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A generalization of Lichtenbaum's theorem on the cohomological dimension of algebraic varieties

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The cohomological dimension of a scheme X, denoted by cd X, is the largest integer i such that there exists a quasi-coherent sheaf F on X with $H^i(X, F) \neq 0$. Let $Y \subset X$ be a locally closed subscheme, i.e. an open subscheme of a closed subscheme of X. We denote by $H^i_Y(X, -)$ the i-th local cohomology functor with support in Y [G, p. 2] and we denote by $\mathrm{cd}(X, Y)$ the cohomological dimension of X with respect to Y, that is the largest integer i such that there exists a quasicoherent sheaf F on X with $H^i_Y(X, F) \neq 0$. Clearly, $H^i_X(X, F) = H^i(X, F)$, so $\mathrm{cd}\, X = \mathrm{cd}(X, X)$. Hence the cohomological dimension with respect to a locally closed subscheme is a generalization of the usual cohomological dimension.

A theorem of Grothendieck says that if X is Noetherian, $\mathrm{cd}(X, Y) \leqq \dim X$ [G, 1.12]. A result conjectured by Lichtenbaum and widely known as Lichtenbaum's theorem says that if X is separated and of finite type over a field, then $\mathrm{cd}\, X < \dim X$ if and only if every top-dimensional irreducible component of X is non-proper. Proofs of this result in various degrees of generality were given by Grothendieck [G, 6.9], Kleiman [K] and Hartshorne [H, 3.2]. The purpose of this note is to prove the following generalization of Lichtenbaum's theorem (for an étale analog see [L, 3.7]:

Theorem. *Let X be a separated scheme of finite type over a field and let Y be a locally closed subscheme of X. Then $\mathrm{cd}(X, Y) < \dim X$ if and only if every connected component of the preimage of Y in every top-dimensional irreducible component of the normalization of X_{red} is non-proper.*

We need the following auxiliary result which is a generalization of [H, pp. 405–406]:

Proposition. *Let X be a Noetherian scheme and $Y \subset X$ a locally closed subscheme.*

(i) $\mathrm{cd}(X, Y) = \mathrm{cd}(X, Y_{\mathrm{red}}) = \mathrm{cd}(X_{\mathrm{red}}, Y_{\mathrm{red}})$

(ii) *If $X_1 X_2, \ldots X_s$ are the irreducible components of X, then* $\mathrm{cd}(X, Y) = \max\{\mathrm{cd}(X_j, Y \cap X_j)\}$

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(iii) *If* $f:X' \to X$ *is a finite surjective map and* $Y' = f^{-1}(Y)$, *then* $\operatorname{cd}(X, Y) = \operatorname{cd}(X', Y')$.

Proof. (i) Local cohomology functors with support in Y depend only on X and the underlying topological space of Y. Since Y and Y_{red} have the same underlying topological space, $\operatorname{cd}(X, Y) = \operatorname{cd}(X, Y_{\text{red}})$. Every quasicoherent sheaf on X_{red} is a quasicoherent sheaf on X, so $\operatorname{cd}(X, Y_{\text{red}}) \geqq \operatorname{cd}(X_{\text{red}}, Y_{\text{red}})$. Every quasicoherent sheaf on X has a filtration whose quotients are quasicoherent sheaves on X_{red}, so $\operatorname{cd}(X, Y_{\text{red}}) \leqq \operatorname{cd}(X_{\text{red}}, Y_{\text{red}})$. This proves (i)

(ii) We use induction on s. If $s=1$, there is nothing to prove. Assume $s>1$ and (ii) proven for $s-1$. Let i_j: $X_j \to X$ be the natural closed embedding. The composition of functors $H^0_Y(X, i_{j*}(-)) = H^0_{Y \cap X_j}(X_j, -)$ leads to the spectral sequence $E_2^{p,q} = H^p_Y(X, R^q i_{j*}(-)) \Rightarrow H^{p+q}_{Y \cap X_j}(X_j, -)$. But $R^q i_{j*} = 0$ for $q>0$. Hence for any quasicoherent sheaf F on X_j we have that $H^p_{Y \cap X_j}(X_j, F) = H^p_Y(X, i_{j*}F)$, so $\operatorname{cd}(X, Y) \geqq \max\{\operatorname{cd}(X_j, X_j \cap Y)\}$. Let F be a quasicoherent sheaf on X. The kernel K of the canonical surjection $F \to i_{s*} i_s^* F$ is a quasicoherent sheaf supported on the union of the first $s-1$ irreducible components of X. The long exact sequence of local cohomology groups associated with the short exact sequence $0 \to K \to F \to i_{s*} i_s^* F \to 0$ together with the induction hypothesis show that $\operatorname{cd}(X, Y) \leqq \max\{\operatorname{cd}(X_j, X_j \cap Y)\}$. This proves (ii).

