reduction of β_F. On the other hand, γ_f can be defined also if f is not a gradient map. In addition, $\mathrm{BI}(f) \neq 0$ yields a global connected branch of zeroes of f bifurcating from $\mathbb{R}^k \times \{0\}$. This need not be true if $\mathscr{C}(S, K) \neq 0$ but $\mathrm{BI}(f) = 0$. In fact it is very well possible that the critical points of F (the zeroes of $f = -\nabla F$) remain in an arbitrarily small neighborhood of a bifurcation point $(\lambda_0, 0)$, say. In this case Theorem 3.4 implies only that there must exist an unbounded family of orbits of the flow associated to $\dot{x} = -\nabla F_\lambda(x)$ connecting the small stationary solutions near $(\lambda_0, 0)$. Such examples can be constructed for $k = 1$ using the methods of [P].

5 Proof of Theorem 3.1

Let $\pi: E \to \Lambda$ represent the Conley index of S over Λ and $\beta: \Lambda \to E$ be the canonical base point section. We have to show the following. For any $\lambda_0 \in \Lambda$ there exists a neighborhood $K = K(\lambda_0)$ and maps $f: E_K = \pi^{-1}(K) \to K \times F$, $g: K \times F \to E_K$, where $F = E_{\lambda_0}$. These maps have to preserve the base points, i.e., $f(\beta(\lambda)) = (\lambda, \beta(\lambda_0))$ and $g(\lambda, \beta(\lambda_0)) = \beta(\lambda)$ for any $\lambda \in K$. Moreover, there exist homotopies $H: E_K \times I \to E_K$ and $\bar{H}: K \times F \times I \to K \times F$ such that

— $H_t(\beta(\lambda)) = H(\beta(\lambda), t) = \beta(\lambda) \quad \forall \lambda \in K, t \in I$

— $H_0 = \mathrm{id}_{E_K}, \quad H_1 = g \circ f$

— $\bar{H}_t(\lambda, \beta(\lambda_0)) = (\lambda, \beta(\lambda_0)) \quad \forall \lambda \in K, t \in I$

— $\bar{H}_0 = \mathrm{id}_{K \times F}, \qquad \bar{H}_1 = f \circ g$.

Since this is a local statement we may assume that the space $p: X \to \Lambda$ over Λ is a product $X = \Lambda \times Y$ with Y locally compact. We identify each fibre X_λ with Y and write φ_λ for the flow on Y coming from X_λ. Similarly, if (N, M) is an index pair for S over Λ we consider the sets S_λ, N_λ and M_λ as subsets of Y. We fix (N, M) and take $E = N/\sim$ as in Defintion 2.4. We need the following lemma due to Salamon.

5.1 Lemma. *There exists a compact, contractible neighborhood K of λ_0 in Λ and times $T_1 > 2T_0 > 0$ such that the following statements hold for the sets*

$$\mathscr{O} := \mathrm{int} \bigcap_{\lambda \in K} (N_\lambda - M_\lambda) \subset Y \qquad \text{and} \qquad \mathscr{P} := \mathrm{cl} \bigcup_{\lambda \in K} (N_\lambda - M_\lambda) \subset Y .$$

(a) *$K \times \mathscr{O}$ and $K \times \mathscr{P}$ are neighborhoods of $S_K = S \cap K \times Y$ and $K \times \mathscr{P}$ is isolating.*

(b) *For all* $y \in Y$, $\lambda, \kappa \in K$ *and* $T \geqq T_1$ *the following holds.*
(i) *If* $\varphi_\lambda(y, [-T_0, T_0]) \subset \mathscr{P}$ *then* $y \in \mathscr{O}$.
(ii) *If* $\varphi_\lambda(y, [0, T]) \subset \mathrm{cl}(N_\kappa - M_\kappa)$ *and* $\varphi_\lambda(y, [T_0, T]) \not\subset \mathscr{O}$ *then* $\varphi_\kappa(\varphi_\lambda(y, T), [0, T]) \cap M_\kappa \neq \varnothing$.

