Math. Z. 208, 437–462 (1991)

Mathematische Zeitschrift



# Integral representations on weakly pseudoconvex domains

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Received September 19, 1990; in final form February 12, 1991

## 1 Introduction

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^\infty$-smooth boundary. For strictly pseudoconvex domains $\Omega$ Lieb and Range have given in [13] $\mathscr{C}^k$-estimates for the $\bar{\partial}$-equation. Let

$$\Phi(\zeta, z) = \sum_{i=1}^{n} P_i(\zeta, z)(\zeta_i - z_i)$$

be the Henkin/Ramirez barrier function. For $\zeta \notin \Omega$ and $z \in \bar{\Omega}$ $\Phi$ has only zeroes for $\zeta = z$. So the barrier form

$$w(\zeta, z) = \sum_{i=1}^{n} \frac{P_i(\zeta, z)}{\Phi(\zeta, z)} d\zeta_i$$

is well defined for $\zeta \notin \Omega$ and $z \in \Omega$. The Lieb/Range solution operator is of the following form. There exists a kernel $\mathscr{D}(\zeta, z)$ on $(U/\Omega) \times \Omega$, such that for all $(0, q)$-forms $f$, which are of class $\mathscr{C}^k$ on $\mathbf{C}^n$, supported in a neighborhood $U$ of $\bar{\Omega}$ and $\bar{\partial}$-closed on $\Omega$, a solution on $\Omega$ of the equation

$$\bar{\partial} u = f$$

is given by

$$T_q(f)(z) := \int_{U/\Omega} \bar{\partial} f(\zeta) \wedge \mathscr{D}(\zeta, z) + \int_U f(\zeta) \wedge B_{n,q}(\zeta, z).$$

$B_{n,q}$ is the Bochner-Martinelli kernel and $\mathscr{D}$ is a sum of terms

$$\text{const.} \left(\frac{\partial_\zeta \|\zeta - z\|^2}{\|\zeta - z\|^2}\right) \wedge w \wedge \left(\bar{\partial}_\zeta \left(\frac{\partial_\zeta \|\zeta - z\|^2}{\|\zeta - z\|^2}\right)\right)^i \wedge (\bar{\partial}_\zeta w)^j \wedge \left(\bar{\partial}_z \left(\frac{\partial_\zeta \|\zeta - z\|^2}{\|\zeta - z\|^2}\right)\right)^{q-1}$$

with $i + j = n - 1 - q$.

This method would also work on any domain, which has a barrier function $\Phi$ with good estimates. In the literature there are many integral representations on special weakly pseudoconvex domains using appropriate barrier functions (given for example by Range, Bruna/Castillo, Fornaess, Berndtsson/Andersson, etc.). But it is well-known, that there exist pseudoconvex domains without support function (see [12]). The special form of the first integral in the above integral formula suggests the following idea. Let $\delta = \delta(\zeta)$ be the distance of $\zeta$ from $\bar{\Omega}$. Then there is a constant $c_k$, such that we have

$$\|\bar{\partial} f(\zeta)\| \leqq c_k |f|_k \, \delta(\zeta)^{k-1}.$$

(We denote by $|\cdot|_k$ the $\mathscr{C}^k$-norm.) So whenever it is possible to control the $z$-derivatives of $w(\zeta, z)$ and $\bar{\partial}_\zeta w(\zeta, z)$ in terms of powers of $\delta^{-1}$, then it is also possible for sufficiently large $k$, to give $\mathscr{C}^{l(k)}$-estimates for $u$ with a certain loss of regularity. This idea is due to Aizenberg/Dautov [1] and Lieb/Range [13]. Nearly at the same time it has been Chaumat/Chollet [2, 3, 4] and Range [16], which used an idea of Skoda [17] to construct barrier forms, or what is the same, a decomposition of unity

$$1 = \sum_{i=1}^{n} (\zeta_i - z_i) \, w_i(\zeta, z)$$

for special domains. Range applied this to pseudoconvex domains of finite type in $\mathbf{C}^2$. Chaumat and Chollet constructed barrier forms for so called s-H-convex domains, which are pseudoconvex domains with a special Stein neighborhood base. For example, a convex domain is 1-H-convex, which is also called uniformly H-convex, and a bigger $s$ indicates a worse neighborhood base. In their case Chaumat and Chollet were able to control the derivatives of $w$ in terms of $\delta^{-1}$. The loss of regularity increases when $s$ increases.

There are some criteria, which guarantee 1-H-convexity, given for example by Diederich/Fornaess, Catlin and Sibony. On the other hand, no example of a smoothly bounded 2-H-convex pseudoconvex domain is known, which is not 1-H-convex. Probably not every smoothly bounded domain with Stein neighborhood base is s-H-convex and moreover there exist pseudoconvex domains without such a base [5]. The method of Chaumat and Chollet heavily relies on the fact, that they can apply Hörmanders theory on a Stein neighborhood of $\bar{\Omega}$. This method fails for general pseudoconvex domains and we have to look for a more intrinsic method. The famous result of Kohn [9, 10] about the Neumann problem with weights suggests the reformulation of Skoda's theory in order to get a barrier form as a solution of a non-coercive elliptic boundary value problem with weights.

This leads us to the notion of the $\mathscr{L}$-complex and will give us a 'canonical' solution of the above decomposition problem. Recently a paper of K. Diederich and T. Ohsawa [6] has appeared, in which problems about canonical solutions for the Cauchy Riemann complex on families of domains are studied.

Now we shall describe the $\mathscr{L}$-complex. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^\infty$-smooth boundary and let $\varphi$ be a plurisubharmonic function on $\Omega$. We denote by $L_{p,q}(\Omega, \varphi)$ the $L^2$-weighted Hilbert space according to Hörmander.

Let $\zeta \notin \bar{\Omega}$ be a fixed point and $t>0$ be a fixed real number. Set for $z \in \Omega$ and $\alpha \geqq 1$:

$$g(z,\zeta)=\|z-\zeta\|^2,\ \varphi(z,\zeta)=\log g(z,\zeta),$$
$$\varphi_1(z,\zeta,t)=(1+t)(n-1)\,\alpha\,\log g(z,\zeta)+t\,\|z\|^2,$$
$$\varphi_2(z,\zeta,t)=\varphi_1(z,\zeta,t)+\varphi(z,\zeta)=((1+t)(n-1)\,\alpha+1)\log g(z,\zeta)+t\,\|z\|^2.$$

Let

$$\bar{\partial}\colon\ L^2_{p,q}(\Omega,\varphi_i)\to L^2_{p,q+1}(\Omega,\varphi_i),\quad i=1,2$$

be the densely defined closed $\bar{\partial}$-operator and

$$\bar{\partial}^*_{\zeta,t,i}\colon\ L^2_{p,q+1}(\Omega,\varphi_i)\to L^2_{p,q}(\Omega,\varphi_i),\quad i=1,2$$

its adjoint. Note that the domains of definition $\mathrm{Dom}(\bar{\partial}^*_{\zeta,t,i})$ and $\mathrm{Dom}(\bar{\partial})$ are independent of $\zeta$, $t$, $i$ and coincide with the corresponding domains for the unweighted operators.

We are now concerned with operators on the following spaces

$$H_1^q:=(L^2_{0,q}(\Omega,\varphi_1))^n,\qquad H_2^q:=L^2_{0,q}(\Omega,\varphi_2),$$
$$W^q:=H_1^q\times H_2^{q-1},$$

with $H_2^{-1}=0$.

$$T\colon\begin{cases} H_1^q & \to H_2^q\\ (f_1,\dots,f_n)\mapsto\sqrt{1+t}\,\sum\limits_{i=1}^n(\zeta_i-z_i)\,f_i\end{cases}$$

$$\mathscr{L}_q\colon\begin{cases} W^q & \to W^{q+1}\\ (a,b)\mapsto(\bar{\partial}a,\ Ta\text{-}\bar{\partial}b)\end{cases}$$

where $a=(a_1,\dots,a_n)\in H_1^q$, $b\in H_2^{q-1}$ and $\bar{\partial}a=(\bar{\partial}a_1,\dots,\bar{\partial}a_n)$. $\mathscr{L}_q$ is a densely defined closed operator. We call the following complex the $\mathscr{L}$-complex:

$$0\to W^0\xrightarrow{\mathscr{L}_0}W^1\xrightarrow{\mathscr{L}_1}W^2\to\dots\to W^n\xrightarrow{\mathscr{L}_n}W^{n+1}\to 0.$$

Now the important relation is true

$$\mathscr{L}_{q+1}\circ\mathscr{L}_q=0,\quad q\geqq 0.$$

$T$ is continuous and the Hilbert space adjoint is given by

$$T^*=\begin{cases} H_2^q\to H_1^q\\ b\ \ \mapsto\sqrt{1+t}\left(\dfrac{\bar{\zeta}_1-\bar{z}_1}{g(z,\zeta)}\,b,\dots,\dfrac{\bar{\zeta}_n-\bar{z}_n}{g(z,\zeta)}\,b\right).\end{cases}$$

So we conclude

$$T\,T^*=(1+t)\,i\mathrm{d}_{H_2^q}.$$

The Hilbert space adjoint of $\mathscr{L}_q$ is given by

$$\mathscr{L}_q^*: \begin{cases} W^{q+1} \to W^q \\ (a,b) \mapsto (\bar\partial^*_{\zeta,t,1}\, a + T^* b,\ -\bar\partial^*_{\zeta,t,2}\, b). \end{cases}$$

$\mathrm{Dom}(\mathscr{L}_q)$ and $\mathrm{Dom}(\mathscr{L}_q^*)$ are independent of the weight functions and coincide with the corresponding domains of $\bar\partial$: $W^q \to W^{q+1}$ and $\bar\partial^*$: $W^{q+1} \to W^q$, where the operators are applied componentwise and $\bar\partial$ and $\bar\partial^*$ are the unweighted operators.

Note that

$$\begin{aligned} \bar\partial^*{}_{\zeta,t,i} f &= e^{\varphi_i} \bar\partial^*(e^{-\varphi_i} f) \\ &= \bar\partial^* f - \sigma(\bar\partial^*, d\varphi_i) f. \end{aligned}$$

Here $\sigma$ denotes the symbol of $\bar\partial^*$ coming from the differentiation of the weight function (see also [7]). $\sigma$ is a differential operator of order zero. So the main part of the differential operators $\mathscr{L}_q$, respectively $\mathscr{L}_q^*$, is given by

$$\bar{\bar\partial}\colon (a,b) \mapsto (\bar\partial a, -\bar\partial b),$$

respectively

$$\bar{\bar\partial}^*\colon (a,b) \mapsto (\bar\partial^* a, -\bar\partial^* b).$$

These operators are independent of $\zeta$ and $t$. Because of this fact the $\mathscr{L}$-complex has similar properties as the weighted $\bar\partial$-complex of Kohn.