(iii) Since $R^q f_* = 0$ for $q>0$, the spectral sequence $E_2^{p,q} = H^p_Y(X, R^q f_*(-)) \Rightarrow H^{p+q}_{Y'}(X', -)$ shows that $H^p_{Y'}(X', F) = H^p_Y(X, f_* F)$ for any quasicoherent sheaf F on X', so $\operatorname{cd}(X', Y') \leqq \operatorname{cd}(X, Y)$. Let $\operatorname{cd}(X', Y') = m$. It remains to show that for every quasicoherent sheaf F on X and for every $i>m$, $H^i_Y(X, F) = 0$. The rest of the proof of (iii) repeats almost verbatim a corresponding part of the proof of [H, 1.1]. By descending induction on i we may assume that statement is true for $i+1$. Then the category C of quasicoherent sheaves on X for which the statement for i is true, is closed under extensions and quotients. By (i) and (ii) we may assume that X is reduced and irreducible. By Noetherian induction we may assume that every quasi-coherent sheaf F on X with $\operatorname{Supp} F < X$ is in C. Consider the sheaf $G = \operatorname{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_{X'}, F)$. Since G is an $f_* \mathcal{O}_{X'}$-module, it defines a coherent sheaf G' on X' and $G = f_* G'$. Hence $G \in C$. On the other hand there is a natural map α: $G \to F$, given by "evaluation at one". Since f is surfjective, $f_* \mathcal{O}_{X'}$ has rank $d>0$ at the generic point of X, so α is generically surjective. Since $\operatorname{im} \alpha$ is a quotient of G, it belongs to C. Since $\operatorname{coker} \alpha$ has support $< X$, it also belongs to C. Finally, F belongs to C, because there is an extension $0 \to \operatorname{im} \alpha \to F \to \operatorname{coker} \alpha \to 0$ and C is closed under extensions. This finishes the proof of the proposition.

Proof of the theorem. Set $n = \dim X$. By the proposition we can assume X reduced, irreducible and normal. In this case we need to show that $H^n_Y(X, F) = 0$ for all quasicoherent sheaves F on X if and only if every connected component of Y is nonproper. First we prove "only if". Assume $Y = Y_1 \sqcup Y_2$, where Y_1 is proper. Since $H^i_Y(X, F) = H^i_{Y_1}(X, F) \oplus H^i_{Y_2}(X, F)$ for all i, it is enough to show that there exists a quasicoherent sheaf F on X such that $H^n_{Y_1}(X, F) \neq 0$. Let X' be a compactification of X. By Lichtenbaum's theorem there exists a quasicoherent sheaf F on X' such that $H^n(X', F) \neq 0$. Since $X' - Y$ is non-proper, Lichtenbaum's theorem says that $H^n(X' - Y, F) = 0$, so the exact sequence $H^n_Y(X', F) \to H^n(X', F) \to H^n(X' - Y, F) = 0$ shows that $H^n_Y(X', F) \neq 0$. By excision [G, 1.3] $H^n_Y(X', F) \neq 0$. This proves "only if".

