Math. Z. 209, 75–88 (1992)

Mathematische Zeitschrift

Stability of strongly continuous representations of abelian semigroups*

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Received September 21, 1990, in final form April 16, 1991

1 Introduction

Let $\{T(t) : t \geqq 0\}$ be a bounded, strongly continuous, one-parameter semigroup of operators on a Banach space X, and let A be its generator. If $\sigma(A) \cap i\mathbf{R}$ is countable and $P\sigma(A^*) \cap i\mathbf{R}$ is empty, then $\|T(t)x\| \to 0$ as $t \to \infty$, for each x in X. In the norm-continuous case, this was proved in [20]. In the strongly continuous case, two very different proofs were given, independently and simultaneously, in [2] and [14], and a third proof has subsequently been given in [8]. Various extensions of this result have appeared in [4, 5, 15, 16].

Now suppose that T is a bounded representation on X of a suitable locally compact abelian semigroup S. For example, S might be $\mathbf{R}^n_+$, so that $T(t_1, \ldots, t_n) = T_1(t_1) \ldots T_n(t_n)$, where $\{T_j : j = 1, \ldots, n\}$ are commuting bounded C_0-semigroups. Even in the norm-continuous case, there are several possible notions of the (joint) spectrum $\mathrm{Sp}(T)$ of T. However, the stability theorem described above involves only the unitary (purely imaginary) part of the spectrum, where there is little ambiguity (see Propos. 2.2). In [16], it was shown that if T is norm-continuous, the unitary part of $\mathrm{Sp}(T)$ is countable, and the unitary part of $P\sigma(T^*)$ is empty, then $\|T(t)x\| \to 0$ as $t \to \infty$ through the semigroup S.

In this paper, we extend this result to strongly continuous representations. We assume that S is embedded in a locally compact abelian group G, and we take as spectrum an analogue of the spectrum of an isometric representation of G, which can be identified with the Gelfand spectrum of a commutative Banach algebra. We use the method of [14, 16] to construct an isometric representation U of S on a different space E (Propos. 3.1). Then we use a Banach algebra construction of Arens to obtain an isometric representation V of G on another space Y (Propos. 3.2). Knowing that $\mathrm{Sp}(V)$ is non-empty,

* Part of this work was carried out while the authors were visiting the University of Franche-Comté, Besançon, and Hokkaido University, Sapporo, respectively

we can deduce that $\mathrm{Sp}(U)$ is non-empty. For $S = \mathbf{R}^n_+$, this shows that a finite number of commuting C_0-semigroups of isometries have a common sequence of approximate eigenvectors, a fact which may be of independent interest. Using some facts about Silov boundaries, we also show that, if $\mathrm{Sp}(U)$ is countable, then U has an eigenspace which is complemented in E. This in turn leads to our main result (Theor. 4.2) and a more general almost periodicity theorem (Theor. 5.1).

The same constructions enable us to show (Theor. 4.3) that, if f in $L^1(S)$ is of spectral synthesis for the unitary part of $\mathrm{Sp}(T)$ (which may now be uncountable), then $\|T(t)\hat{f}(T)\| \to 0$ as $t \to \infty$. This fact was first proved in [9] for $S = N$, and then in [8, 18] (independently) for $S = \mathbf{R}_+$, and in [18] when T is norm-continuous.

In its original form, this article contained Section 2 and some incomplete versions of Proposition 4.1 and Theorems 4.2 and 5.1. At that time, we were unable to prove Corollary 3.3. The idea of using the Arens construction was introduced later by the second author to establish Theorem 4.3 for norm-continuous representations. He is publishing this in a separate article [18]. The first author saw how the same idea could be used to obtain both Corollary 3.3 (and hence Theor. 4.2 and 5.1 in their full generality) and Theorem 4.3 for strongly continuous representations. In a complicated situation, it was decided to rewrite the present paper jointly, while allowing [18] to go forward for publication elsewhere.

2 The Spectrum

Let S be a subsemigroup of a locally compact, abelian, group G. Let S^* be the space of continuous, bounded, characters of S; thus S^* consists of the non-zero, continuous, bounded, homomorphisms of S into the multiplicative semigroup $\mathbf{C}$. Let

$$S^*_u = \{\chi \in S^* : |\chi(s)| = 1 \quad \text{for all } s \text{ in } S\} .$$

We assume that S is measurable with non-empty interior S° in G, and we consider S with the restriction of Haar measure on G. For f in $L^1(S)$, χ in S^*, let

$$\hat{f}(\chi) = \int_S f(t)\chi(t)\,dt .$$

We assume that the functions $\hat{f}$ $(f \in L^1(S))$ separate the points of S^* from each other and from 0. For example, this is satisfied if S° is dense in S. Without loss of generality, we assume that $G = S - S$. Then each χ in S^*_u extends uniquely to a character in the dual group $\hat{G}$, so we may identify S^*_u with a subset of $\hat{G}$.

Note that S satisfies Følner's condition: there is a net (Ω_α) of compact subsets of S such that $\frac{|(\Omega_\alpha + t)\triangle\Omega_\alpha|}{|\Omega_\alpha|} \to 0$ uniformly for t in compact subsets of S, where $|\Omega|$ is the Haar measure of Ω. The locally compact, abelian, semigroup S° satisfies the condition [17, pp. 131, 145], and it follows easily that S satisfies it. Moreover if S is σ-compact, then the net may be chosen to be a sequence.

