Math. Z. 208, 41–55 (1991)

Mathematische Zeitschrift

On the creation of conjugate points

Rafael Oswaldo Ruggiero

IMPA, Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110. Jardîm Botânico, Rio de Janeiro RJ 22460, Brasil

0 Introduction

Let M be a compact manifold and let $\psi_t\colon M \to M$ be a C^∞ flow acting on M. The flow ψ_t is said to be an *Anosov* flow if there exist constants $C>0$, $0<\lambda<1$ and subspaces E_p^s, E_p^u, E_p of $T_p M$, $\forall p \in M$ such that:

(a) $E_p^s \oplus E_p^u \oplus E_p = T_p M$

$$E_p = \frac{\partial}{\partial t}(\psi_t(p))_{t=0} \text{ and}$$

$$D\psi_t|_{E_p^s} = E^s_{\psi_t(p)},\ D\psi_t|_{E_p^u} = E^u_{\psi_t(p)}$$

(b) $\|D\psi_t(v)\| \leqq C\lambda^t \|v\|, \forall t \geqq 0, \forall v \in E_p^s$

$$\|D\psi_{-t}(v)\| \leqq C\lambda^t \|v\|, \forall t \geqq 0, \forall v \in E_p^u.$$

Geodesic flows of manifolds of negative curvature are well known examples of such flows [1]. Recall that given a Riemannian metric g of M, the geodesic flow $\varphi_t\colon T_1 M \to T_1 M$ – with $T_1 M$ being the unit tangent bundle of M – is defined as $\varphi_t(p, v) = (\gamma(t), \gamma'(t))$, where $\gamma(t)$ is the unit geodesic of M with initial conditions $\gamma(0) = p$, $\gamma'(0) = v$.

A Riemannian metric of M has no conjugate points if there exists $p \in M$ such that the exponential map $\exp_p\colon T_p M \to M$ is non-singular. Let $B(M)$ be the set of metrics with no conjugate points of M, and let $A(M)$ be the set of metrics of M whose geodesic flows are Anosov flows. The set $A(M)$ is open in the C^k topology for every $k \in N$ as was showed by Anosov [1]. On the other hand Klingenberg [8] proved that $A(M) \subset B(M)$, so if M admits a metric whose geodesic flow is Anosov then the interior of $B(M)$ in the C^k topology is non-empty for every $k \in N$. The main result of this work is the following:

Theorem A. *The interior of $B(M)$ in the C^2 topology coincides with $A(M)$.*

This statement is straightforward from the following result:

Theorem B. *Let (M, g) be a Riemannian manifold with no conjugate points whose geodesic flow is not Anosov. Then, given $\varepsilon>0$ there exists a metric g_ε – which is conformal to g – having conjugate points and such that*

$$\|g - g_\varepsilon\|_{C^2} < \varepsilon.$$

The main idea of the proof of Theorem B is the fact that Eberlein's characterization [4] of metrics in $B(M)-A(M)$ is not an open condition in the C^2 topology for the set of metrics with no conjugate points. In fact, we show that if $g \in B(M)-A(M)$ then there exists a geodesic γ (which is just the one given in Eberlein's theorem [4]) such that for every $\varepsilon > 0$ there are $T > 0$ and a non-trivial, sectionally C^2 vector field $X:[-T, T] \to R^n$ defined along γ with $X(-T)=X(T)=0$ such that

$$I_{[-T,T]}(X, X) \leqq \varepsilon$$

where $I_{[a,b]}$ is the index form of γ. From this fact we deduce that performing arbitrarily small perturbations of g in the C^2 topology, augmenting the curvature along a compact segment of γ, we get geodesics of those metrics and non-trivial vector fields defined along them having both two different zeros and negative index. So from Morse theory we conclude that those metrics have conjugate points. We do not know if Theorem B holds for any C^k topology with $k > 2$. The obstructions appearing on these cases are similar to Pugh's closing lemma problems [11]. Indeed, the metrics g_ε in Theorem B are obtained from local perturbations of the metric g. Since the set of recurrent geodesics in the unit tangent bundle is of total Lebesgue measure, we must be very careful in controlling the intersections of γ with the support of g_ε in order to guarantee a "global" increase of the curvature along perturbations of γ. This control is achieved by choosing very special shapes for the supports of the g_ε. This is the main step toward the proof of Theorem B. And it is just at this stage when we loose some regularity in the proximity of the perturbations g_ε to g. It is interesting to remark that if the geodesic γ is either closed or non-recurrent, then Theorem B holds in the C^k topology for every $k \in N$. In the first three sections we shall construct a family of perturbations of g satisfying certain particular properties, and in Section four we shall prove Theorem B. I am specially grateful to the referee for his useful remarks and suggestions concerning this work.