Set

$$\Lambda^q_{\zeta,t} = \mathscr{L}_{q-1}\, \mathscr{L}^*_{q-1} + \mathscr{L}^*_q\, \mathscr{L}_q,$$

with

$$\mathrm{Dom}\, \Lambda^q_{\zeta,t} = \{f \in W^q \mid f \in \mathrm{Dom}\, \mathscr{L}^*_{q-1} \cap \mathrm{Dom}\, \mathscr{L}_q,\ \mathscr{L}_q f \in \mathrm{Dom}\, \mathscr{L}^*_q,\ \mathscr{L}^*_{q-1} f \in \mathrm{Dom}\, \mathscr{L}_{q-1}\}.$$

For the applications we have in mind, only $\Lambda = \Lambda^1_{\zeta,t}$ is interesting. We shall show, that the kernel of $\Lambda$ vanishes for $t > 0$. From functional analytic reasons it follows that

$$\mathscr{N} = \mathscr{N}^1_{\zeta,t} = (\Lambda^1_{\zeta,t})^{-1}$$

is a bounded operator on $W^1$. The same reasoning as in the case of the Neumann problem shows, that we can solve for a given $\alpha$ with $\mathscr{L}_1\, \alpha = 0$ the system:

$$\mathscr{L}_0\, \beta = \alpha$$

by the canonical solution

$$\beta = \mathscr{L}_0^*\, \mathscr{N}_{\zeta,t}\, \alpha,$$

which is orthogonal to the kernel of $\mathscr{L}_0$.

So let $\alpha = (a,b) = (a_1, \ldots, a_n, b) \in \ker \mathscr{L}_1$, i.e. $\bar\partial a = 0$, $Ta = \bar\partial_z b$.

Then $\beta = (w, 0) = (w_1, \ldots, w_n, 0)$ solves

$$\begin{aligned} \bar\partial_z w &= a, \\ Tw &= b. \end{aligned}$$

Applying this to $\alpha=(0, \sqrt{1+t})$, we obtain a solution $w=w(z, \zeta, t)$ of

$$\bar{\partial}_z w=0,$$
$$\sum_{i=1}^{n} (\zeta_i - z_i) w_i = 1.$$

If for a given Sobolev norm with respect to $z$ there exists a $t$, independent of $\zeta$, such that $\mathcal{N}$ and $\mathcal{L}_0^* \mathcal{N}$ respect this norm and if we additionally can control derivatives of $\zeta$ and $z$ in terms of $\delta^{-1}$, then we can insert this barrier form $w$ into the solution operator described above. In our case however, we cannot explicitly calculate the loss of regularity. We only know, that it is finite. Consequently this method leads only to an integral representation in the $\mathscr{C}^\infty$-category.

*Main results*

Before stating the theorems we need some definitions. We denote by $(\phi, \psi)_i$, $i=1,2$ the inner product in the Hilbert space $L^2_{p,q}(\Omega, \varphi_i) \cdot \|\phi\|_i^2 := (\phi, \phi)_i$. For $x=(a, b) \in W^q$, $y=(\tilde{a}, \tilde{b}) \in W^q$ we set

$$(x, y) := \sum_{i=1}^{n} (a_i, \tilde{a}_i)_1 + (b, \tilde{b})_2,$$
$$\|x\|^2 := \sum_{i=1}^{n} \|a_i\|_1^2 + \|b\|_2^2.$$

Let $x=(a, b) \in W^q$ be $\mathscr{C}^\infty$-smooth up to the boundary of $\Omega$. For each nonnegative integer $s$ we define the weighted Sobolev norm, namely

$$\|x\|_s^2 := \sum_{i=1}^{n} \|a_i\|_{1,s}^2 + \|b\|_{2,s}^2,$$

with

$$\|c\|_{i,s}^2 = \sum \left\| \frac{\partial^{u+v} c}{\partial z_1^{u_1} \dots \partial z_n^{u_n} \partial \bar{z}_1^{v_1} \dots \partial \bar{z}_n^{v_n}} \right\|_i^2$$

for $c \in L^2_{p,q}(\Omega, \varphi_i) \cap \mathscr{C}^\infty_{(p,q)}(\bar{\Omega}), i=1,2$. Here the sum is over all nonnegative integers, $u_i$, $v_i$, with $u_1 + \dots + u_n = u$, $v_1 + \dots + v_n = v$, $u+v \leqq s$. Note, that in the norms the weights are involved.

We denote by $W_s^q$ the completion of $W^q \cap \mathscr{C}^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\|_s$. We shall need also pointwise norms for nonnegative integers $k$, $s$, depending on point $\zeta$.

$$\|x\|_{s,(k)} = \sum_{|u|+|v| \leqq k} \left\| \frac{\partial^{|u|+|v|} x(z, \zeta)}{\partial \zeta_1^{u_1} \dots \partial \zeta_n^{u_n} \partial \bar{\zeta}_1^{v_1} \dots \partial \bar{\zeta}_n^{v_n}} \right\|_s^2.$$

Let $U$ denote a fixed sufficiently small neighborhood of $\bar{\Omega}$.

**Theorem 1** *There exists an increasing sequence $0<K_0<K_1<\dots$ of real numbers, such that for every smooth mapping $\alpha$: $U\setminus\bar{\Omega}\to W^1$ and every non-negative integer $s$ and $t\geqq K_s$ we have the estimate*

$$\|\mathcal{N}_{\zeta,t}\,\alpha(\cdot,\zeta)\|_s\leqq\frac{c(t,s)}{\delta(\zeta)^{2s}}\|\alpha(\cdot,\zeta)\|_s,$$

$$\|\mathcal{L}_0^*\,\mathcal{N}_{\zeta,t}\,\alpha(\cdot,\zeta)\|_s\leqq\frac{c(t,s)}{\delta(\zeta)^{2s}}\|\alpha(\cdot,\zeta)\|_s,$$

$$\|\mathcal{L}_0\,\mathcal{L}_0^*\,\mathcal{N}_{\zeta,t}\,\alpha(\cdot,\zeta)\|_s\leqq\frac{c(t,s)}{\delta(\zeta)^{2s}}\|\alpha(\cdot,\zeta)\|_s.$$

**Theorem 2** *There exists an increasing sequence $0<K_0<K_1<\dots$ of real numbers, such that for every smooth mapping $\alpha$: $U\setminus\bar{\Omega}\to W^1$ and every non-negative integers $k$, $s$ we have the estimate*

$$\|\mathcal{N}_{\zeta,t}\,\alpha(z,\zeta)\|_{s,(k)}\leqq c(t,s,k)\sum_{k_1+k_2\leqq k}\delta^{-\kappa(s,k_1)}\|\alpha\|_{s+k_1,(k_2)}$$

*and*

$$\|\mathcal{L}_0^*\,\mathcal{N}_{\zeta,t}\,\alpha(z,\zeta)\|_{s,(k)}\leqq c(t,s,k)\sum_{k_1+k_2\leqq k}\delta^{-\kappa(s,k_1)-1}\|\alpha\|_{s+k_1,(k_2)}$$

*for $t\geqq K_{s+k}$, with $\kappa(S,a)=(2s+2+a)(a+1)-2$.*

**Theorem 3** *For each positive integer $r$ there exists an increasing sequence $0<t_0<\dots$ of real numbers and a $\mathscr{C}^\infty$-smooth maping $w$: $\bar{\Omega}\times(U\setminus\bar{\Omega})\to\mathbf{C}^n$, which solves*

$$\bar{\partial}_z\,w(z,\zeta)=0,\qquad\sum_{i=1}^{n}(\zeta_i-z_i)\,w_i(z,\zeta)=1,$$

*and fulfills the estimates*

$$|D_\zeta^a\,w(\cdot,\zeta)|_s\leqq\frac{c(s,r)}{\delta(\zeta)^{t_s}},$$

*for $a\leqq r$, and all $s$ where $D_\zeta^a$ is an arbitrary differential operator of type*

$$\frac{\partial^{u+v}}{\partial\zeta_1^{u_1}\dots\partial\zeta_n^{u_n}\,\partial\bar{\zeta}_1^{v_1}\dots\partial\bar{\zeta}_n^{v_n}},$$

*with $u_1+\dots+u_n=u$, $v_1+\dots+v_n=v$, $u+v=a$.*

When we insert the barrier form $w$ into the solution operator $T_q(f)$, which will be constructed in Sect. 5, we obtain the following integral representations.

**Theorem 4** *Let $\Omega\subset\subset\mathbf{C}^n$ be a pseudoconvex domain with smooth boundary. Then there exist for $q=0,1,\dots,n$ linear integral operators $R_q$, $T_q$, with $R_q=0$ for $q>0$, $R_0$: $\mathscr{C}^\infty(\bar{\Omega})\to\mathscr{C}^\infty(\bar{\Omega})\cap\mathcal{O}(\Omega)$, $T_q$: $\mathscr{C}^\infty_{0,q}(\bar{\Omega})\to\mathscr{C}^\infty_{0,q-1}(\bar{\Omega})$, $T_{-1}=0$, such that for $f\in\mathscr{C}^\infty_{0,q}(\bar{\Omega})$ we have*

$$f=R_q(f)+\bar{\partial}T_q(f)+T_{q+1}(\bar{\partial}f).$$

For $\bar{\partial}$-closed forms we obtain a solution operator analogous to the operator of Lieb/Range resp. Chaumat/Chollet. In the case of non-$\bar{\partial}$-closed forms, we can use an idea of Peters [15], substituting $E\bar{\partial}f - \bar{\partial}Ef$ for $\bar{\partial}f$ (cf. the proof of Theorem 4, $E$ is a Seeley continuation operator). Then we obtain a homotopy formula for $f$. There are some obvious consequences of the existence of a barrier function.

**Corollary 1** *If $f$ depends smoothly on some additional parameters, then also $R_q(f)$ and $T_q(f)$ depend smoothly on these parameters.*

Once that for a weakly pseudoconvex domain with smooth boundary a barrier form is constructed, there exist well-known techniques for real transversal intersections of such domains (see for example [14]). So let the domain $\Omega$ be a real transversal intersection of finitely many weakly pseudoconvex domains with smooth boundary.

**Corollary 2** *For $q=1, \ldots, n$ there exist operators $T_q$: $\mathscr{C}^\infty_{(0,q)}(\bar{\Omega}) \cap \ker(\bar{\partial}) \to \mathscr{C}^\infty_{(0,q-1)}(\bar{\Omega})$, such that we have*

$$\bar{\partial} T_q(f) = f.$$

Actually one can prove the existence of a homotopy formula. The following corollary is trivial, when a Stein neighborhood base exists.

**Corollary 3** *Let $V$ be a fixed neighborhood of $\bar{\Omega}$. Then there exist for $q=1, \ldots, n$ operators $T_q$: $\mathscr{C}^1_{(0,q)}(V) \cap \ker(\bar{\partial}) \to \mathscr{C}^1_{(0,q-1)}(\bar{\Omega})$, with $\bar{\partial} T_q(f) = f$ on $\bar{\Omega}$ and*

$$|T_q(f)|_{k+\lambda, \Omega} \leqq c_{k,\lambda} |f|_{k,V}$$

*for $\lambda < 1$ and $k = 1, 2, \ldots$.*

Because $f$ is already defined in a neighborhood of $\bar{\Omega}$, we can construct a solution operator analogous to the one in Theorem 4, but with $\bar{\partial}Ef = 0$ in a smaller neighborhood of $\bar{\Omega}$. Then the regularity of the solution is given by the Bochner-Martinelli integral. The other term is $\mathscr{C}^\infty$-smooth up to the boundary.

Before coming to the proofs I want to thank Anne-Marie Chollet and Jaques Chaumat from Orsay (Paris) for their patience in many discussions concerning their articles.

## 2 Generalization of the Skoda estimate

Let $\phi, \psi \in W^2 \cap \operatorname{Dom} \mathscr{L}_1 \cap \operatorname{Dom} \mathscr{L}_0^*$ and set

$$\mathscr{Q}(\phi, \psi) = (\mathscr{L}_0^* \phi, \mathscr{L}_0^* \psi) + (\mathscr{L}_1 \phi, \mathscr{L}_1 \psi).$$

First we have to estimate $\mathscr{Q}(x, x)$.