In view of the results of [16], we shall not primarily be interested in the case when S is discrete. Indeed, the basic example which we have in mind is $S = \mathbf{R}^n_+ = [0, \infty)^n$. Note, however, that we do not assume that $0 \in S$, so our results also cover examples such as $S = (0, \infty)^n$ or $S = [1, \infty)^n$. In each of these cases, we shall identify S^* with $\mathbf{C}^n_-$, where $\mathbf{C}_- = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \leqq 0\}$. Here a character χ in S^* is identified with $z = (z_1, \dots, z_n)$, where

$$\chi(t_1, \dots, t_n) = \chi(t) = e^{t \cdot z} = \exp(t_1 z_1 + \cdots + t_n z_n) .$$

Thus $S_u^* = i\mathbf{R}^n$.

Let $T : S \to \mathscr{B}(X)$ be a representation of S on a Banach space X. Thus T is a strongly continuous homomorphism of S into the Banach algebra $\mathscr{B}(X)$ of bounded linear operators on X. We shall always assume that T is bounded, so that there is a constant M such that $\|T(t)\| \leqq M$ for all t in S. We shall also consider the adjoint operators $T^*(t) = T(t)^*$ on X^*, but we note that T^* may not be a representation of S, as strong continuity may fail.

A character χ in S^* is said to be an *eigenvalue of* T if there exists a non-zero vector x in X such that $T(t)x = \chi(t)x$ for all t in S; an *eigenvalue of* T^* if there exists a non-zero functional ϕ in X^* such that $T^*(t)\phi = \chi(t)\phi$ for all t; an *approximate eigenvalue* of T if there exists a net (x_α) in X with $\|x_\alpha\| = 1$ such that $\|T(t)x_\alpha - \chi(t)x_\alpha\| \to 0$ uniformly for t in each compact subset of S; an *ω-approximate eigenvalue of* T if there exists a sequence (x_n) in X with $\|x_n\| = 1$ such that $\|T(t)x_n - \chi(t)x_n\| \to 0$ for each t in S. We shall denote by $P\sigma(T)$, $P\sigma(T^*)$, $A\sigma(T)$ and $A_\omega\sigma(T)$ the sets of eigenvalues of T, eigenvalues of T^*, approximate eigenvalues of T, and ω-approximate eigenvalues of T, respectively. For a single operator U (bounded or unbounded) on X, we shall denote the spectrum, point spectrum, and approximate point spectrum of U by $\sigma(U)$, $P\sigma(U)$, and $A\sigma(U)$, respectively.

For f in $L^1(S)$, let

$$\hat{f}(T) = \int_S f(t)T(t)\,dt .$$

(The integral exists as a strongly convergent Bochner integral.) The map $f \mapsto \hat{f}(T)$ is a homomorphism between the Banach algebras $L^1(S)$ and $\mathscr{B}(X)$. The *spectrum* of T is defined to be

$$\operatorname{Sp}(T) = \left\{\chi \in S^* : |\hat{f}(\chi)| \leqq \|\hat{f}(T)\| \quad \text{for all } f \text{ in } L^1(S)\right\} .$$

Note that, if $S = G$, then $\operatorname{Sp}(T)$ is the finite L-spectrum of T [12] or the Arveson spectrum of T [6].

We shall be primarily interested in the unitary part of the spectrum. Thus we let $\operatorname{Sp}_u(T) = \operatorname{Sp}(T) \cap S_u^*$, $P\sigma_u(T) = P\sigma(T) \cap S_u^*$, etc. If T is bounded, $\chi \in P\sigma_u(T)$, $x \in X \setminus (0)$ with $T(t)x = \chi(t)x$ $(t \in S)$, $\phi \in X^*$ with $\phi(x) = 1$, and $\phi_\alpha = |\Omega_\alpha|^{-1} \int_{\Omega_\alpha} \overline{\chi(t)} T^*(t)\phi\,dt$, where (Ω_α) is a Følner net, then any limit point ψ of (ϕ_α) satisfies $T^*(t)\psi = \chi(t)\psi$ and $\psi(x) = 1$. Thus $P\sigma_u(T) \subseteq P\sigma_u(T^*)$.

To understand the notion of spectrum, it is helpful first to consider the example of multipliers.

Example 2.1 Let $S = \mathbf{R}^n_+$, $X = L^p(\Omega, \mu)$ for some measure space (Ω, μ) and some $1 \leqq p < \infty$, and let $h : \Omega \to \mathbf{C}^n_-$ be a measurable function. Define

$$(T(t)g)(\omega) = e^{t \cdot h(\omega)} g(\omega) \qquad (g \in X, \omega \in \Omega) .$$

For f in $L^1(\mathbf{R}^n_+)$,

$$\widehat{f}(T)g = \widetilde{f}(-h)g,$$

where $\widetilde{f}$ is the Laplace transform of f:

$$\widetilde{f}(z) = \int_{\mathbf{R}^n_+} e^{-t \cdot z} f(t)\, dt \qquad (z \in \mathbf{C}^n_+) .$$

Thus

$$\|\widehat{f}(T)\| = \operatorname*{ess\,sup}_{\omega \in \Omega} |\widetilde{f}(-h(\omega))|,$$

and, for z in $\mathbf{C}^n_-$,

$$z \in \mathrm{Sp}(T) \quad \Leftrightarrow \quad |\widetilde{f}(-z)| \leqq \operatorname*{ess\,sup}_{\omega \in \Omega} |\widetilde{f}(-h(\omega))| \qquad \text{for all } f \text{ in } L^1(\mathbf{R}^n_+) .$$

Since $\widetilde{f}$ is holomorphic in $\mathbf{C}^n_-$, it follows from the Maximum Modulus Principle that $\mathrm{Sp}(T)$ contains any compact subset of $\mathbf{C}^n_-$ whose boundary is contained in the essential range of h.