A proof can be found in [S, Lemma 6.6]. □

Now take K and $T \geqq T_1$ as in Lemma 5.1.
For $\lambda, \kappa \in K$ and $y \in E_\lambda$ we say that y satisfies $A(\lambda, \kappa)$ if

— $\varphi_\lambda(y, [0, 2T]) \subset N_\lambda - M_\lambda$ and
— $\varphi_\lambda(y, [T, 3T]) \subset N_\kappa - M_\kappa$ and
— $\varphi_\kappa(\varphi_\lambda(y, 3T), [0, T]) \subset N_\kappa - M_\kappa$.

Then we define $f\colon E_K \to K \times F$ and $g\colon K \times F \to E_K$ as follows

$$f(y) := \begin{cases} (\lambda, \varphi_{\lambda_0}(\varphi_\lambda(y, 3T), T)) & \text{if } y \in E_\lambda \text{ satisfies } A(\lambda, \lambda_0) \\ (\lambda, \beta(\lambda_0)) & \text{otherwise} \end{cases}$$

$$g(\lambda, y) := \begin{cases} \varphi_\lambda(\varphi_{\lambda_0}(y, 3T), T) & \text{if } y \text{ satisfies } A(\lambda_0, \lambda) \\ \beta(\lambda) & \text{otherwise .} \end{cases}$$

We have to verify the continuity of f and g. Since $A(\lambda, \lambda_0)$ is an open condition f is continuous in $y \in E_\lambda$ if y satisfies $A(\lambda, \lambda_0)$. For the same reason f is continuous in $y \in E_\lambda$ if

— $\varphi_\lambda(y, [0, 2T]) \not\subset \mathrm{cl}(N_\lambda - M_\lambda)$ or
— $\varphi_\lambda(y, [T, 3T]) \not\subset \mathrm{cl}(N_{\lambda_0} - M_{\lambda_0})$ or
— $\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), [0, T]) \not\subset \mathrm{cl}(N_{\lambda_0} - M_{\lambda_0})$.

Therefore it suffices to show that if $y \in E_\lambda$ does not satisfy any of these conditions nor $A(\lambda, \lambda_0)$ then $\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), T) \in M_{\lambda_0}$. Remember that M_{λ_0} corresponds to $\beta(\lambda_0) \in N_{\lambda_0}/M_{\lambda_0}$. We have to consider three cases.

1. case: $\varphi_\lambda(y, [0, 2T]) \not\subset N_\lambda - M_\lambda$,
2. case: $\varphi_\lambda(y, [T, 3T]) \not\subset N_{\lambda_0} - M_{\lambda_0}$,
3. case: $\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), [0, T]) \not\subset N_{\lambda_0} - M_{\lambda_0}$.

In the third case we get immediately $\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), T) \in M_{\lambda_0}$.
Since $\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), [0, T]) \subset \mathrm{cl}(N_{\lambda_0} - M_{\lambda_0}) \subset N_{\lambda_0}$ it suffices to show in the first two cases that

$$\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), [0, T]) \cap M_{\lambda_0} \neq \varnothing .$$

According to Lemma 5.1(b) this is the case if

$$\varphi_\lambda(\varphi_\lambda(y, 2T), [0, T]) = \varphi_\lambda(y, [2T, 3T]) \subset \mathrm{cl}(N_{\lambda_0} - M_{\lambda_0})$$

and

$$\varphi_\lambda(\varphi_\lambda(y, 2T), [T_0, T]) = \varphi_\lambda(y, [2T + T_0, 3T] \not\subset \mathscr{O} .$$

Since $\varphi_\lambda(y, [T, 3T]) \subset \mathrm{cl}(N_{\lambda_0} - M_{\lambda_0})$ the first condition holds. In the first case we get $\varphi_\lambda(y, 2T) \in M_\lambda$, hence $\varphi_\lambda(y, 3T) \notin N_\lambda - M_\lambda$. Thus we are done because $\mathscr{O} \subset N_\lambda - M_\lambda$. In the second case we get immediately $\varphi_\lambda(y, 3T) \in M_{\lambda_0}$, thus $\varphi_\lambda(y, 3T) \notin \mathscr{O}$. This proves the continuity of f.