**Lemma 1** *Let $x = (a, b) \in W^1 \cap \operatorname{Dom} \mathscr{L}_1 \cap \operatorname{Dom} \mathscr{L}_0^*$. Then the following estimate is true:*

$$\mathscr{Q}(x, x) \geqq t \|x\|^2 + (1 - 1/\alpha) \|b\|_2^2.$$

*Proof.*

$$\|\mathscr{L}_1(x)\|^2 = \|\bar\partial a\|_1^2 + \|Ta - \bar\partial b\|_2^2,$$
$$\|\mathscr{L}_0^*(x)\|^2 = \|\bar\partial_{\zeta,t}^* a + T^* b\|_1^2.$$
$$\mathscr{Q}(x,x) = \|\bar\partial a\|_1^2 + \|\bar\partial_{\zeta,t}^* a\|_1^2 + \|\bar\partial b\|_2^2 + \|Ta\|_2^2 + \|T^* b\|_1^2$$
$$+2\,\mathrm{Re}(\bar\partial_{\zeta,t}^* a, T^* b)_1 - 2\,\mathrm{Re}(Ta, \bar\partial b)_2.$$
$$\|T^* b\|_1^2 = (b, TT^* b)_2 = (1+t)\|b\|_2^2.$$
$$(\bar\partial_{\zeta,t}^* a, T^* b)_1 = (a, \bar\partial T^* b)_1,$$
$$(Ta, \bar\partial b)_2 = (a, T^* \bar\partial b)_1.$$

Hence

$$\mathscr{Q}(x,x) \geqq \|\bar\partial a\|_1^2 + \|\bar\partial_{\zeta,t}^* a\|_1^2 + (1+t)\|b\|_2^2 + 2\,\mathrm{Re}(a, [\bar\partial, T^*]\, b)_1,$$

with $[A,B] := AB - BA$.

Set $T_0 := (\sqrt{1+t})^{-1} T$, $T_0^* := (\sqrt{1+t})^{-1} T^*$. In his paper Skoda used the fact, that $b$ is holomorphic. But instead of the Skoda term $\bar\partial T_0^* b$ in our case we have $[\bar\partial, T_0^*]\, b$ for a not necessarily holomorphic $b$. Since the result is the same we can take over the Skoda estimate. Note that in the following calculations the Kohn weight function $t|z|^2$ plays the role of the additional plurisubharmonic weight $\psi$ occurring in the Skoda weight function. There is also a difference in the meaning of the function $g$!

Following the ideas of Skoda we obtain for the crucial term

$$2|\mathrm{Re}(a, \sqrt{1+t}\,[\bar\partial, T_0^*]\, b)_1| = 2|\mathrm{Re}(\sqrt{1+t}\, a, [\bar\partial, T_0^*]\, b)_1|$$
$$\leqq \frac{1}{\alpha}\int_\Omega |b|^2 e^{-\varphi_2} dV + \alpha(1+t)\int_\Omega g^{-1}\left|\sum_{i,k=1}^n e^{\varphi}\frac{\partial((\zeta_i - z_i)e^{-\varphi})}{\partial z_k} a_{ik}\right|^2 e^{-\varphi_1} dV.$$

Here $dV$ denotes the volume element and

$$a_i = \sum_{k=1}^n a_{ik}\, d\bar\zeta_k, \qquad \text{for } i = 1, 2, \ldots, n.$$

For $a \in \mathrm{Dom}\,\bar\partial_{\zeta,t}^* \cap \mathrm{Dom}\,\bar\partial$ we need the Hörmander estimate for $i = 1, \ldots, n$

$$\|\bar\partial a_i\|_1^2 + \|\bar\partial_{\zeta,t}^* a_i\|_1^2 \geqq \int_\Omega \sum_{k,l=1}^n \frac{\partial^2 \varphi_1}{\partial z_k \partial \bar z_l} a_{ik}\overline{a_{il}}\, e^{-\varphi_1} dV.$$

This yields

$$\mathscr{Q}(x,x) \geqq \left(1 - \frac{1}{\alpha} + t\right)\|b\|_2^2$$
$$+\int_\Omega\left[\sum_{i,k,l=1}^n \frac{\partial^2\varphi_1}{\partial z_k \partial \bar z_l} a_{ik}\overline{a_{il}} - \frac{\alpha(1+t)}{g}\left|\sum_{i,k=1}^n e^{\varphi}\frac{\partial}{\partial z_k}((\zeta_i - z_i)e^{-\varphi})\, a_{ik}\right|^2\right] e^{-\varphi_1} dV.$$

Because of

$$\frac{\partial^2\varphi_1}{\partial z_k \partial \bar z_l} = (n-1)\,\alpha(1+t)\frac{\partial^2 \log(g)}{\partial z_k \partial \bar z_l} + t\,\delta_{kl}$$

and the Skoda estimate [17, (2.12)]

$$(n-1)\,\alpha \sum_{i,k,l=1}^{n} \frac{\partial^2 \log(g)}{\partial z_k\, \partial \bar{z}_l}\, a_{ik}\, \overline{a_{il}} \geqq \frac{\alpha}{g} \left| \sum_{i,k=1}^{n} e^{\varphi} \frac{\partial}{\partial z_k}((\zeta_i - z_i)\, e^{-\varphi})\, a_{ik} \right|^2$$

we obtain

$$\mathcal{Q}(x, x) \geqq t\, \|a\|_1^2 + \left(1 - \frac{1}{\alpha} + t\right) \|b\|_2^2.$$

Now the lemma follows for $x \in \operatorname{Dom} \mathcal{L}_1 \cap \operatorname{Dom} \mathcal{L}_0^*$.

*Remark.* $t$ will be large for our applications, so we do not need $\alpha > 1$. For sake of simplicity we choose

$$\alpha = 1.$$

We need the following notations

$$\Box_{\zeta,t} = \bar{\partial}\, \bar{\partial}_{\zeta,t}^* + \bar{\partial}_{\zeta,t}^*\, \bar{\partial},$$
$$\Lambda_{\zeta,t} = \mathcal{L}_0\, \mathcal{L}_0^* + \mathcal{L}_1^*\, \mathcal{L}_1.$$

An easy calculation shows for $x = (a, b)$

$$\Lambda_{\zeta,t}\, x = (\Box_{\zeta,t}\, a + [\bar{\partial}, T^*]\, b + T^*\, Ta,\ \Box_{\zeta,t}\, b + [T, \bar{\partial}_{\zeta,t}^*]\, a + T T^*\, b),$$

and $\operatorname{Dom} \Lambda_{\zeta,t}$ is given by the set

$$\{x = (a, b) \in W^1 \mid a_i, b \in \operatorname{Dom} \bar{\partial}^* \cap \operatorname{Dom} \bar{\partial}, \bar{\partial} a_i, \bar{\partial} b \in \operatorname{Dom} \bar{\partial}^*, \bar{\partial}^* a_i, \bar{\partial}^* b \in \operatorname{Dom} \bar{\partial}\}$$
$$= \{x \in W^1 \mid x \in \operatorname{Dom} \mathcal{L}_1 \cap \operatorname{Dom} \mathcal{L}_0^*, \mathcal{L}_1\, x \in \operatorname{Dom} \mathcal{L}_1^*, \mathcal{L}_0^*\, x \in \operatorname{Dom} \mathcal{L}_0\}.$$

With this definition $\Lambda = \Lambda_{\zeta,t}$ becomes a densely defined self-adjoint operator on $W^1$. The main part of $\Lambda$ is given by $n+1$ copies of the usual unweighted complex Laplacian $\Box$. It follows, that the system of $\Lambda_{\zeta,t}$ is elliptic.

*Convention.* Let $A$ and $B$ be two quadratic matrices, where the entries are differential operators of degree $r$ and $s$. Then in general it is not true, that the degree of $[A, B]$ is smaller than $r+s$. But if one of the matrices has a *scalar* form, that means, it is a multiple of the unit matrix, then we can apply this rule. The commutator of two scalar matrices is again scalar. This will be important in the following considerations.

## 3 A priori estimates

Lemma 1 shows, that $\Lambda_{\zeta,t}$ is injective on its domain of definition in $W^1$. It follows, that there exists a unique bounded operator $\mathcal{N} = \mathcal{N}_{\zeta,t}$: $W^1 \to W^1$, with

$$\operatorname{Im} \mathcal{N} \subset \operatorname{Dom} \Lambda_{\zeta,t},$$

and for all $\alpha \in W^1$, $\psi \in \operatorname{Dom} \mathscr{L}_1 \cap \operatorname{Dom} \mathscr{L}_0^*$:

$$\mathscr{Q}(\mathscr{N}\alpha, \psi) = (\alpha, \psi)$$
$$A_{\zeta,t}\mathscr{N} = \mathrm{id}_{W^1},$$
$$\mathscr{N} A_{\zeta,t} = \mathrm{id}_{\operatorname{Dom} A}.$$

Let $\alpha \in W^1 \cap \mathscr{C}^\infty(\bar{\Omega})$. Suppose we are given $\phi \in W^1 \cap \mathscr{C}^\infty(\bar{\Omega}) \cap \operatorname{Dom} \mathscr{L}_0^* \cap \operatorname{Dom} \mathscr{L}_1$, with

$$\mathscr{Q}(\phi, \psi) = (\alpha, \psi),$$

for all $\psi \in \operatorname{Dom} \mathscr{L}_0^* \cap \operatorname{Dom} \mathscr{L}_1$ and fixed $\zeta$ and $t$. We want to show, that for a given nonnegative integer $s$ there exists a constant $K_s > 0$, such that if $t > K_s$, then the following estimate is true

$$\|\phi\|_s \leqq \frac{\text{const.}(t)}{\delta(\zeta)^{2s}} \|\alpha\|_s.$$

The estimate is trivial for $s=0$. For the other cases we follow the ideas of Kohn in [9]. But in our situation it is crucial, that we will be able to choose the constant $K_s$ independently of $\zeta$! Let $\Omega$ be given by a smooth defining function $r$ with $\Omega = \{r < 0\}$. Let $P \in b\Omega$ be a boundary point, $V$ a sufficiently small neighborhood of $P$ with a special boundary coordinate system

$$t_1, t_2, \ldots, t_{2n-1}, r.$$

$t_1, t_2, \ldots, t_{2n-1}$ are the tangential and $r$ is the normal coordinate. If $\alpha$ is a $(2n)$-tuple of nonnegative integers we denote by

$$D^\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \ldots \partial t_{2n-1}^{\alpha_{2n-1}} \partial r^{\alpha_{2n}}}.$$

For $\alpha_{2n} = 0$ we write $D_b^\alpha$. We shall apply these operators to forms and matrices of forms componentwise. If $\phi \in \operatorname{Dom} \mathscr{L}_0^* \cap \mathscr{C}^\infty(\bar{\Omega})$, then $D_b^\alpha \phi \in \operatorname{Dom} \mathscr{L}_0^*$. Let $\varphi$ be a $\mathscr{C}^\infty$-smooth function on $V \cap \bar{\Omega}$ with compact support in $V$.