When $n = 1$, T is a C_0-semigroup whose generator A is given by $Ag = hg$, whenever $hg \in X$. The spectrum $\sigma(A)$ of A is the essential range of h. Thus $\sigma(A) \subseteq \mathrm{Sp}(T)$, but the inclusion may be strict.

For general n,

$$\mathrm{Sp}_u(T) = \{iy \in i\mathbf{R}^n : iy \quad \text{is in the essential range of } h\},$$
$$P\sigma(T) = P\sigma(T^*) = \{z \in \mathbf{C}^n_- : \mu\{\omega : h(\omega) = z\} > 0\} .$$

It follows that if $\mathrm{Sp}_u(T)$ is countable and $P\sigma_u(T)$ (or $P\sigma\widehat{u}(T^*)$) is empty, then, for almost all ω, $\mathrm{Re}\, h_j(\omega) < 0$ for some j. Hence T is stable, in the sense that

$$\lim_{t_j \to \infty} \|T(t)g\| = 0 \qquad (g \in X) .$$

Thus for fixed t in $(0, \infty)^n$, the one-parameter semigroup $T_t(\tau) = T(\tau t)$ $(\tau \geqq 0)$ is stable.

Now, take $n = 2$,

$$\Omega = \{(x_1 + iy_1, x_2 + iy_2) \in \mathbf{C}^2_- : x_1 x_2 (1 + y_1^2)(1 + y_2^2) \geqq 1\},$$
$$\mu = \text{ four-dimensional Lebesgue measure,}$$
$$h(z_1, z_2) = (z_1, z_2) .$$

Then $\mathrm{Sp}(T) = \Omega$, $\mathrm{Sp}_u(T)$ is empty, T is stable, but, for each one-parameter subgroup T_t $(t \in \mathbf{R}^2_+)$, the spectrum of the generator is $\mathbf{C}_-$. So the stability theorem for C_0-semigroups [2, 14] is not applicable. In this example, the one-parameter semigroups $\tau \mapsto T(\tau, 0)$, $\tau \mapsto T(0, \tau)$ are both stable, but if we take $\Omega' = \Omega \cup (i\mathbf{R} \times \{-1\}) \cup (\{-1\} \times i\mathbf{R})$, μ' to coincide with μ on Ω and with one-dimensional Lebesgue measure on $i\mathbf{R} \times \{-1\}$ and on $\{-1\} \times i\mathbf{R}$, and h to be the identity function, then the same properties hold without the subsemigroups $T(\tau, 0)$, $T(0, \tau)$ being stable.

Proposition 2.2 *Let T be a bounded representation of S. Then $A_\omega\sigma(T)$, $A\sigma(T)$ and $P\sigma(T^*)$ are all contained in* $\mathrm{Sp}(T)$. *Moreover, $A\sigma_u(T) = \mathrm{Sp}_u(T)$, and, if S is σ-compact, then $A_\omega\sigma_u(T) = \mathrm{Sp}_u(T)$.*

Proof. Suppose first that $\chi \in A_\omega\sigma_u(T)$, so there is a sequence (x_n) in X with $\|x_n\| = 1$ such that $\|T(t)x_n - \chi(t)x_n\| \to 0$. Then, for f in $L^1(S)$,

$$\|\hat{f}(T)x_n - \hat{f}(\chi)x_n\| = \left\| \int_S (T(t)x_n - \chi(t)x_n) f(t)\, dt \right\| \to 0$$

by the Dominated Convergence Theorem. Thus

$$\|\hat{f}(T)\| \geqq \lim_{n\to\infty} \|\hat{f}(T)x_n\| = \lim_{n\to\infty} \|\hat{f}(\chi)x_n\| = |\hat{f}(\chi)| .$$

Next, suppose that $\chi \in A\sigma(T)$, and consider f in $L^1(S)$ with compact support K. There is a net of unit vectors (x_α) such that $\|T(t)x_\alpha - \chi(t)x_\alpha\| \to 0$ uniformly on K, so $\|\hat{f}(T)x_\alpha - \hat{f}(\chi)x_{\hat{\alpha}}\| \to 0$. Hence $\|\hat{f}(T)\| \geqq |\hat{f}(\chi)|$. Since the functions of compact support are dense in $L^1(S)$ and $|\hat{f}(\chi)| \leqq \|f\|_1$, $\|\hat{f}(T)| \leqq M\|f\|_1$, it follows that $\chi \in \mathrm{Sp}(T)$.

Next, suppose that $\chi \in P\sigma(T^*)$, so there exists ϕ in X^* with $\|\phi\| = 1$ and $T^*(t)\phi = \chi(t)\phi$ for all t. Then

$$\hat{f}(T)^*\phi = \int_S f(t)T^*(t)\phi\, dt = \hat{f}(\chi)\phi ,$$

so $|\hat{f}(\chi)| \leqq \|\hat{f}(T)^*\| = \|\hat{f}(T)\|$. Thus $\chi \in \mathrm{Sp}(T)$.

Finally, suppose that $\chi \in \mathrm{Sp}_u(T)$. For each compact subset Ω of S, define f_Ω in $L^1(S)$ by

$$f_\Omega(t) = \begin{cases} \dfrac{\overline{\chi(t)}}{|\Omega|} & \text{if } t \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

By assumption, $\|\hat{f}_\Omega(T)\| \geqq |\hat{f}_\Omega(\chi)| = 1$. Hence there exists y_Ω in X such that $\|y_\Omega\| \leqq 2$ and $\|x_\Omega\| = 1$, where

$$x_\Omega = \frac{1}{|\Omega|} \int_\Omega \overline{\chi(t)} T(t) y_\Omega\, dt .$$

Then

$$\|T(t)x_\Omega - \chi(t)x_\Omega\| = \frac{1}{|\Omega|}\left\| \int_{\Omega+t} \overline{\chi(s)}\chi(t)T(s)y_\Omega\, ds - \int_{\Omega} \overline{\chi(s)}\chi(t)T(s)y_\Omega\, ds \right\|$$
$$\leqq \frac{M|(\Omega+t)\triangle\Omega|}{|\Omega|}.$$

It follows from Følner's condition that $\chi \in A\sigma(T)$ ($\chi \in A_\omega\sigma(T)$ if S is σ-compact).