The continuity of g is proved similarly.

Finally, we have to show that $g \circ f$ and $f \circ g$ are homotopic to the identities on E_K and $K \times F$, respectively. For $y \in E_\lambda \subset E_K$

$$g \circ f(y) = \varphi_\lambda(\varphi_{\lambda_0}(\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), T), 3T), T)$$

$$= \varphi_\lambda(\varphi_{\lambda_0}(\varphi_\lambda(y, 3T), 4T), T)$$

if this lies in $N_\lambda - M_\lambda$ and $g \circ f(y) = \beta(\lambda)$ otherwise. Since K is contractible there exists a deformation $h: K \times I \to K$ with $h(\kappa, 0) = \lambda_0$ and $h(\kappa, 1) = \kappa$. We set for $y \in E_\lambda$

$$H(y, s) := \varphi_\lambda(\varphi_{h(\lambda, s)}(\varphi_\lambda(y, 3T), 4T), T)$$

if this lies in $N_\lambda - M_\lambda$ and $H(y, s) = \beta(\lambda)$ otherwise. Obviously, $H(y, 0) = g \circ f(y)$ and $H(y, 1) = \varphi_\lambda(y, 8T)$ which is homotopic to the identity. The proof that H is continuous and that f, g respect the basepoints is left to the reader as well as the proof that $f \circ g$ is fibre homotopic to the identity. In fact, using some results of Dold it is not necessary to prove that $f \circ g \simeq \mathrm{id}$. One can argue as follows. Since $g \circ f \simeq \mathrm{id}$ the space $E_K \to K$ over K is dominated by the trivial space $K \times F \to K$ over K ([D, 1.3]). This implies that $E_K \to K$ has the WCHP ([D, Corollary 5.3]). And since K is contractible it follows that $E_K \to K$ is fibre homotopy equivalent to the trivial space $K \times F$ ([D, Theorem 6.4]). □

The maps f, g and H are modifications of maps constructed by Salamon. We refer the reader in particular to Theorem 6.7 and Corollary 6.8 of [S].

6 Proof of the Theorems 3.3 and 3.4

The very short and nearly formal proofs of this section indicate the suitability of our definitions. We start with the following result.

6.1 Proposition. *Let $\pi: E \to \Lambda$ have the (based) WCHP and let $f, g: K \to \Lambda$ be homotopic. Then f^*E and g^*E are (based) fibre homotopy equivalent.*

Proof. Choose a homotopy $h: K \times I \to \Lambda$ between $h_0 = f$ and $h_1 = g$. Then f^*E is isomorphic to the part of h^*E over $K \times \{0\}$ and g^*E is isomorphic to the part of h^*E over $K \times \{1\}$. Furthermore, $h^*E \to K \times I$ has the WCHP since this property is preserved by pull backs. Now we can apply Corollary 6.6 of [D] which says that $h^*E | K \times \{0\}$ and $h^*E | K \times \{1\}$ are fibre homotopy equivalent. The fibre homotopy equivalences defined in [D] preserve the base points if π has the based WCHP. □

To prove Theorem 3.3 let $i: K \hookrightarrow \Lambda$ denote the inclusion and $r: \Lambda \to K$ the retraction. Since $i \circ r$ is homotopic to the identity we can apply Proposition 6.1 and get

$$\mathscr{C}(S, \Lambda) = (i \circ r)^* \mathscr{C}(S, \Lambda) = r^* i^* \mathscr{C}(S, \Lambda) = r^* \mathscr{C}(S, K) . \qquad \square$$

To prove Theorem 3.4 suppose there exists a compact isolated invariant set T of X such that $T \cap X_K = S$. Then $\mathscr{C}(T, K) = \mathscr{C}(S, K)$ is not fibre homotopy trivial.

On the other hand, $\mathscr{C}(T, K) = i^*\mathscr{C}(T, \Lambda)$ is fibre homotopy trivial according to Proposition 6.1 since $i\colon K \hookrightarrow \Lambda$ is homotopic to the constant map. $\square$

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