**Definition.** A finite sum of terms

$$\lambda(z, \zeta, t) \frac{(\zeta_1 - z_1)^{i_1} \ldots (\zeta_n - z_n)^{i_n} (\bar{\zeta}_1 - \bar{z}_1)^{j_1} \ldots (\bar{\zeta}_n - \bar{z}_n)^{j_n}}{g(z, \zeta)^a},$$

where $\lambda$ is a $\mathscr{C}^\infty$-smooth matrix on $\Omega \times (\bar{U} \setminus \Omega) \times \mathbf{R}_+$ and $i_\nu, j_\nu \geqq 0$, will be called of *type* $\mathscr{A}_k$, $k \geqq 0$, if

$$2a - (i_1 + \ldots + i_n + j_1 + \ldots + j_n) \leqq k.$$

A scalar matrix of type $\mathscr{A}_k$ is denoted by $\mathscr{D}_k$. A diagonally acting differential operator of order $s$, which is independent of $\zeta$ and $t$ we denote by $U^s$. The following formulae are immediate

$$|\mathscr{A}_k(z,\zeta,t)| \leqq \frac{c(t)}{\delta(\zeta)^k},$$
$$\mathscr{L}_0^* = \tilde{\partial}^* + \mathscr{A}_1 = U^1 + \mathscr{A}_1,$$
$$[\mathscr{L}_0^*, D^\alpha] = [\tilde{\partial}^*, D^\alpha] + \mathscr{A}_{1+|\alpha|} = U^\alpha + \mathscr{A}_{1+|\alpha|}.$$

For $\mathscr{L}_1$ the formulae are similar.

$$(D_b^\alpha)^* = \sum_{\substack{u \subset \alpha \\ \rho + |u| = |\alpha|}} \mathscr{A}_\rho D_b^u, \qquad \mathscr{A}_0 = \mathscr{D}_0 = U^0 = \pm 1.$$

**Lemma 2** *For each nonnegative integer $s$ and real positive number $t$ there exist constants $K_s$, $K_s^1$, $K_s^2$, such that the following is valid. Let $\alpha \in W^1 \cap \mathscr{C}^\infty(\bar{\Omega})$ and $\phi \in W^1 \cap \mathscr{C}^\infty(\bar{\Omega}) \cap \operatorname{Dom} \mathscr{L}_1 \cap \operatorname{Dom} \mathscr{L}_0^*$ be given, such that for fixed $\zeta$ and $t$*

$$\mathscr{Q}(\phi, \psi) = (\alpha, \psi)$$

*for all $\psi \in \operatorname{Dom} \mathscr{L}_1 \cap \operatorname{Dom} \mathscr{L}_0^*$. Let $\varphi$ be a smooth cut-off function supported in the neighborhood, where the tangential derivatives are defined. Then we have the following inequality*

$$\mathscr{Q}(\varphi D_b^\beta \phi, \varphi D_b^\beta \phi) \leqq K_s \|\phi\|_s^2 + K_s^1(t) \sum_{\rho=1}^{s} \frac{1}{\theta^{1+\rho}} \|\phi\|_{s-\rho}^2 + K_s^2(t) \sum_{\rho=0}^{s} \frac{1}{\theta^\rho} \|\alpha\|_{s-\rho}^2,$$

*with $|\beta| = s$, $\theta = \delta(\zeta)^2$. $K_s$ is a constant depending on $s$, whereas $K_s^1(t)$ and $K_s^2(t)$ are constants depending on $s$ and $t$. None of these constants depends on $\alpha$ or $\zeta$.*

*Proof.* We shall use the following convention to shorten the notations. The calculations will be of that kind, that for any term which contains $\mathscr{L}_0^*$ there exists a twin term of the same kind with a $\mathscr{L}_1$ instead. We shall only write down the $\mathscr{L}_0^*$-terms and the symbol $\oplus$ then indicates, that we have to complete the formula by the corresponding $\mathscr{L}_1$-term.

We show the lemma by induction over $s$.

$s = 0$.

$$\begin{aligned}
\mathscr{Q}(\varphi\phi, \varphi\phi) &= (\mathscr{L}_0^* \varphi\phi, \mathscr{L}_0^* \varphi\phi) \oplus \\
&= (\mathscr{L}_0^* \phi, \mathscr{L}_0^* \varphi^2 \phi) + (\mathscr{L}_0^* \phi, [\varphi, \mathscr{L}_0^*] \varphi\phi) + ([\varphi, \mathscr{L}_0^*] \phi, \mathscr{L}_0^* \varphi\phi) \oplus \\
&= (\varphi\alpha, \varphi\phi) + (\mathscr{L}_0^* \varphi\phi, [\varphi, \mathscr{L}_0^*] \phi) \\
&\quad + \|[\varphi, \mathscr{L}_0^*] \phi\|^2 + ([\mathscr{L}_0^*, \varphi] \phi, \mathscr{L}_0^* \varphi\phi) \oplus \\
&= W_1 + W_2 + W_3 + W_4 \oplus.
\end{aligned}$$

$$|W_1| \leqq \varepsilon \|\phi\|^2 + C_\varepsilon \|\alpha\|^2,$$

for small $\varepsilon > 0$. Analogously

$$|W_2| + |W_4| \leqq \varepsilon \mathscr{Q}(\varphi\phi, \varphi\phi) + C_\varepsilon \|\phi\|^2,$$
$$|W_3| \leqq c \|\phi\|^2.$$

For $\varepsilon<1$ the conclusion follows.

$s \Rightarrow s+1$: $|\beta|=|(\beta_1, \ldots, \beta_{2n})|=s+1$.

$$\begin{aligned}\mathcal{Q}(\varphi D_b^\beta \phi, \varphi D_b^\beta \phi) &= (\varphi D_b^\beta \alpha, \varphi D_b^\beta \phi)+([\mathcal{L}_0^*, \varphi D_b^\beta]\, \phi, \mathcal{L}_0^* \varphi D_b^\beta \phi) \\ &\quad +(\mathcal{L}_0^* \phi, [(\varphi D_b^\beta)^*, \mathcal{L}_0^*]\, \varphi D_b^\beta \phi) \oplus \\ &= W_1+W_2+W_3 \oplus. \\ |W_1| &\leqq c\,\|\phi\|_{s+1}^2+c\,\|\alpha\|_{s+1}^2, \\ |W_2| &\leqq \varepsilon \mathcal{Q}(\varphi D_b^\beta \phi, \varphi D_b^\beta \phi)+C_\varepsilon W_{21},\end{aligned}$$

with

$$W_{21}=\|[\mathcal{L}_0^*, \varphi D_b^\beta]\, \phi\|^2.$$

Now

$$\begin{aligned}[\mathcal{L}_0^*, \varphi D_b^\beta] &= [\tilde{\partial}^*, \varphi D_b^\beta]+[\mathcal{A}_1, \varphi D_b^\beta] \\ &= [\tilde{\partial}^*, \varphi D_b^\beta]+\sum_{\substack{|\nu| \geqq 1 \\ |\gamma|=s+1-|\nu|}} \mathcal{A}_{1+|\nu|} D^\gamma \phi.\end{aligned}$$

The first term on the right side is of order $s+1$ and independent of $\zeta$ and $t$. So we obtain

$$\begin{aligned}|W_{21}| &\leqq c\,\|\phi\|_{s+1}^2+\sum_{\substack{\nu \geqq 1 \\ |\gamma|=s+1-\nu}} \|\mathcal{A}_{1+\nu} D^\gamma \phi\|^2 \\ &\leqq c\,\|\phi\|_{s+1}^2+\sum_{\substack{\nu \geqq 1 \\ |\gamma|=s+1-\nu}} \frac{c(t)}{\delta^{2(1+\nu)}}\|\phi\|_{s+1-\nu}^2.\end{aligned}$$

It remains $W_3$.

$$W_3=(\mathcal{L}_0^* \phi, X \varphi D_b^\beta \phi),$$

with

$$X=[(\varphi D_b^\beta)^*, \mathcal{L}_0^*].$$

$$(\varphi D_b^\beta)^*=(\pm D_b+\mathcal{A}_1)^\beta \varphi=\sum_{|\nu| \geqq 0}^{s+1} \mathcal{A}_{|\nu|} D_b^{\beta-\nu} \varphi,$$

with $\mathcal{A}_0=\pm 1=\text{const}$.

Hence $W_3=A+B+C$, with

$$\begin{aligned}A &= (\mathcal{L}_0^* \phi, [[(D_b^\beta)^*, \varphi], \mathcal{L}_0^*]\, \varphi D_b^\beta \phi), \\ B &= (\mathcal{L}_0^* \phi, [\varphi, \mathcal{L}_0^*](D_b^\beta)^* \varphi D_b^\beta \phi), \\ C &= (\varphi \mathcal{L}_0^* \phi, [(D_b^\beta)^*, \mathcal{L}_0^*]\, \varphi D_b^\beta \phi).\end{aligned}$$

Set

$$\begin{aligned}Y_1 &= [[(D_b^\beta)^*, \varphi], \mathcal{L}_0^*], \\ Y_2 &= [[[(D_b^\beta)^*, \varphi]. \mathcal{L}_0^*], \varphi].\end{aligned}$$

Then

$$A=(\varphi \mathcal{L}_0^* \phi, Y_1 D_b^\beta \phi)+(\mathcal{L}_0^* \phi, Y_2 D_b^\beta \phi).$$

$Y_2$ is much easier to handle and gives better estimates than $Y_1$. So we concentrate on the first term.

$$(\varphi \mathcal{L}_0^* \phi, Y_1 D_b^\beta \phi) = \sum_{\substack{u \cup v \subset \beta \\ |u|+\rho=s}} (\varphi \mathcal{L}_0^* \phi, D_b^u [\dots [Y_1, D_b^{v_1}], D_b^{v_2}], \dots, D_b^{v_\rho}] D_b^e \phi)$$

$$= \sum_{\substack{u \cup v \subset \beta \\ |u|+\rho=s}} ((D_b^u)^* \varphi \mathcal{L}_0^* \phi, S D_b^e \phi),$$

with

$$S = [\dots [Y_1, D_b^{v_1}], D_b^{v_2}], \dots], D_b^{v_\rho}]$$

and $e=(0, \dots, 1, \dots, 0)$, such that $u+e \leqq \beta$.

Now it is easy to see, that

$$Y_1 = U^s + \sum_{\substack{\kappa \geqq 1 \\ |v|+\kappa=s}} \mathcal{A}_{1+\kappa} D^v,$$

$$S = U^s + \sum_{\substack{\kappa \geqq 1 \\ |v|+\kappa=s}} \mathcal{A}_{1+\kappa+\rho} D^v.$$

So we have to estimate terms of type ($|u|+\rho=s$):

$$((D_b^u)^* \varphi \mathcal{L}_0^* \phi, U^{s+1} \phi + \sum_{\substack{\kappa \geqq 1 \\ |w|+\kappa=s+1}} \mathcal{A}_{1+\kappa+\rho} D^w \phi)$$

$$= (\sum_{\substack{\pi \geqq 1 \\ |v|+\pi=|u|}} \mathcal{A}_\pi D_b^v \varphi \mathcal{L}_0^* \phi, U^{s+1} \phi + \sum_{\substack{\kappa \geqq 1 \\ |w|+\kappa=s+1}} \mathcal{A}_{1+\kappa+\rho} D^w \phi)$$

($\mathcal{A}_0 = U^0$).