Corollary 2.3 *If* $S = \mathbf{R}_+$, *and* A *is the infinitesimal generator of* T, *then* $\sigma(A) \subseteq \mathrm{Sp}(T)$, *and* $\mathrm{Sp}_u(T) = \sigma(A) \cap i\mathbf{R}$.

In the main stability results, we shall be concerned only with the unitary part of the spectrum. Proposition 2.2 shows that, for these purposes, it matters not at all whether we consider the full spectrum which we have introduced, or merely the approximate point spectrum. The advantage of considering $\mathrm{Sp}(T)$ is given by Proposition 2.4, which allows us to use Banach algebra techniques.

Given a bounded representation T of S on X, let $\mathscr{A}_T$ be the closure of $\{\widehat{f}(T) : f \in L^1(S)\}$ in $\mathscr{B}(X)$ for the norm topology. Then $\mathscr{A}_T$ is a commutative Banach algebra. As $\mathscr{A}_T$ may be non-unital, we shall also introduce the notation $\widetilde{\mathscr{A}}_T$ for $\mathscr{A}_T + \mathbf{C}\cdot I_X$.

Proposition 2.4 *There is a bijective correspondence between points* χ *in* $\mathrm{Sp}(T)$ *and characters* ϕ *of* $\mathscr{A}_T$, *given by:* $\phi(\widehat{f}(T)) = \widehat{f}(\chi)$.

Proof. If $\chi \in \mathrm{Sp}(T)$ and $\phi_\chi(\widehat{f}(T)) = \widehat{f}(\chi)$, then, by definition of $\mathrm{Sp}(T)$, ϕ_χ is bounded, and therefore extends to a multiplicative functional on $\mathscr{A}_T$. By assumption on S, ϕ_χ is non-zero, and the map $\chi \mapsto \phi_\chi$ is injective.

Suppose that ϕ is a character of $\mathscr{A}_T$. There is a character ψ of $L^1(S)$ given by $\psi(f) = \phi(\widehat{f}(T))$. Exactly as for groups [19, p.8], it can be seen that every character of $L^1(S)$ arises from a character of S, so there exists χ in S^* such that $\psi(f) = \widehat{f}(\chi)$. Thus $\chi \in \mathrm{Sp}(T)$ and $\phi = \phi_\chi$.

In the sequel, we shall identify $\mathrm{Sp}(T)$ with the character space $\widehat{\mathscr{A}}_T$. Thus we write $\chi(\widehat{f}(T))$ for $\widehat{f}(\chi)$. In the topology induced from $\widehat{\mathscr{A}}_T$, $\mathrm{Sp}(T)$ is locally compact, and the identification of $\mathrm{Sp}_u(T)$ with a subset of $\widehat{G}$ is a homeomorphism [see 19, p.10]). Let Γ_T be the Silov boundary of $\mathscr{A}_T$.

Proposition 2.5 *For* χ *in* Γ_T, *there is a net* (x_α) *in* X *with* $\|x_\alpha\| = 1$ *such that* $\|T(t)x_\alpha - \chi(t)x_\alpha\| \to 0$ *for all* t *in* S *and* $\|\widehat{f}(T)x_\alpha - \widehat{f}(\chi)x_\alpha\| \to 0$ *for all* f *in* $L^1(S)$.

Proof. By Zelazko's Theorem [22] [see also 7], there is a net (U_α) in $\mathscr{A}_T$ with $\|U_\alpha\| = 1$ such that $\|\widehat{f}(T)U_\alpha - \widehat{f}(\chi)U_\alpha\| \to 0$ for all f in $L^1(S)$. Choose y_α in X such that $\|y_\alpha\| \leqq 2$ and $\|U_\alpha y_\alpha\| = 1$. Let $x_\alpha = U_\alpha y_\alpha$, so that $\|x_\alpha\| = 1$. For any f in $L^1(S)$,

$$\|\widehat{f}(T)x_\alpha - \widehat{f}(\chi)x_\alpha\| \leqq \|\widehat{f}(T)U_\alpha - \widehat{f}(\chi)U_\alpha\|\ \|y_\alpha\| \to 0.$$

For fixed t in S, let

$$f_t(s) = \begin{cases} f(s-t) & \text{if } s-t \in S \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\widehat{f}_t(\chi) = \int_{S+t} \chi(s) f(s-t)\, ds = \int_S \chi(s+t) f(s)\, ds = \chi(t)\widehat{f}(\chi), \quad \widehat{f}_t(T) = T(t)\widehat{f}(T).$$

Thus

$$\|T(t)\widehat{f}(T)x_\alpha - \chi(t)\widehat{f}(\chi)x_\alpha\| \to 0 .$$

But

$$\|T(t)\widehat{f}(T)x_\alpha - \widehat{f}(\chi)T(t)x_\alpha\| \;\to\; 0 .$$

Hence

$$|\widehat{f}(\chi)| \; \|T(t)x_\alpha - \chi(t)x_\alpha\| \;\to\; 0 .$$

Choosing any f in $L^1(S)$ such that $\widehat{f}(\chi) \neq 0$, it follows that $\|T(t)x_\alpha - \chi(t)x_\alpha\| \to 0$.