1 The equation $x''(t)+K(t)\,x(t)=0$

The purpose of this section is to show that the property of having conjugate points is an open property in the set of metrics of a given manifold. We start by recalling some canonical features of Morse theory of Riemannian manifolds. Given $m \in N$ consider the set of pairs $A(m)=\{(h, H)\}$ – where $h: R^m \times R^m \to R$ is an inner product and $H \in R^{m^2}$ is a symmetric linear operator with respect to h – endowed with the induced C^0 topology. For a given continuous curve $H(t)$ of linear operators of R^{m^2} we say that the equation

$$x''(t)+H(t)\,x(t)=0$$

has conjugate points if there exists $a, b \in R$, $a \neq b$, such that the equation has a non-trivial solution $J(t)$ with $J(a)=J(b)=0$. We say that a and b are conjugate.

Let $C^2_*([a, b], R^m)$ be the space of continuous, piecewise C^2 functions $X:[a, b] \to R^m$ such that $X(a)=X(b)=0$. Associated to each continuous function $c:[a, b] \to A(m)$, $c(t)=h(t), H(t))$, we define the bilinear form

$$I_{c,[a,b]}: C^2_*([a, b], R^m) \times C^2_*([a, b], R^m) \to R$$

by the following formula:

$$I_{c,[a,b]}(X, Y) = -\int_a^b h(t)(X''(t)+HX(t), Y(t))\,dt + \sum_i h(t_i)(X'^{+}(t_i) - X'^{-}(t_i), Y(d_i))$$

where t_i, $i=1,2,\ldots,l$ is the set of points on which $X(t)$ is not differentiable and

$$X'^{+}(t) = \lim_{S\to t^{+}} X'(s), \qquad X'^{-}(t) = \lim_{S\to t^{-}} X'(s).$$

We shall restrict our study to the subset Γ of continuous curves in $A(m)$ satisfying the following two properties:

(1) For every $c\colon [a,b]\to A(m)$ belonging to Γ the family of index forms associated to the restrictions of c to each $[d,e]\subset[a,b]$ is a family of symmetric bilinear forms.

(2) For every $c\subset\Gamma$ there exists $\varepsilon=\varepsilon(c)$ such that the index form associated to the restriction of c to every interval $[d,e]$ of length less than ε is positive definite in the set $C^2_*([d,e], R^m)$.

In these conditions the Morse theorem holds in Γ, i.e.,

Theorem 1.1 *Let $c\in\Gamma$ be a continuous curve, $c(t)=(h(t), H(t))$ defined in an open interval. Then the equation $x''(t)=H(t)\,x(t)$ has conjugate points if and only if there exists $[a,b]$, $a\neq b$, such that the form $I_{c,[a,b]}$ is degenerated in $C^2_*([a,b], R^m)$. For every $t>a$ in the domain of c the index of $I_{c,[a,t]}$ (i.e., the subspace of $C^2_*([a,t], R^m)$ on which $I_{c,[a,t]}$ is negative definite) is finite and if $t>b$ is close to b it equals the dimension of the kernel of $I_{c,[a,b]}$.*

Remark that Theorem 1.1 implies that for every continuous curve $c=(h,H)\in\Gamma$ the fact that the equation $x''+Hx=0$ has conjugate points is equivalent to the existence of a, t, $a<t$ with the property that the form $I_{c,[a,t]}$ has non-zero index. Denote as $\|\ \|_\infty$ the sup norm for functions. From the upper semicontinuity of the index in the set of quadratic forms and Theorem 1.1 it is not hard to prove the following fact:

Corollary 1.1 *Let $c\colon R\to A(m)$, $c(t)=(h(t), H(t))$ be a continuous curve in Γ such that the equation $x''(t)+H(t)\,x(t)=0$ has conjugate points $a\neq b$. Then for every $D>0$ there exists $\varepsilon>0$ such that if $\alpha(t)=(q(t), Q(t))\in\Gamma$ satisfies*

(1) $\|q(t)-h(t)\|_\infty<\varepsilon\,\forall t\in[a,b]$.