Set

$$A_1 = (\mathcal{A}_\pi D_b^v \varphi \mathcal{L}_0^* \phi, U^{s+1} \phi),$$
$$A_2 = (\mathcal{A}_\pi D_b^v \varphi \mathcal{L}_0^* \phi, \mathcal{A}_{1+\kappa+\rho} D^w \phi),$$

with $\pi \geqq 0$, $\pi + |v| = u$, $\kappa \geqq 1$, $|w|+\kappa = s+1$, $|u|+\rho=s$, $\mathcal{A}_0 = U^0$.

$$|A_1| \leqq c \|\phi\|_{s+1}^2 + \frac{1}{\theta^\pi} \|D_b^v \varphi \mathcal{L}_0^* \phi\|^2.$$

$$\|D_b^v \varphi \mathcal{L}_0^* \phi\|^2 \leqq \|[D_b^v \varphi, \mathcal{L}_0^*] \phi\|^2 + \|\mathcal{L}_0^* [D_b^v, \varphi] \phi\|^2 + \|\mathcal{L}_0^* D_b^v \varphi \phi\|^2$$
$$\leqq c \|\phi\|_{|v|}^2 + c(t) \sum_{\lambda \geqq 0} \frac{1}{\theta^{\lambda+1}} \|\phi\|_{|v|-\lambda}^2 + \mathcal{Q}(\varphi D_b^v \phi, \varphi D_b^v \phi).$$

Hence

$$|A_1| \leqq c \|\phi\|_{s+1}^2 + c(t) \sum_{\lambda \geqq 0} \frac{1}{\theta^{\lambda+1}} \|\phi\|_{s-\lambda}^2 + c(t) \sum_{\substack{\pi \geqq 0 \\ |v|+\pi \leqq s}} \frac{1}{\theta^\pi} \mathcal{Q}(\varphi D_b^v \phi, \varphi D_b^v \phi).$$

$$A_2 = (\mathcal{A}_\pi D_b^v \bar{\partial}^* \phi, \mathcal{A}_{1+\kappa+\rho} D^w \phi) + \sum_{\lambda+|\sigma|=|v|} (\mathcal{A}_{\pi+1+\lambda} D^\sigma \phi, \mathcal{A}_{1+\kappa+\rho} D^w \phi),$$

with $\mathscr{A}_0 = U^0$.

(a) $|v| = s \Rightarrow |u| = s$, $\pi = 0$, $\rho = 0 \Rightarrow$

$$|(\mathscr{A}_0\, D_b^v\, \tilde{\partial}^*\phi,\ \mathscr{A}_{1+\kappa}\, D^w\phi)| \leqq c\,\|\phi\|_{s+1}^2 + \frac{c(t)}{\theta^{1+\kappa}}\|\phi\|_{s+1-\kappa}^2.$$

(b) $|v| < s$

$$|(\mathscr{A}_\pi\, D_b^v\, \tilde{\partial}^*\phi,\ \mathscr{A}_{1+\kappa+\rho}\, D^w\phi)| \leqq \frac{c(t)}{\delta^{\pi+\rho+\kappa+1}}\|\phi\|_{s+1-\rho-\pi}\|\phi\|_{s+1-\kappa}$$

$$\leqq \frac{c(t)}{\theta^{\pi+\rho}}\|\phi\|_{s+1-\rho-\pi}^2 + \frac{c(t)}{\theta^{\kappa+1}}\|\phi\|_{s+1-\kappa}^2.$$

$$|(\mathscr{A}_{\pi+\lambda+1}\, D^\sigma\phi,\ \mathscr{A}_{\kappa+\rho+1}\, D^w\phi)| \leqq \frac{c(t)}{\delta^{\pi+\kappa+\rho+\lambda+2}}\|\phi\|_{s-\pi-\lambda-\rho}\|\phi\|_{s+1-\kappa}$$

$$\leqq \frac{c(t)}{\theta^{\pi+\rho+\lambda+1}}\|\phi\|_{s-\pi-\lambda-\rho}^2 + \frac{c(t)}{\theta^{\kappa+1}}\|\phi\|_{s+1-\kappa}^2.$$

As summary we have got

$$|A| \leqq c\,\|\phi\|_{s+1}^2 + c(t)\sum_{\rho\geqq 1}\frac{1}{\theta^{1+\rho}}\|\phi\|_{s+1-\rho}^2 + c(t)\sum_{\substack{\pi\geqq 0\\ |v|+\pi\leqq s}}\frac{1}{\theta^\pi}\mathscr{Q}(\varphi D_b^v\phi,\ \varphi D_b^v\phi).$$

Now we estimate term $B$.

$$B = (\mathscr{L}_0^*\phi,\ U^0(\varphi D_b^\beta)^* D_b^\beta\phi).$$

We only look at the worst, case, that is

$$\begin{aligned}
B^* &= (\mathscr{L}_0^*\phi,\ (\varphi D_b^\beta)^* U^0 D_b^\beta\phi)\\
&= (\varphi D_b^\beta\,\mathscr{L}_0^*\phi,\ U^0 D_b^\beta\phi)\\
&= ([\varphi D_b^\beta, \mathscr{L}_0^*]\,\phi,\ U^0 D_b^\beta\phi) + (\mathscr{L}_0^*\,\varphi D_b^\beta\phi,\ U^0 D_b^\beta\phi)\\
&= B_1^* + B_2^*
\end{aligned}$$

$$|B_2^*| = \varepsilon\mathscr{Q}(\varphi D_b^\beta\phi,\ \varphi D_b^\beta\phi) + C_\varepsilon\|\phi\|_{s+1}^2,$$
$$|B_1^*| \leqq c\,\|\phi\|_{s+1}^2 + \|[\varphi D_b^\beta, \mathscr{L}_0^*]\,\phi\|^2.$$

The second term on the right we can handle in the usual way and obtain

$$|B| \leqq \varepsilon\mathscr{Q}(\varphi D_b^\beta\phi,\ \varphi D_b^\beta\phi) + C\,\|\phi\|_{s+1}^2 + c(t)\sum_{\rho\geqq 1}\frac{1}{\theta^{1+\rho}}\|\phi\|_{s+1-\rho}^2.$$

Now we come to the most complicated and most tedious part, namely case $C$. Elementary commutator operations give

$$C = \sum_{\substack{|u|+\rho=s+1\\ \rho\geqq 1}} (D_b^u\,\varphi\mathscr{L}_0^*\phi,\ Z_\rho\,\varphi D_b^\beta\phi),$$

with

$$Z_\rho = [[\ldots[(D_b^{v_1})^*, \mathscr{L}_0^*], (D_b^{v_2})^*], \ldots, (D_b^{v_\rho})^*].$$

Hence $C$ is a sum of terms

$$\begin{aligned}C_{u,\rho}&=(D_b^u\,\varphi\mathscr{L}_0^*\,\phi, Z_\rho\,\varphi D_b^\beta\,\phi)\\&=(D_b^u\,\varphi\mathscr{L}_0^*\,\phi, Z_\rho[\varphi, D_b^e]\,D_b^{\beta-e}\,\phi)+(D_b^u\,\varphi\mathscr{L}_0^*\,\phi,[Z_\rho, D_b^e]\,\varphi D_b^{\beta-e}\,\phi)\\&\quad+((D_b^e)^*\,D_b^u\,\varphi\mathscr{L}_0^*\,\phi, Z_\rho\,\varphi D_b^{\beta-e}\,\phi)\\&=C_1+C_2+C_3,\end{aligned}$$

with $e:=(0,\ldots,1,\ldots,0)$ ($i$-th component), such that $q+u_i\leqq\beta_i$. The case $C_1$ is much easier to handle than the other terms, so we skip over it. Evidently we have

$$Z_\rho=U^1+\mathscr{A}_{1+\rho},\qquad [Z_\rho, D_b^e]=U^1+\mathscr{A}_{2+\rho}.$$

Set $C_2=C_{21}+C_{22}$, with

$$\begin{aligned}C_{21}&=(D_b^u\,\varphi\mathscr{L}_0^*\,\phi, U^1\,D_b^{\beta-e}\,\phi),\\C_{22}&=(D_b^u\,\varphi\mathscr{L}_0^*\,\phi,\mathscr{A}_{2+\rho}\,D_b^{\beta-e}\,\phi),\end{aligned}$$

$|u|+\rho=s+1$, $\rho\geqq 1$.

$$|C_{21}|\leqq c\,\|\phi\|_{s+1}^2+c(t)\sum_{\rho\geqq 1}\frac{1}{\theta^{1+\rho}}\|\phi\|_{s-\rho}^2,$$

for arbitrary $u$ and

$$|C_{22}|\leqq c\,\|\phi\|_{s+1}^2+c(t)\sum_{\rho\geqq 1}\frac{1}{\theta^{1+\rho}}\|\phi\|_{s-\rho}^2,$$

for $|u|=s$.

Now let $\rho\geqq 2$:

$$\begin{aligned}C_{22}&=|(\mathscr{A}_\rho\,D_b^u\,\varphi\mathscr{L}_0^*\,\phi,\mathscr{A}_2\,D_b^{\beta-e}\,\phi)|\\&\leqq\frac{c(t)}{\theta^2}\|\phi\|_s^2+c(t)\sum\frac{1}{\theta^{1+\rho+\kappa}}\|\phi\|_{s+1-\rho-\kappa}^2.\end{aligned}$$

This yields

$$|C_2|\leqq c\,\|\phi\|_{s+1}^2+c(t)\sum_{\rho=1}^{s+1}\frac{1}{\theta^{1+\rho}}\|\phi\|_{s+1-\rho}^2.$$

It remains case $C_3$.

$$C_3=((D_b^e+\mathscr{A}_1)\,D_b^u\,\varphi\mathscr{L}_0^*\,\phi,(U^1+\mathscr{A}_{1+\rho})\,\varphi D_b^{\beta-e}\,\phi),$$

with $|u|+\rho=s+1$, $\rho\geqq 1$, $u+e\leqq\beta$. Now $C_3=C_{31}+C_{32}+C_{33}+C_{34}$, with

$$\begin{aligned}C_{31}&=(D_b^e\,D_b^u\,\varphi\mathscr{L}_0^*\,\phi, U^1\,\varphi D_b^{\beta-e}\,\phi),\\C_{32}&=(D_b^e\,D_b^u\,\varphi\mathscr{L}_0^*\,\phi,\mathscr{A}_{1+\rho}\,\varphi D_b^{\beta-e}\,\phi),\\C_{33}&=(\mathscr{A}_1\,D_b^u\,\varphi\mathscr{L}_0^*\,\phi, U^1\,\varphi D_b^{\beta-e}\phi),\\C_{34}&=(\mathscr{A}_1\,D_b^u\,\varphi\mathscr{L}_0^*\,\phi,\mathscr{A}_{1+\rho}\,\varphi D_b^{\beta-e}\,\phi).\end{aligned}$$

$$\begin{aligned}|C_{31}|&\leqq C_\varepsilon\|\phi\|_{s+1}^2+\varepsilon\,\|[D_b^{e+u}\,\varphi,\mathscr{L}_0^*]\,\phi\|^2\\&\quad+\varepsilon\,\|\mathscr{L}_0^*\,[D_b^{l+u},\varphi]\,\phi\|^2+\varepsilon\,\|\mathscr{L}_0^*\,\varphi D_b^{l+u}\phi\|^2.\end{aligned}$$

The terms with the brackets can be estimated in the usual way. $e+u=\beta$ for $|u|=s$, so we obtain

$$|C_{31}| \leqq c\|\phi\|_{s+1}^2 + c(t) \sum_{\rho \geqq 1} \frac{1}{\theta^{\rho+1}} \|\phi\|_{s+1-\rho}^2 + c(t) \sum_{|u| \leqq s} \mathcal{Q}(\varphi D_b^u \phi, \varphi D_b^u \phi) + \varepsilon \mathcal{Q}(\varphi D_b^\beta \phi, \varphi D_b^\beta \phi).$$

Case $C_{32}$: If $|u|=s$, then $\rho=1$. Hence

$$|C_{32}| \leqq \frac{C_\varepsilon(t)}{\theta^2} \|\phi\|_s^2 + \varepsilon \|D_b^{e+u} \varphi \mathcal{L}_0^* \phi\|^2.$$