3 Two constructions

In this section, we describe two constructions which reduce the study of asymptotic behaviour of bounded representations of semigroups to that of isometric representations of groups. The first construction was used in [14]; it was called the *limit isometric representation* in [18].

Let T be a bounded representation of a semigroup S on X. We regard S as being ordered by: $s \preceq t$ if $t - s \in S$. We put

$$X_s(T) = \left\{ x \in X : \inf_{t \in S} \|T(t)x\| = 0 \right\} = \left\{ x \in X : \lim_{t \in S} \|T(t)x\| = 0 \right\}.$$

Proposition 3.1 *Let T be a bounded representation of S on a Banach space X. There exist a Banach space E, a bounded linear map Q of X into E with dense range, and a representation U of S by isometries on E, with the following properties:*

(1) *If $x \in X$ and $Qx = 0$, then $x \in X_s(T)$;*
(2) *$QT(t) = U(t)Q \quad (t \in S)$;*
(3) $\mathrm{Sp}(U) \subseteq \mathrm{Sp}(T)$, $P\sigma(U^*) \subseteq P\sigma(T^*)$.

Proof. Replacing the norm by the equivalent norm

$$\|x\|_T = \sup\Big(\{\|T(t)x\| : t \in S\} \cup \{\|x\|\} \Big),$$

we may assume that $\|T(t)\| \leqq 1$. Let

$$\ell(x) = \inf_{t \in S} \|T(t)x\| = \lim_{t \in S} \|T(t)x\| .$$

Then ℓ is a seminorm on X, with $\ell(x) \leqq \|x\|$ and $\ell^{-1}(0) = X_s(T)$. Let $\tilde{\ell}$ be the induced norm on $X/X_s(T)$, and $(E, \|\cdot\|_E)$ be the completion of $(X/X_s(T), \tilde{\ell})$. Let $Q : X \to X/X_S(T) \subseteq E$ be the canonical map. It is clear that Q is bounded with dense range, and (1) is satisfied.

It is also clear that $\ell(T(t)x) = \ell(x)$ for all t and x. Hence $T(t)$ induces an isometry $U(t)$ on E satisfying (2). Moreover,

$$\tilde{\ell}(U(t)Qx - U(s)Qx) \leqq \|T(t)x - T(s)x\|,$$

so U is strongly continuous.

For f in $L^1(S)$, $\hat{f}(U)Q = Q\hat{f}(T)$. Hence, for x in X,

$$\begin{aligned}\tilde{\ell}(\hat{f}(U)Qx) = \ell(\hat{f}(T)x) = \inf_{t\in S} \|T(t)\hat{f}(T)x\| &\leqq \inf_{t\in S} \|\hat{f}(T)\| \ \|T(t)x\| \\ &= \|\hat{f}(T)\|\ell(x) \\ &= \|\hat{f}(T)\|\tilde{\ell}(Qx)\,.\end{aligned}$$

Thus, $\|\hat{f}(U)\| \leqq \|\hat{f}(T)\|$. Hence, $\mathrm{Sp}(U) \subseteq \mathrm{Sp}(T)$. Moreover, if $\psi \in E^*$, $\psi \neq 0$, and $U^*(t)\psi = \chi(t)\psi$, then $Q^*\psi \in X^*$, $Q^*\psi \neq 0$, and $T^*(t)(Q^*\psi) = \chi(t)Q^*\psi$. Thus $P\sigma(U^*) \subseteq P\sigma(T^*)$.

The second construction is a variant of one used in [18].

Proposition 3.2 *Let U be a representation of S by isometries on a Banach space E. There exist a Banach space Y containing $\mathscr{A}_U$ and a representation V of G by isometries on Y such that*

(1) $\hat{g}(V)(\hat{f}(U)) = \hat{g}(U)\hat{f}(U) \quad (f, g \in L^1(S))$;
(2) $\mathrm{Sp}(V) \subseteq \mathrm{Sp}_u(U)$.

Proof. Let $\mathscr{A}_U^+$ be the Banach subalgebra of $\mathscr{B}(E)$ generated by $\mathscr{A}_U$, $\{U(t) : t \in S\}$, and the identity operator.

Let $t_1, t_2, \ldots, t_n \in S$, $A_0, A_1, \ldots, A_k \in \mathscr{A}_U^+$, $m_{ij} \in \mathbf{N}$ $(i = 0, 1, \ldots, k;$ $j = 1, 2, \ldots, n)$, and suppose that

$$A_0 U(m_{01}t_1 + \cdots + m_{0n}t_n) = \sum_{i=1}^{k} A_i U(m_{i1}t_1 + \cdots + m_{in}t_n)\,.$$

Then

$$\begin{aligned}\|A_0\| &= \sup\{\|A_0x\| : \|x\| = 1\} \\ &= \sup\{\|U(m_{01}t_1 + \cdots + m_{0n}t_n)A_0x\| : \|x\| = 1\} \\ &\leqq \sum_{i=1}^{k} \sup\{\|U(m_{i1}t_1 + \cdots + m_{in}t_n)A_ix\| : \|x\| = 1\} \\ &= \sum_{i=1}^{k} \|A_i\|\,.\end{aligned}$$

It follows from a construction of Arens [3, Theor. 3.93] that there is a commutative Banach algebra $\mathscr{B}$, with identity, containing $\mathscr{A}_U^+$, in which each $U(t)$ is invertible. For t_1, t_2 in S, define

$$U(t_1 - t_2) = U(t_1)U(t_2)^{-1}.$$

Then $\{U(t) : t \in G\}$ is a well-defined group of elements of norm one in $\mathscr{B}$.