Now we are going to prove the other implication. Assume all connected components of Y are non-proper. We are going to argue by induction on the number of proper irreducible components of Y, which we denote by m. Let $V \supset Y$ be an open subscheme of X such that Y is closed in V. By excision [G, 1.3] the functors $H^i_Y(X, -)$ and $H^i_Y(V, -|V)$ are isomorphic, so upon replacing X by V we may assume that Y is closed in X. Assume $m=0$, i.e. every irreducible component of Y is non-proper. Let F be a quasicoherent sheaf on X. There is a spectral sequence $E_2^{p,q} = H^p(X, \mathscr{H}^q_Y(X,F)) \Rightarrow H^{p+q}_Y(X,F)$, where $\mathscr{H}^q_Y(X,F)$ (which in some sources is denoted by $\underline{H}^q_Y(X,F)$) is the q-th local cohomology sheaf of F with support in Y. The stalk of $\mathscr{H}^q_Y(X,F)$ at a point $x \in X$ is $H^q_{Y'}(\operatorname{Spec} \mathscr{O}_{X,x}, F|\operatorname{Spec} \mathscr{O}_{X,x})$, where Y' is the preimage of Y in $\operatorname{Spec} \mathscr{O}_{X,x}$. If $\dim \mathscr{O}_{X,x} < q$, that is $\dim \bar{x} > n-q$, Grothendieck's result [G, 1.12] shows that this stalk is 0. Since $\mathscr{O}_{X,x}$ is normal, its completion is a domain. So, if $\dim \mathscr{O}_{X,x} = q$, that is, $\dim \bar{x} = n-q$, Hartshorne's local vanishing theorem [H, 3.1] shows that this stalk may be non-zero only if Y' is the closed point of $\operatorname{Spec} \mathscr{O}_{X,x}$, i.e. x is the generic point of an irreducible component of Y. We conclude that the sheaf $\mathscr{H}^q_Y(X,F)$ is supported in dimension $\leqq n-q$ and the $(n-q)$-dimensional points of the support of $\mathscr{H}^q_Y(X,F)$ are generic points of irreducible components of Y. The sheaf $\mathscr{H}^q_Y(X,F)$ is the direct limit of subsheaves S_t such that $S_0 = 0$ and S_t/S_{t-1} is a quasicoherent sheaf on Y. Namely, S_t is the annihilator of $\mathscr{I}^t$, where $\mathscr{I}$ is the defining sheaf of ideals of Y. Every quasicoherent sheaf is the direct limit of its coherent subsheaves. Every coherent subsheaf of S_t/S_{t-1} is supported on a closed subscheme Y'' of Y such that $\dim Y'' \leqq n-q$ and every irreducible component of Y'' of dimension $(n-q)$ is an irreducible component of Y, hence non-proper. Lichtenbaum's theorem says that $\operatorname{cd} Y'' < (n-q)$. Since cohomology commutes with direct limits, $H^{n-q}(Y, S_i/S_{i-1}) = 0$ for all i, so $H^{n-q}(X, S_t) = 0$ for all t and $H^{n-q}(X, \mathscr{H}^q(X,F)) = 0$ for all q. The above spectral sequence shows that $H^n_Y(X,F) = 0$. This finishes the case that $m=0$.

Now assume $m > 0$. Let Y_1 be the union of the proper irreducible components of Y and Y_2 the union of the non-proper irreducible components. Mayer-Vietoris gives the following exact sequence $H^n_{Y_1 \cap Y_2}(X,F) \to H^n_{Y_1}(X,F) \oplus H^n_{Y_2}(X,F) \to H^n_Y(X,F) \to 0$. Since $H^n_{Y_2}(X,F) = 0$, this sequence takes the form

$$(*) \qquad H^n_{Y_1 \cap Y_2}(X,F) \xrightarrow{f} H^n_{Y_1}(X,F) \longrightarrow H^n_Y(X,F) \longrightarrow 0.$$

On the other hand by [G, 1.9] we have the exact sequence

$$(**) \qquad H^n_{Y_1 \cap Y_2}(X,F) \xrightarrow{f} H^n_{Y_1}(X,F) \longrightarrow H^n_{Y_1 - (Y_1 \cap Y_2)}(X - Y_1 \cap Y_2, F) \longrightarrow 0.$$

The map $f: H^n_{Y_1 \cap Y_2}(X,F) \to H^n_{Y_1}(X,F)$ is the same in both $(*)$ and $(**)$, because in both cases it is induced by the natural inclusion $H^0_{Y_1 \cap Y_2}(X, -) \hookrightarrow H^0_{Y_1}(X, -)$. The scheme $Y_1 - (Y_1 \cap Y_2)$ has no proper connected components, for if it did, the preimage of such a component would be a proper connected component of Y which does not exist by assumption. The scheme $X - Y_1 \cap Y_2$ is normal, n-dimensional and the number of proper irreducible components of $Y_1 - (Y_1 \cap Y_2)$ is less than m. Hence by induction $H^n_{Y_1 - (Y_1 \cap Y_2)}(X - Y_1 \cap Y_2, F) = 0$, so $(**)$ shows that f is surjective. Hence $(*)$ shows that $H^n_Y(X,F) = 0$, as required. The proof is finished.

References

[G] Grothendieck, A.: Local cohomology. (Notes by R. Hartshorne) (Lect. Notes Math., vol. 41) Berlin Heidelberg New York: Springer 1966

[H] Hartshorne, R.: Cohomological dimension of algebraic varieties. Ann. Math. **88**, 403–450 (1968)

[K] Kleiman, S.: On the vanishing of $H^n(X, F)$ for an n-dimensional variety. Proc. Am. Math. Soc. **18** 940–944 (1967)

[L] Lyubeznik, G.: Étale cohomological dimension and the topology of algebraic varieties. (to appear) 1991