The second term on the right we treat like before.
Now assume $|u|<s$. Hence $\rho \geqq 2$.

$$\begin{aligned} |C_{32}| &\leqq \frac{c(t)}{\delta^{1+\rho}} \|D_b^{e+u} \varphi \mathcal{L}_0^* \phi\| \, \|\phi\|_s \\ &\leqq \frac{c(t)}{\theta^{\rho-1}} \{\|[D_b^{e+u} \varphi, \mathcal{L}_0^*] \phi\|^2 + \|\mathcal{L}_0^* [D_b^{e+u}, \varphi] \phi\|^2\} \\ &\quad + \frac{c(t)}{\theta^{\rho-1}} \|\mathcal{L}_0^* \varphi D_b^{e+u} \phi\|^2 + \frac{c(t)}{\theta^2} \|\phi\|_s^2. \end{aligned}$$

Only the estimates for the first term are not immediate:

$$[D_b^{e+u} \varphi, \mathcal{L}_0^*] = [D_b^{e+u}, \bar{\partial}^*] + \sum_{\substack{|w|+\kappa=|u|+1 \\ \kappa \geqq 1}} \mathcal{A}_{\kappa+1} D^w.$$

Consequently

$$\frac{1}{\theta^{\rho-1}} \|[D_b^{e+u} \varphi, \mathcal{L}_0^*] \phi\|^2 \leqq c(t) \left\{ \frac{1}{\theta^{\rho-1}} \|\phi\|_{s+2-\rho}^2 + \sum_{\kappa \geqq 1} \frac{1}{\theta^{\rho+\kappa}} \|\phi\|_{s+2-\rho-\kappa}^2 \right\}.$$

So we obtain

$$\begin{aligned} |C_{32}| &\leqq c\|\phi\|_{s+1}^2 + \varepsilon \mathcal{Q}(\varphi D_b^\beta \phi, \varphi D_b^\beta \phi) \\ &\quad + c(t) \left\{ \sum_{\rho \geqq 1} \frac{1}{\theta^{\rho+1}} \|\phi\|_{s+1-\rho}^2 + \sum_{\substack{|u|+\rho=s+1 \\ \rho \geqq 1}} \frac{1}{\theta^\rho} \mathcal{Q}(\varphi D_b^u \phi, \varphi D_b^u \phi) \right\}. \end{aligned}$$

$C_{33}$ we can handle like $C_{31}$. In the left factor of the inner product of $C_{33}$ there is one derivative less than in $C_{31}$, but a factor $\mathcal{A}_1$ more. The factor on the right side of the inner product is the same as in $C_{31}$. The same argumentation is valid for $C_{32}$ and $C_{34}$. Hence collecting all terms and choosing $\varepsilon$ small enough, we have the result:

$$\mathscr{Q}(\varphi D_b^\beta \phi, \varphi D_b^\beta \phi) \leqq c \|\phi\|_{s+1}^2 + c(t) \sum_{\rho \geqq 1} \frac{1}{\theta^{\rho+1}} \|\phi\|_{s+1-\rho}^2$$

$$+ c(t) \sum_{\substack{|u|+\rho=s+1 \\ \rho \geqq 1}} \frac{1}{\theta^\rho} \mathscr{Q}(\varphi D_b^u \phi, \varphi D_b^u \phi) + c \|\alpha\|_{s+1}^2.$$

Using the inductive hypothesis for $\mathscr{Q}(\varphi D_b^u \phi, \varphi D_b^u \phi)$ we obtain the desired result. Lemma 2 is therefore proved.

**Lemma 3** *With the same hypotheses as in Lemma 2, there exist for each nonnegative integer s constants $c_s$, $c_s^1(t)$, $c_s^2(t)$, such that*

$$\|\varphi \phi\|_s^2 \leqq c_s \sum_{|\beta|=s} \|\varphi D_b^\beta \phi\|^2 + c_s^1(t) \sum_{\rho \geqq 1} \frac{1}{\theta^\rho} \|\phi\|_{s-\rho}^2 + c_s^2(t) \sum_{\rho \geqq 0} \frac{1}{\theta^\rho} \|\alpha\|_{s-\rho-1}^2.$$

*Proof.* Obviously it is sufficient estimating $\|\varphi D^u \phi\|$, with $|u|=s$. We apply induction over $s$. The cases $s=0$ and $D^u = D_b^u$ are trivial.

Let $D^u = \dfrac{\partial}{\partial r} D_b^\gamma$, $|\gamma| = s-1$. In the sequel we denote a matrix of functions, which is independent of $\zeta$ and $t$ by $U$. Because $\bar{\partial} + \bar{\partial}^*$ is an elliptic system, the boundary of $\Omega$ is noncharacteristic. Therefore we have locally

$$\frac{\partial}{\partial r} = U\bar{\partial} + U\bar{\partial}^* + \sum_{|\sigma|=1} U D_b^\sigma.$$

This yields

$$\frac{\partial}{\partial r} = U\mathscr{L}_1 + U\mathscr{L}_0^* + \sum_{|\sigma|=1} U D_b^\sigma + \mathscr{A}_1.$$

Applying this to $\varphi D_b^\gamma \phi$, we obtain

$$\frac{\partial}{\partial r}(\varphi D_b^\gamma \phi) = U\mathscr{L}_1 \varphi D_b^\gamma \phi + U\mathscr{L}_0^* \varphi D_b^\gamma \phi + \sum_{|\sigma|=1} U D_b^\sigma \varphi D_b^\gamma \phi + \mathscr{A}_1 D_b^\gamma \phi.$$

This equality implies

$$\|\varphi D^u \phi\|^2 \leqq c \mathscr{Q}(\varphi D_b^\gamma \phi, \varphi D_b^\gamma \phi) + c \sum_{|\beta|=s} \|\varphi D_b^\beta \phi\|^2 + \frac{C(t)}{\theta} \|\phi\|_{s-1}^2.$$

Lemma 2 yields the conclusion.

Now let

$$D^u = \frac{\partial^k}{\partial r^k} D_b^\gamma, |\gamma| = s-k, k \geqq 2.$$

The unweighted Laplacian is a determined elliptic system, so there exist matrices $U$, with

$$\frac{\partial^2}{\partial r^2}=U\square+\sum_{|\sigma|=2} UD_b^\sigma+\sum_{|\mu|=1} U\frac{\partial}{\partial r}D_b^\mu+U\frac{\partial}{\partial r}+\sum_{|\nu|=1} UD_b^\nu.$$

Because of

$$\Lambda_{\zeta,t}=\square+\mathscr{A}_1\frac{\partial}{\partial r}+\sum_{|\nu|=1}\mathscr{A}_1 D_b^\nu+\mathscr{A}_2$$

we have

$$\frac{\partial^2\phi}{\partial r^2}=U\alpha+\sum_{|\sigma|=2} UD_b^\sigma\phi+\sum_{|\mu|=1} U\frac{\partial}{\partial r}D_b^\mu\phi+\mathscr{A}_1\frac{\partial\phi}{\partial r}+\sum_{|\nu|=1}\mathscr{A}_1 D_b^\nu\phi+\mathscr{A}_2\phi.$$

Differentiating this equation and using induction gives

$$\begin{aligned}\frac{\partial^k\phi}{\partial r^k}=&\sum_{i\leqq k-2} U\frac{\partial^i\alpha}{\partial r^i}+\sum_{|\beta|=s} UD_b^\beta\phi+\sum_{|\mu|=k-1} U\frac{\partial}{\partial r}D_b^\mu\phi\\&+\sum_{\substack{i\geqq 1\\|u|=k-i}}\mathscr{A}_i D_\beta^u\phi+\sum_{\substack{i\geqq 1\\|v|=k-i-1}}\mathscr{A}_i\frac{\partial}{\partial r}D_b^v\phi.\end{aligned}$$

Applying $D_b^\gamma$ and using the inductive hypothesis, we obtain the conclusion of the lemma.

**Lemma 4** *Same hypotheses as in Lemma 2. For each nonnegative integer s there exist constants $K_s$ and $C_s(t)$, such that for $t\geqq K_s$ we have*

$$\|\phi\|_s\leqq\frac{C_s(t)}{\delta^{2s}}\|\alpha\|_s.$$

*Proof.* The assertion of Lemma 2 is also valid in a neighborhood $V$ which is contained in the interior of $\Omega$. Here $D_b^\beta$ can be replaced by a general derivative $D^\beta$, $|\beta|=s$. This is immediate from the proof. Actually there are better estimates because of interior regularity, but we cannot use them.

So let $\{\varphi_i\}$ be a partition of unity of $\bar{\Omega}$, the supports chosen small enough, such that the above lemmata for $\varphi=\varphi_i$ apply. Then we have

$$\begin{aligned}t\|\phi\|_s^2&\leqq tC_s^1\sum_i\|\varphi_i\phi\|_s^2\\&\leqq tC_s^2\sum_i C_s^i\sum_{|\beta|=s}\|\varphi_i D_b^\beta\phi\|^2+tC_s^2\sum_i C_s^{1,i}(t)\sum_{|\rho|\geqq 1}\frac{1}{\theta^\rho}\|\phi\|_{s-\rho}^2\\&\quad+tC_s^2\sum_i C_s^{2,i}(t)\sum_{|\rho|\geqq 0}\frac{1}{\theta^\rho}\|\alpha\|_{s-\rho-1}^2.\end{aligned}$$

Now Lemma 2 and Lemma 3 yield

$$t\|\phi\|_s^2 \leqq (\tilde{K}_s C_s^2 \sum_i C_s^i)\|\phi\|_s^2 + \tilde{C}_s^1(t) \sum_{\rho \geqq 1} \frac{1}{\theta^{1+\rho}} \|\phi\|_{s-\rho}^2 + \tilde{C}_s^2(t) \sum_{\rho \geqq 0} \frac{1}{\theta^\rho} \|\alpha\|_{s-\rho}^2.$$

We choose $K_s \geqq \tilde{K}_s C_s^2 \sum_i C_s^i + 1$. Then we obtain

$$\|\phi\|_s^2 \leqq \tilde{C}_s^1(t) \sum_{\rho \geqq 1} \frac{1}{\theta^{1+\rho}} \|\phi\|_{s-\rho}^2 + \tilde{C}_s^2(t) \sum_{\rho \geqq 0} \frac{1}{\theta^\rho} \|\alpha\|_{s-\rho}^2.$$

The lemma now follows by a simple induction argument over $s$.

**Lemma 5** *Same hypotheses as in Lemma 4, $t \geqq K_s$. Then for each nonnegative integer $s$ there exists a constant $C_s(t)$, depending on $t$, with*

$$\|\mathscr{L}_0^* \phi\|_s + \|\mathscr{L}_1 \phi\|_s + \|\mathscr{L}_0 \mathscr{L}_0^* \phi\|_s \leqq \frac{C_s(t)}{\delta^{2s}} \|\alpha\|_s.$$

*Proof.* The proof is a simultaneous induction over $s$ for $\mathscr{L}_0^* \phi$, $\mathscr{L}_1 \phi$, $\mathscr{L}_0 \mathscr{L}_0^* \phi$ and $\mathscr{L}_1^* \mathscr{L}_1 \phi = \alpha - \mathscr{L}_0 \mathscr{L}_0^* \phi$.