Let Y be the closed linear span in $\mathscr{B}$ of

$$\{U(t)\hat{f}(U) : t \in G, f \in L^1(S)\}.$$

For s in G, define $V(s) : Y \to Y$ by: $V(s)(A) = U(s)A$ $(A \in Y)$. Then V is a strongly continuous representation of G by isometries on Y.

Let $f, g \in L^1(S), t \in G$. Then

$$\begin{aligned}\hat{g}(V)\Big(U(t)\hat{f}(U)\Big) &= \int_S g(s)V(s)\Big(U(t)\hat{f}(U)\Big)\,ds\\ &= \int_S g(s)U(t)U(s)\hat{f}(U)\,ds\\ &= U(t)\int_S g(s)U(s)\hat{f}(U)\,ds,\end{aligned}$$

where the integrals are convergent in the norm of $\mathscr{B}$. But, for x in E,

$$\Big(\int_S g(s)U(s)\hat{f}(U)\,ds\Big)x = \int_S g(s)U(s)\Big(\hat{f}(U)x\Big)\,ds = \Big(\hat{g}(U)\hat{f}(U)\Big)x,$$

where the second integral is convergent in the norm of E. Thus

$$\hat{g}(V)\Big(U(t)\hat{f}(U)\Big) = U(t)\hat{g}(U)\hat{f}(U) = \hat{g}(U)U(t)\hat{f}(U)\,. \qquad (*)$$

Now (1) follows immediately on taking $t = 0$.

Let $\chi \in \mathrm{Sp}(V)$, so that

$$|\hat{g}(\chi)| \leqq \|\hat{g}(V)\| \quad (g \in L^1(G)).$$

For g in $L^1(S)$, it follows from (*) that

$$\hat{g}(V)(A) = \hat{g}(U)A \quad (A \in Y),$$

so $\|\hat{g}(V)\| \leqq \|\hat{g}(U)\|$. Thus

$$|\hat{g}(\chi)| \leqq \|\hat{g}(U)\| \qquad (g \in L^1(S)),$$

so $\chi \in \mathrm{Sp}(U)$. Since V is a representation of the group G, $\mathrm{Sp}(V)$ consists of unitary characters.

Corollary 3.3 *Let U be a representation of S by isometries on a non-zero Banach space E. Then* $\mathrm{Sp}(U)$ *is non-empty. Moreover,* $A\sigma_u(U)$ *is non-empty. If S is σ-compact, then* $A_\omega\sigma_u(U)$ *is non-empty.*

Proof. The space Y constructed in Proposition 3.2 is non-zero, so the spectral theory of isometric representations of groups [6, 12] shows that $\mathrm{Sp}(V)$ is non-empty. Hence, $\mathrm{Sp}_u(U)$ is non-empty, by Proposition 3.2(2). The results now follow from Corollary 2.2.

Corollary 3.3 was previously known in the cases when T is norm-continuous, in particular when S is discrete, (essentially due to Lyubich, [12]), and when $S = \mathbf{R}_+$ [14], but we believe it to be a new result for $S = \mathbf{R}^n_+$. We shall need it to prove our main result, Theorem 4.2. From that result, it will follow that Corollary 3.3 is valid, not only when U is isometric, but whenever U is not stable. In the case when $S = \mathbf{R}_+$, a particularly elementary proof of this is contained in [2]. In that case, we can formulate Corollary as follows.

Corollary 3.4 *Let* $\{U_j(\tau) : \tau \geqq 0\}$ $(j = 1, \dots, n)$ *be commuting C_0-semigroups of isometries on a non-zero Banach space E. There exists a sequence (x_r) of unit vectors in E and real numbers $\lambda_1, \dots, \lambda_n$ such that*

$$\|U_j(\tau)x_r - e^{i\lambda_j\tau}x_r\| \to 0$$

as $\tau \to \infty$, uniformly for τ in compact subsets of $\mathbf{R}_+$, $j = 1, \dots, n$.

4 Stability

The following proposition is needed for the proofs of Theorems 4.2 and 5.1. In fact, the latter theorem will give more information about the situation of this proposition.

Proposition 4.1 *Let U be a representation of S by isometries on a non-zero Banach space E, and suppose that* $\mathrm{Sp}_u(U)$ *is countable. Then there exist a non-zero projection P in $\widetilde{\mathscr{A}}_U$ and a character χ_0 in* $\mathrm{Sp}_u(U)$ *such that $U(t)P = \chi_0(t)P$ for all t in S. In particular, χ_0 is an eigenvalue of both U and U^*.*

Proof. As in Section 2, we identify $\mathrm{Sp}(U)$ with $\widehat{\mathscr{A}}_U$, and we consider the Silov boundary Γ_U. By Corollary 3.3, $\mathrm{Sp}(U)$ is non-empty, so $\mathscr{A}_U$ is not radical, and Γ_U is non-empty. By Proposition 2.5, each χ in Γ_U satisfies:

$$|\chi(t)| = \lim_\alpha \|\chi(t)x_\alpha\| = \lim_\alpha \|U(t)x_\alpha\| = 1$$

for some net (x_α) of unit vectors. Thus Γ_U is contained in $\mathrm{Sp}_u(U)$, so Γ_U is countable. Since Γ_U is locally compact, it follows that Γ_U has an isolated point χ_0. By [21, p.55], χ_0 is isolated in $\widehat{\mathscr{A}}_U = \mathrm{Sp}(U)$. By Silov's Idempotent Theorem, there exists P in $\widetilde{\mathscr{A}}_U$ such that $P^2 = P$, $\chi_0(P) = 1$, and $(P\widetilde{\mathscr{A}}_U)\widehat{} = \{\chi_0 | P\widetilde{\mathscr{A}}_U\}$.

Take t in S and λ in $A\sigma(U(t)|PE)$. Choose f in $L^1(S)$ such that $\hat{f}(\chi_0) \neq 0$, and let $g = f_t - \lambda f$, where f_t is as in the proof of Proposition 2.5. There is a sequence (x_n) in PE such that $\|x_n\| = 1$ and $\|U(t)x_n - \lambda x_n\| \to 0$. Then

$$\|\hat{g}(U)x_n\| = \|\hat{f}(U)(U(t)x_n - \lambda x_n)\| \to 0.$$

Thus $P\widehat{g}(U)$ is not invertible in $P\widetilde{\mathcal{A}_U}$, so there is a character χ in $(P\widetilde{\mathcal{A}_U})^{\wedge}$ such that $\chi(P\widehat{g}(U)) = 0$. Since $\chi = \chi_0 | P\widetilde{\mathcal{A}_U}$, it follows that

$$0 = \chi_0(P\widehat{g}(U)) = \widehat{g}(\chi_0) = \widehat{f}(\chi_0)(\chi_0(t) - \lambda) .$$

Hence $\lambda = \chi_0(t)$.

Thus $A\sigma(U(t)|PE) = \{\chi_0(t)\}$. Since $\partial\sigma(U(t))|PE) \subseteq A\sigma(U(t)|PE)$, it follows that $\sigma(U(t)|PE) = \{\chi_0(t)\}$. Since $U(t)|PE$ is isometric, it now follows that $|\chi_0(t)| = 1$, $U(t)|PE$ is invertible, and hence, by Gelfand's Theorem (for a short proof, see [1]) that $U(t) \mid PE = \chi_0(t) I_{PE}$, so $U(t)P = \chi_0(t)P$.

If we choose a non-zero vector x in PE, then $U(t)x = \chi_0(t)x$ for all t, so χ_0 is an eigenvalue for U. If we choose a non-zero functional ψ in $(PE)^*$ and put $\phi = P^*\psi$, then $U^*(t)\phi = \chi_0(t)\phi$, so χ_0 is an eigenvalue for U^*.

Remark. Proposition 4.1, and hence Theorems 4.2 and 5.1, remain valid if the assumption that $\mathrm{Sp}_u(U)$ is countable is replaced by the weaker condition that $\mathrm{Sp}_u(U)$ is scattered, that is, each subset of $\mathrm{Sp}_u(U)$ has an isolated point. If S is second countable, then G, $\widehat{G}$ and $\mathrm{Sp}_u(U)$ are also second countable, in which case $\mathrm{Sp}_u(U)$ is scattered (if and) only if it is countable.

The following theorem was proved in [16] in the case when T is norm-continuous and in [2, 14] when $S = \mathbf{R}_+$.

Theorem 4.2 *Let T be a bounded representation of S on a Banach space X, and suppose that $\mathrm{Sp}_u(T)$ is countable and $P\sigma\widehat{u}(T^*)$ is empty. Then $\lim_{t\in S} \|T(t)x\| = 0$, for all x in X.*

Proof. Let E, Q, and U be as in Proposition 3.1, and suppose that $E \neq (0)$. Then Proposition 4.1 shows that $P\sigma\widehat{u}(T^*)$ is non-empty, a contradiction. Hence $E = (0)$, and the result follows from Proposition 3.1(1).

The following theorem was first proved in [9] when $S = \mathbf{N}$, then in [8, 18] when $S = \mathbf{R}_+$, and in [18] when S is norm-continuous.

Theorem 4.3 *Let T be a bounded representation of S on a Banach space X, let $f \in L^1(S)$ and suppose that f is of spectral synthesis for $\mathrm{Sp}_u(T)$. Then $\lim_{t\in S} \|T(t)\widehat{f}(T)\| = 0$.*

Proof. Let Q and U be as in Proposition 3.1 and V be as in Proposition 3.2. Since $\mathrm{Sp}(V) \subseteq \mathrm{Sp}_u(U) \subseteq \mathrm{Sp}_u(T)$, f is of spectral synthesis for $\mathrm{Sp}(V)$. It follows from the spectral theory of isometric representations of groups [6, 12] that $\widehat{f}(V) = 0$. By Proposition 3.2(1), $\widehat{f}(U)\widehat{g}(U) = 0$ for all g in $L^1(S)$. But, if t_0 is an interior point of S, then taking a limit in the strong operator topology, we find that $\widehat{f}(U)U(t_0) = 0$. Since $U(t_0)$ is isometric, it follows that $\widehat{f}(U) = 0$.

Now let $x \in X$. By Proposition 3.1(2),

$$Q\widehat{f}(T)x = \widehat{f}(U)Qx = 0 .$$

By Proposition 3.1(1), $\lim_{t\in S} \|T(t)\widehat{f}(T)x\| = 0$. This establishes convergence of $T(t)\widehat{f}(T)$ to 0 in the strong operator topology.