We use the same local set-up as in the proof of Lemma 2. Because of the ellipticity of the system $\tilde{\partial} + \tilde{\partial}^*$ we can express

$$\frac{\partial}{\partial r} = U \mathscr{L}_{\nu-1}^* + U \mathscr{L}_\nu + \sum_{|\sigma|=1} U D_b^\sigma + \mathscr{A}_1$$

when applied to smooth elements of $\operatorname{Dom} \mathscr{L}_{\nu-1}^* \cap \operatorname{Dom} \mathscr{L}_\nu \cap W^\nu$, $\nu = 0, 1, 2$. Here we denote by $U$ matrices, independent of $\zeta$ and $t$, which pick out the correct coefficients of the operators. Higher normal derivatives we can reduce by the formulae given in the proof of Lemma 3 to derivatives, which contain at most one normal direction. Therefore it is easy to see, that it suffices to only treat purely tangential directions. Let $\varphi$ be a cut-off function as in the proof of Lemma 2 and let $D = D_b^\gamma$, with $|\gamma| = s$.

$$\|\varphi D \mathscr{L}_0^* \phi\|^2 \leqq \|[\varphi D, \mathscr{L}_0^*] \phi\|^2 + \mathscr{Q}(\varphi D \phi, \varphi D \phi).$$

Evaluating the Lie bracket and Lemma 2 gives the conclusion. The same procedure works for $\mathscr{L}_1$.

$$\begin{aligned} X^2 &:= (D \mathscr{L}_0 \mathscr{L}_0^* \mathscr{N} \alpha, D \mathscr{L}_0 \mathscr{L}_0^* \mathscr{N} \alpha) \\ &= A + B, \end{aligned}$$

with

$$A = ([D, \mathscr{L}_0] \mathscr{L}_0^* \mathscr{N} \alpha, D \mathscr{L}_0 \mathscr{L}_0^* \mathscr{N} \alpha),$$

$$B = (\mathscr{L}_0 D \mathscr{L}_0^* \mathscr{N} \alpha, D \mathscr{L}_0 \mathscr{L}_0^* \mathscr{N} \alpha).$$

$$|A| \leqq \varepsilon X^2 + C_\varepsilon \|[D, \mathscr{L}_0] \mathscr{L}_0^* \mathscr{N} \alpha\|^2.$$

The second expression can already be estimated by the first part of the proof.

$$\begin{aligned} B &= (\mathscr{L}_0 D\mathscr{L}_0^* \mathscr{N}\alpha, D\alpha) - (\mathscr{L}_0 D\mathscr{L}_0^* \mathscr{N}\alpha, D\mathscr{L}_1^* \mathscr{L}_1 \mathscr{N}\alpha) \\ &= B_1 - B_2. \\ B_1 &= ([\mathscr{L}_0, D]\, \mathscr{L}_0^* \mathscr{N}\alpha, D\alpha) + (D\mathscr{L}_0 \mathscr{L}_0^* \mathscr{N}\alpha, D\alpha). \end{aligned}$$

Therefore we can estimate

$$|B_1| \leqq \varepsilon X^2 + C_\varepsilon(\|D\alpha\|^2 + \|[\mathscr{L}_0, D]\, \mathscr{L}_0^* \mathscr{N}\alpha\|^2).$$

The second term we can estimate by the first part of the proof. The second factor of $B_2$ is element of Dom $\mathscr{L}_0^*$, so

$$\begin{aligned} B_2 &= (D\mathscr{L}_0^* \mathscr{N}\alpha, \mathscr{L}_0^* D\mathscr{L}_1^* \mathscr{L}_1 \mathscr{N}\alpha) \\ &= (D\mathscr{L}_0^* \mathscr{N}\alpha, \mathscr{L}_0^* [D, \mathscr{L}_1^*]\, \mathscr{L}_1 \mathscr{N}\alpha) \\ &= (\mathscr{L}_0 D\mathscr{L}_0^* \mathscr{N}\alpha, [D, \mathscr{L}_1^*]\, \mathscr{L}_1 \mathscr{N}\alpha) \\ &= ([\mathscr{L}_0, D]\, \mathscr{L}_0^* \mathscr{N}\alpha, [D, \mathscr{L}_1^*]\, \mathscr{L}_1 \mathscr{N}\alpha) + (D\mathscr{L}_0 \mathscr{L}_0^* \mathscr{N}\alpha, [D, \mathscr{L}_1^*]\, \mathscr{L}_1 \mathscr{N}\alpha). \end{aligned}$$

Consequently we obtain

$$|B_2| \leqq \varepsilon X^2 + C_\varepsilon(\|[D, \mathscr{L}_1^*]\, \mathscr{L}_1 \mathscr{N}\alpha\|^2 + \|[\mathscr{L}_0, D]\, \mathscr{L}_0^* \mathscr{N}\alpha\|^2).$$

So if $\varepsilon$ is chosen small enough, the proof goes through for every expression.

Now the a priori estimates are finished. It is well known, how to achieve Theorem 1 by elliptic regularisation. Because in our case the kernel of $\Lambda_{\zeta,t}$ is trivial, the constructed regular solution for $\Lambda_{\zeta,t}\,\phi = \alpha$ is necessarily the canonical solution. Theorem 1 is therefore proven.

## 4 Regularity

So far we have treated the point $\zeta$ as a fixed parameter. It is our aim, however, to construct a barrier form, which is regular in $z$ and $\zeta$.

The domain of definition of $\Lambda_{\zeta,t}$ is independent of the weight function. An explicit calculation gives the following representation

$$\Lambda_{\zeta,t}\,\phi = \square\,\phi + \sum_{j=1}^{n} \mathscr{A}_1 \frac{\partial \phi}{\partial \bar{z}_j} + \mathscr{A}_2\,\phi.$$

From this independence it is easily seen, that the operators $\{\partial/\partial \bar{z}_j\}_j$ are defined on Dom $\Lambda_{\zeta,t}$ (compare also [7]). We shall study the following situation. For a given $s$ and $t \geqq K_s$

$$\mathscr{N}: W_s^1 \to \text{Dom}\, \Lambda_{\zeta,t} \cap W_s^1$$

is a bounded operator because of Theorem 1. On the other hand

$$\Lambda_{\zeta,t}\,\mathscr{N} = i\mathrm{d}_{W_s^1}.$$

The first equation shows, that the dependence on $\zeta$ and $t$ is given by the factors $\mathscr{A}_1$ and $\mathscr{A}_2$, whereas the differential operations involved are constant.

So let

$$\alpha:\begin{cases} U/\bar{\Omega} \to W_s^1 \cap \mathscr{C}^\infty(\bar{\Omega}) \\ \zeta \quad \mapsto \alpha_\zeta(\cdot) \end{cases}$$

be a $\mathscr{C}^\infty$-smooth mapping. Let $D_\zeta^k$ be a differential operator of order $k$ with respect to $\zeta$. Set

$$\gamma_\zeta = \mathscr{N} \alpha_\zeta.$$

Then we obtain

$$D_\zeta^1 \mathscr{N} = -\mathscr{N}(D_\zeta^1 \Lambda_{\zeta,t}) \mathscr{N} + \mathscr{N} D_\zeta^1.$$

The second term gives a differentiation of $\alpha$ and is therefore benign. Inductively the main term of $D_\zeta^k \mathscr{N}$ is given by

$$\sum_{r \leq k} \sum_{\substack{k_i \geq 1 \\ k_1 + \ldots + k_r = k}} a(k_1, \ldots, k_r) \mathscr{N}(D_\zeta^{k_1} \Lambda_{\zeta,t}) \mathscr{N} \ldots \mathscr{N}(D_\zeta^{k_r} \Lambda_{\zeta,t}) \mathscr{N},$$

with constants $a(k_1, \ldots, k_r)$. Here $D_\zeta^1 \Lambda$ means $D_\zeta^1$ applied to $\Lambda$. The final estimates for the leading term will absorb the remaining terms.

*Remark.* A priori it is not clear, if the above formalism is applicable. But if one works carefully with small variations $h$ of $\zeta$ in a neighborhood of a given $\zeta_0$, one can see successively, that everything goes through. For example, the continuity in the $\zeta$-variable, with respect to Sobolev norms in $z$, follows from the identity

$$\gamma_{\zeta+h} - \gamma_\zeta = \mathscr{N}_{\zeta,t}(\alpha_{\zeta+h} - \alpha_\zeta) - \mathscr{N}_{\zeta,t}(\Lambda_{\zeta+h,t} - \Lambda_{\zeta,t}) \gamma_{\zeta+h}$$

and Theorem 1. Here we loose one derivative with respect to $z$.

Because of

$$D_\zeta^a \gamma_\zeta = \sum_{a_1 + a_2 = a} \text{const.}(a_1, a_2)(D_\zeta^{a_1} \mathscr{N})(D_\zeta^{a_2} \alpha_\zeta)$$

we obtain

$$\begin{aligned} \|D_\zeta^a \gamma_\zeta\|_s &\leq c_s \sum_{a_1 + a_2 = a} \|(D_\zeta^{a_1} \mathscr{N})(D_\zeta^{a_2} \alpha_\zeta)\|_s \\ &\leq c_s \sum_{a_1 + a_2 = a} \sum_{r \leq a_1} \sum_{\substack{k_i \geq 1 \\ k_1 + \ldots + k_r = a_1}} \|\mathscr{N}(D_\zeta^{k_1} \Lambda) \mathscr{N} \ldots \mathscr{N}(D_\zeta^{k_r} \Lambda) \mathscr{N}(D_\zeta^{a_2} \alpha)\|_s. \end{aligned}$$

Now

$$D_\zeta^k \Lambda_{\zeta,t} = \sum_{j=1}^n \mathscr{A}_{1+k} \frac{\partial}{\partial \bar{z}_j} + \mathscr{A}_{2+k}.$$

We suppose $t \geq K_{s+a}$. From Theorem 1 we conclude inductively

$$\begin{aligned} \|D_\zeta^a \gamma_\zeta\|_s &\leq c(t) \sum_{\substack{a_1 + a_2 = a \\ r \leq a_1}} \sum_{\substack{k_i \geq 1 \\ k_1 + \ldots + k_r = a_1}} \frac{\|\mathscr{N} D_\zeta^{k_2} \Lambda \mathscr{N} \ldots \mathscr{N} D_\zeta^{k_r} \Lambda \mathscr{N}(D_\zeta^{a_2} \alpha)\|_{s+1}}{\delta^{2s+(1+k_1)}} \\ &\leq c(t) \sum_{\substack{a_1 + a_2 = a \\ r \leq a_1}} \sum_{\substack{k_i \geq 1 \\ k_1 + \ldots + k_r = a_1}} \frac{\|D_\zeta^{a_2} \alpha\|_{s+r}}{\delta^{2s+r+k_1+(2(s+1)+k_2)+\ldots+(2(s+r-1)+k_r)+2(s+r)}} \\ &\leq c(t) \sum_{a_1 + a_2 = a} \frac{1}{\delta^{(2s+2+a_1)(a_1+1)-2}} \|D_\zeta^{a_2} \alpha\|_{s+a_1}. \end{aligned}$$

With $\kappa(s,a):=(2s+2+a)(a+1)-2$ the first part of Theorem 2 is shown. The proof for $\beta_\zeta = \mathscr{L}_0^* \mathscr{N} \alpha_\zeta$ goes along the same line.

$$\mathscr{L}_0^* = \bar{\partial}^* + \mathscr{A}_1,$$

$$D_\zeta^b \mathscr{L}_0^* = \mathscr{A}_{1+b}, b>0.$$

$$\begin{aligned}\|D_\zeta^b \beta_\zeta\|_s &\leqq c \sum_{b_1+b_2=b} \|(D_\zeta^{b_1} \mathscr{L}_0^*)(D_\zeta^{b_2} \gamma_\zeta)\|_s \\ &\leqq c \sum_{b_1+b_2=b} \|\mathscr{A}_{1+b_1}(D_\zeta^{b_2}\gamma_\zeta)\|_s + c\|\bar{\partial}^* D_\zeta^b \gamma_\zeta\|_s \\ &\leqq c(t)\left\{\sum_{b_1+b_2=b} \frac{1}{\delta^{1+b_1}}\|D_\zeta^{b_2}\gamma_\zeta\|_s + \|\bar{\partial}^* D_\zeta^b \gamma_\zeta\|_s\right\}.\end{aligned}$$

Now $D_\zeta^b \gamma_\zeta$ is a sum of terms

$$X = \mathscr{N}(D_\zeta^{k_1} A_{\zeta,t})\mathscr{N} \ldots \mathscr{N}(D_\zeta^{k_r} A_{\zeta,t})\mathscr{N}(D_\zeta^{a_2}\alpha),$$

with $a_1+a_2=b$, $r \leqq a_1$, $k_1+\ldots+k_r = a_1$.

$$\bar{\partial}^* = \mathscr{L}_0^* + \mathscr{A}_1.$$

Because of Theorem 1 we obtain for $t \geqq K_{s+b}$

$$\|\mathscr{L}_0^* X\|_s \leqq c(t) \sum_{a_1+a_2=b} \frac{1}{\delta^{\kappa(s,a_1)}}\|D_\zeta^{a_2}\alpha\|_{s+a_1}.$$

Hence

$$\begin{aligned}\|D_\zeta^b \beta_\zeta\|_s &\leqq c(t)\left\{\sum_{b_1+b_2=b}\frac{1}{\delta^{1+b_1}}\|D_\zeta^{b_2}\gamma_\zeta\|_s + \sum_{b_1+b_2=b}\frac{1}{\delta^{\kappa(s,b_1)}}\|D_\zeta^{b_2}\gamma_\zeta\|_{s+b_1}\right\} \\ &\leqq c(t)\sum_{b_1+b_2=b}\frac{1}{\delta^{\kappa(s,b_1)+1}}\|D_\zeta^{b_2}\alpha_\zeta\|_{s+b_1}.\end{aligned}$$

Theorem 2 is therefore proven.

## 5 The barrier form

For a fixed $t$ we want to solve the system of equations

$$\mathscr{L}_0 \beta = (0, \sqrt{1+t}) \in W^1.$$

The canonical solution is given by

$$\mathscr{L}_0^* \mathscr{N}(0, \sqrt{1+t}) =: (w, 0) \in W^0.$$

Hence $w = w(z, \zeta, t)$ solves

$$\bar{\partial}_z w = 0,$$
$$\sum_{i=1}^{n} (\zeta_i - z_i)\, w_i(z, \zeta, t) = 1.$$

This barrier can be used, for large $t$, constructing integral kernels. However it is more convenient to get rid of the dependence on $t$. This can be achieved by glueing together the solutions having different regularity properties. The proof is almost a copy of the proof in [10]. Therefore we give only a sketch here. Let $r \geqq 1$ be a given integer. By Theorem 2 there exist an increasing sequence $0 < R_0 < R_1 < \dots$, and for each nonnegative integer $s$, constants $c(s, r)$ and a solution $w_s$ of the above system, with

$$\| D_\zeta^k w_s \|_s \leqq \frac{c(s, r)}{\delta^{R_s}} \qquad \text{for } k \leqq r.$$

When we apply the Sobolev lemma, we obtain

$$| D_\zeta^k w_s |_s \leqq \frac{c(s, r)}{\delta^{R_{s+n+1}}} \qquad \text{for } k \leqq r.$$

Now our aim it is to construct a solution $w$ with

$$| D_\zeta^k w |_s \leqq \frac{c(s, r)}{\delta^{R_{s+n+1}}} \qquad \text{for } k \leqq r.$$

As already mentioned above, one can proceed in the same way as in [10]. It is sufficient to construct a sequence $(\tilde{w}_j)_j$ of solutions, such that for all $\zeta$

$$| D_\zeta^k (\tilde{w}_{j+1} - \tilde{w}_j) |_j \leqq 2^{-j} \qquad \text{for } k \leqq r.$$

Then evidently

$$w = \tilde{w}_k + \sum_{\nu=1}^{\infty} (\tilde{w}_{k+\nu} - \tilde{w}_{k+\nu-1})$$

solves our problem. If $\tilde{w}_1, \dots, \tilde{w}_{m-1}$ are already constructed, then we have to correct $w_m$ by a term $V_\varepsilon$, such that for all $\zeta$

$$| D_\zeta^k (w_m - V_\varepsilon - \tilde{w}_{m-1}) |_j \leqq 2^{-m+1} \qquad \text{for } k \leqq r.$$

Here $V_\varepsilon$ has to be a solution of the system

$$\bar{\partial}_z V_\varepsilon = 0,$$
$$\sum_{i=1}^{n} (\zeta_i - z_i)\, V_{\varepsilon, i}(z, \zeta) = 1.$$

With a partition of unity $(\rho_\mu)$ and local translations $\Phi_\varepsilon^\mu(z)=z+\varepsilon a^\mu$ (see [10]) we set with $h=w_m-\tilde{w}_{m-1}$:

$$V_\varepsilon(z)=\sum_\mu \rho_\mu(z)\, h(\Phi_\varepsilon^\mu(z))-\Delta_\varepsilon(z,\zeta).$$

Therefore the unknown term $\Delta_\varepsilon(z,\zeta)$ has to be a solution of the system

$$\bar{\partial}_z \Delta_\varepsilon(z,\zeta)=\sum_\mu \bar{\partial}_z \rho_\mu(z)\, h(\Phi_\varepsilon^\mu(z))=\sum_\mu \bar{\partial}_z \rho_\mu(z)(h(\Phi_\varepsilon^\mu(z))-h),$$

$$\sum_{i=1}^{n} (\zeta_i-z_i)\, \Delta_{\varepsilon,i}(z,\zeta)=0.$$

The terms on the right are smooth up to the boundary in the $z$-variable. The second term on the right shows, that we can make it and its derivatives in $\zeta$ up to order $r$ as small as we wish in terms of powers of $\delta(\zeta)$, when we let $\varepsilon$ depend on $\zeta$, for example

$$\varepsilon=c\,\delta(\zeta)^R$$

with constants $R$ and $c$, $R$ large. Then by the $\mathscr{L}$-complex a solution with the desired properties $\Delta_\varepsilon(z,\zeta)$ of the above system is given. Because of

$$w_m-\tilde{w}_{m-1}-V_\varepsilon=h-V_\varepsilon=\sum_\mu \rho_\mu(h(\Phi_\varepsilon^\mu)-h)$$

this difference can also be chosen as small as we wish.

For our solution operator $T_q$ we only need to choose $r=1$. The proof of Theorem 3 is then obvious.

For the proof of Theorem 4 we proceed as follows. Let

$$w_0=\frac{\partial_\zeta \|\zeta-z\|^2}{\|\zeta-z\|^2}$$

be the Bochner-Martinelli barrier form and let $E\colon \mathscr{C}^0(\bar{\Omega})\to C_c^0(U)$ be a Seeley continuation operator (see [13]). We set $\Delta_{0,1}=\{(\lambda_0,\lambda_1)\in\mathbf{R}^2 \mid \lambda_0, \lambda_1\geqq 0, \lambda_0+\lambda_1=1\}$, $\Delta_0=\{0\}$, $\Delta_1=\{1\}$, $S=b\Omega$, $R=U\setminus\bar{\Omega}$,

$$\eta=\lambda_0 w_0+\lambda_1 w_1.$$

For $q=0,\ldots,n$ we define

$$\mathscr{D}_{n,q}(z,\zeta,\lambda):=c_{n,q}\,\eta\wedge((\bar{\partial}_\zeta+d_\lambda)\,\eta)^{n-q-1}\wedge(\bar{\partial}_z\,\eta)^q,$$

with

$$c_{n,q}=(2\pi i)^{-n}(-1)^{q(q-1)/2}\binom{n-1}{q}.$$

For $f\in\mathscr{C}^\infty_{(0,q)}(\bar{\Omega})$ set

$$R_q(f)=\int_{R\times\Delta_1} (E\bar{\partial}f-\bar{\partial}Ef)\wedge\mathscr{D}_{n,q}(z,\zeta,\lambda),$$

$$T_q(f)=\int_{R\times\Delta_{0,1}} (E\bar{\partial}f-\bar{\partial}Ef)\wedge\mathscr{D}_{n,q-1}(z,\zeta,\lambda)-\int_{U\times\Delta_0} Ef\wedge\mathscr{D}_{n,q-1}(z,\zeta,\lambda).$$

The last term is the Bochner-Martinelli integral. $E\bar{\partial}f-\bar{\partial}Ef$ vanishes on $\bar{\Omega}$, so all integrals are well defined. $w$ is holomorphic in $z$. This implies $R_q=0$ for $q>0$. Set

$$X:=R_q(f)+\bar{\partial}T_q(f)+T_{q+1}(\bar{\partial}f).$$

We show, that $X=f$ on $\Omega$. The following Koppelman formula is well-known

$$\bar{\partial}_z\mathscr{D}_{n,q-1}=(-1)^q(\bar{\partial}_\zeta+d_\lambda)\mathscr{D}_{n,q}.$$

Consequently we have with $g:=E\bar{\partial}f-\bar{\partial}Ef$

$$\begin{aligned}\bar{\partial}_z\int_{R\times\Delta_{0,1}} g\wedge\mathscr{D}_{n,q-1}&=(-1)^q\int_{R\times\Delta_{0,1}} g\wedge(\bar{\partial}_\zeta+d_\lambda)\mathscr{D}_{n,q}\\&=-\int_{R\times\Delta_{0,1}}(\bar{\partial}_\zeta+d_\lambda)(g\wedge\mathscr{D}_{n,q})+\int_{R\times\Delta_{0,1}}(\bar{\partial}_\zeta g)\wedge\mathscr{D}_{n,q}\\&=\int_{S\times\Delta_{0,1}} g\wedge\mathscr{D}_{n,q}-\int_{R\times\Delta_1} g\wedge\mathscr{D}_{n,q}\\&\quad+\int_{R\times\Delta_0} g\wedge\mathscr{D}_{n,q}+\int_{R\times\Delta_{0,1}}\bar{\partial}g\wedge\mathscr{D}_{n,q}.\end{aligned}$$

This implies

$$\begin{aligned}X&=\int_{R\times\Delta_0} g\wedge\mathscr{D}_{n,q}-\bar{\partial}\Big(\int_{U\times\Delta_0} Ef\wedge\mathscr{D}_{n,q-1}\Big)-\int_{U\times\Delta_0} E\bar{\partial}f\wedge\mathscr{D}_{n,q}\\&=-\int_{R\times\Delta_0}(\bar{\partial}_\zeta+d_\lambda)(Ef\wedge\mathscr{D}_{n,q})-\int_{\Omega\times\Delta_0}\bar{\partial}f\wedge\mathscr{D}_{n,q}-\bar{\partial}\Big(\int_{\Omega\times\Delta_0} f\wedge\mathscr{D}_{n,q-1}\Big)\\&=\int_{S\times\Delta_0} f\wedge\mathscr{D}_{n,q}-\int_{\Omega\times\Delta_0}\bar{\partial}f\wedge\mathscr{D}_{n,q}-\bar{\partial}\Big(\int_{\Omega\times\Delta_0} f\wedge\mathscr{D}_{n,q-1}\Big).\end{aligned}$$

The Bochner-Martinelli formula gives the conclusion. Therefore Theorem 4 is proven.

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