To deduce norm-convergence, we perform a similar construction to Proposition 3.2. Consider the representation $\widetilde{T}$ of S on $\mathcal{A}_T$ given by:

$\widetilde{T}(t)(A) = T(t)A$ $(t \in S, A \in \mathscr{A}_T)$. This is strongly continuous, and, as in Proposition 3.2, we find that $\widehat{f}(\widetilde{T})(A) = \widehat{f}(T)A$ and $\mathrm{Sp}(\widetilde{T}) \subseteq \mathrm{Sp}(T)$. This shows that f is of spectral synthesis with respect to $\mathrm{Sp}(\widetilde{T})$, so we may apply the result obtained in the second paragraph of the proof and deduce that

$$\lim_{t\in S} \|T(t)\widehat{f}(T)\widehat{g}(T)\| = \lim_{t\in S} \|\widetilde{T}(t)\widehat{f}(\widetilde{T})(\widehat{g}(T))\| = 0 \qquad (g \in L^1(S)).$$

Taking a suitable net (g_α) in $L^1(S)$ such that $\|(\widehat{g}_\alpha(T) - T(t_0))\widehat{f}(T)\| \to 0$, it follows that $\lim_{t\in S} \|T(t+t_0)\widehat{f}(T)\| = 0$, and hence that $\lim_{t\in S} \|T(t)\widehat{f}(T)\| = 0$.

In [8], it was shown, for $S = \mathbf{R}_+$, that Theorem 4.2 can be deduced from Theorem 4.3 by means of harmonic analysis. Since we do not know how to prove Theorem 4.3 for general S by such methods, and since our proof of Theorem 4.3 involves Banach algebra techniques (in the proof of Propos. 3.2), it seems natural to give, as we have done, a proof of Theorem 4.2 which involves further Banach algebra methods.

5 Almost periodicity

Let T be a bounded representation of S on a Banach space X, and let $X_b(T)$ be the closed linear span of the unitary eigenvectors of T. The representation T is said to be *(weakly) asymptotically almost periodic* if, for each cofinal ultrafilter $\mathscr{U}$ on S, and each x in X, $\lim_{t\in\mathscr{U}} T(t)x$ exists (in the weak topology), i.e. for each net (t_α) in S such that $t_\alpha \to \infty$, there is a subnet (s_β) such that $\lim_\beta T(s_\beta)x$ exists (weakly) for each x in X. Then T is asymptotically almost periodic if and only if $X = X_s(T) \oplus X_b(T)$ [10]. When $S = \mathbf{R}_+$ or $\mathbf{Z}_+$, this is equivalent to T being almost periodic, i.e. each orbit $\{T(t)x : t \in S\}$ is relatively compact. But Example 2.1, for (Ω', μ'), shows that, when $S = \mathbf{R}^2_+$, T may be asymptotically almost periodic, but not almost periodic.

Let $E\sigma_u(T)$ be the set of all χ in S^*_u for which there exists a non-zero functional ϕ in X^* such that $T^*(t)\phi = \chi(t)\phi$ for all t and $\phi(x) = 0$ whenever $T(t)x = \chi(t)x$ for all t. Thus $P\sigma_u(T^*) = P\sigma_u(T) \cup E\sigma_u(T)$. If T is weakly asymptotically almost periodic (in particular, if X is reflexive), then $E\sigma_u(T)$ is empty.

In the case when $S = \mathbf{R}_+$ [15, 5] or T is norm-continuous [16], Lyubich and Phong have shown that a bounded representation T is asymptotically almost periodic if $\mathrm{Sp}_u(T)$ is countable and $E\sigma_u(T)$ is empty. We can apply the methods of [14, 15, 16] to strongly continuous representations in the following way.

Let $L = \overline{X_s(T) \oplus X_b(T)}$. In fact, $L = X_s(T) \oplus X_b(T)$. This can be seen by elementary arguments, but it also follows from the fact that $T|L$ is asymptotically almost periodic, so L has a decomposition. Without loss of generality, we may assume that $\|T(t)\| \leqq 1$, and, as in Proposition 3.1, we obtain a norm ℓ_1 on $X/X_s(T)$ given by:

$$\ell_1(Q_1 x) = \lim_{t\in S} \|T(t)x\|,$$

where Q_1 is the quotient map of X onto $X/X_s(T)$. Since each $T(t)$ maps $X_b(T)$ isometrically onto itself, $\ell_1(Q_1 x) = \|x\|$ if $x \in X_b(T)$. Hence $(L/X_s(T), \ell_1)$ is complete, hence closed in $X/X_s(T)$, so there is a norm $\tilde{\ell}$ on X/L given by:

$$\begin{aligned}\tilde{\ell}(x+L) &= \inf\{\ell_1(Q_1(x-y)) : y \in X_b(T)\} \\ &= \inf\{\|T(t)(x-y)\| : t \in S, y \in X_b(T)\} \\ &= \lim_{t \in S} \inf\{\|T(t)(x-y)\| : y \in X_b(T)\},\end{aligned}$$

since $T(t)$ maps $X_b(T)$ onto $X_b(T)$.

Let E be the completion of $(X/L, \tilde{\ell})$. As in Proposition 3.1, T induces a representation U of S by isometries on E, with $\mathrm{Sp}(U) \subseteq \mathrm{Sp}(T)$. If E is non-zero and $\mathrm{Sp}_u(T)$ is countable, then Proposition 4.1 provides a non-zero functional ψ in E^* and a character χ in S_u^* such that $U^*(t)\psi = \chi(t)\psi$. If $\phi = Q^*\psi$, where $Q : X \to X/L \subseteq E$ is the canonical map, then $T^*(t)\phi = \chi(t)\phi$ and ϕ vanishes on $X_b(T)$. Thus we obtain the following extension of Theorem 4.2.

Theorem 5.1 *Let T be a bounded representation of S on a Banach space X, and suppose that $\mathrm{Sp}_u(T)$ is countable and $E\sigma_u(T)$ is empty. Then T is asymptotically almost periodic.*

Acknowledgements. We are very grateful to Professors W. Arendt and T. Ando for their hospitality, and to CNRS (France) and the Inoue Foundation for Science (Japan) for their financial support.

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