(2) $\|H(r)\|_\infty\leqq D$ and $\|Q(r)-H(r)\|_\infty<\varepsilon\,\forall r\in[a,b+1]$

then the equation $x''(t)+Q(t)\,x(t)=0$ has conjugate points a, $b(\alpha)$, where $b(\alpha)$ is close to b.

From Corollary 1.1 we deduce the persistence of conjugate points under small perturbations of metrics. More precisely we have:

Corollary 1.2 *The set of metrics with conjugate points of a given manifold is an open set in the C^k topology for every $k\geqq 2$.*

Proof. Let g be a metric with conjugate points of a manifold M. Let $\mathscr{R}$ be the associated curvature tensor, i.e., $\mathscr{R}(X,Y)Z=\nabla_Y\nabla_X Z-\nabla_X\nabla_Y Z+\nabla_{[X,Y]}Z$, where ∇ is the Levi-Civita connection of the metric g and X, Y, Z are C^∞ vector fields of M. There exist a geodesic $\gamma(t)$, and real numbers $a \neq b$ such that the equation

$$x''(t)+K(t)\,x(t)=0$$

has conjugate points, where $K(t)\colon N_{\gamma(t)} \to N_{\gamma(t)}$ is a family of linear operators defined in the subspace $N_{\gamma(t)}$ of $T_{\gamma(t)}M$ which is normal to $\gamma'(t)$ as follows: let $\{e_i(t)\}$, $i=0,1,2,\dots,m$ be an orthonormal parallel frame defined along $\gamma(t)$ with $\gamma'(t)=e_0(t)$, and let $K_{ij}(t)=g(\mathscr{R}(\gamma'(t),e_i(t))\,\gamma'(t),e_j(t))\;\forall i,j\neq 0$. Then $K(t)$ is the $m \times m$ matrix whose entries are K_{ij}. It is clear that the curve $c(t)=(g(t),K(t))$ belongs to Γ. Let $\bar g$ be a C^k perturbation of g, $k \geqq 2$. Let $\bar\gamma(t)$ be the geodesic of h with initial conditions $\bar\gamma(0)=\gamma(0)$, $\bar\gamma'(0)=\gamma'(0)$ and let $\{\bar e_i(t)\}$ be the orthonormal parallel frame defined along $\bar\gamma(t)$ with initial conditions $\bar e_i(0)=e_i(0)\;\forall i$. Let $\bar K(t)$ be the curvature operators constructed as above using $\{\bar e_i(t)\}$. If $\bar g$ is close enough to g it is clear that $\bar\gamma$, $\bar{\mathscr{R}}$ and $q(t)=(\bar g(t),\bar K(t))$ will be sufficiently close to γ, $\mathscr{R}$ and $c(t)=(g(t),K(t))$ in a way such that Corollary 1.1 applies. So the equation $x''(t)+\bar K(t)\,x(t)=0$ has conjugate points. □

2 Conformal metric changes

In this section we shall deduce some technical lemmas concerning conformal deformations of a given Riemannian metric. Let (M,g) be a complete Riemannian manifold of dimension n and let $\gamma(t)$ be a unit geodesic of (M,g). Here we use the notation g_p to designate the metric g at the point $p\in M$. A metric $\bar g$ of M is said to be conformal to g if there exists a function $h\colon M\to R$, $h>0$ of class C^∞ such that for every $p\in M$ we have $\bar g_p=h(p)\,g_p$. Suppose that the geodesic segment $\gamma\colon[-L,L]\to M$ is imbedded in M. We first find conditions for h such that this geodesic segment of g is also a geodesic segment of $\bar g$. Let ∇ be the Levi-Civita connection of g, and let $\bar\nabla$ be the corresponding one of $\bar g$. For a given C^∞ function $f\colon M\to R$ let $\operatorname{grad}(f)_p\in X^\infty(M)$ be the gradient vector field of f at p with respect to the metric g. We know that the connection of $\bar g$ can be written in terms of g and ∇. In fact, if we write $\bar g_p=e^{2\sigma(p)}g_p$ where $\sigma(p)=\frac{1}{2}\log h(p)$ we have:

$$(\bar\nabla_X Y)_p=(\nabla_X Y)_p+g_p(\operatorname{grad}(\sigma),X)_p\,Y_p+g_p(\operatorname{grad}(\sigma),Y)_p\,X_p-g_p(X,Y)\operatorname{grad}(\sigma)_p$$

where X and Y are C^∞ vector fields defined in a neighborhood of p in M. It is easy to see that this formula does not depend on the differentiable extensions of X_p and Y_p to any neighborhood of p. Consider a parallel, orthonormal frame $E(t)=\{e_i(t)\}$ $i=0,1,2,\dots,n-1$ defined along $\gamma(t)$ with $\gamma'(t)=e_0(t)$. Since the segment $\{\gamma(t),|t|\leqq L\}$ is imbedded there exists an open neighborhood V of this segment which can be parametrized by a Fermi coordinate system of γ associated to the frame $E(t)$, i.e., there exist $\delta_0>0$ and an injective map $\phi\colon V\to R^n$, $\phi(q)=(x_0(q),x_1(q),\dots,x_{n-1}(q))$, $|x_0(q)|<L$, $\left|\left(\sum_{i=1}^{n-1}x_i^2\right)^{\frac{1}{2}}\right|\leqq\delta_0\;\forall q\in V$, where $x_i(q)$ is defined by

$$q=\exp_{\gamma(x_0(q))}\left[x_1(q)\,e_1(x_0(q))+x_2(q)\,e_2(x_0(q))+\dots+x_{n-1}(q)\,e_{n-1}(x_0(q))\right].$$

Here, $\exp_a$ is the exponential map at the point $a \in M$. Remark that the coordinates are given by the arclengths of $\gamma(t)$ and the geodesics which are tangent to $e_i(t)$. Moreover, if $R_\alpha = \{q \in M, d(q, \gamma(t)) < \alpha, |t| \leqq L\}$ then there exists $b > 0$ such that for every $0 < \delta < \delta_0$ we have

$$\phi^{-1}\left\{(x_0, x_1, \ldots, x_{n-1}), |x_0| \leqq L, \left|\left(\sum_{i=1}^{n-1} x_i^2\right)^{\frac{1}{2}}\right| \leqq b\delta\right\} \subset R_\delta.$$

Lemma 2.1 *Let $\bar{g}_p = h(p)\, g_p$, where $h: M \to R$ is a differentiable function and let $\gamma(t)$ be a geodesic of (M, g). If $\operatorname{grad}(h)_{\gamma(t)} = 0$ and $h(\gamma(t)) = 1\, \forall |t| \leqq L$ then $\gamma(t)$ is a geodesic of $\bar{g}$ for every $|t| \leqq L$. Moreover, the parameter t is the arclength of the segment $\{\gamma(t), |t| \leqq L\}$ for both g and $\bar{g}$.*

Proof. Using the coordinate system ϕ constructed as before, the gradient of a function $f: \{\gamma(t), |t| \leqq L\} \to R$ can be written as

$$\operatorname{grad}(f)_{\gamma(t)} = \sum_{i=0}^{n-1} g(\operatorname{grad}(f)_{\gamma(t)}, e_i(t))\, e_i(t).$$

Writing $h = e^{2\sigma}$ we get, by the conformal connection formula

$$\bar{\nabla}_{\gamma'(t)} \gamma'(t) = \nabla_{\gamma'(t)} \gamma'(t) + 2 g_{\gamma(t)}(\operatorname{grad}(\sigma), \gamma')\, \gamma'(t) - g_{\gamma(t)}(\gamma', \gamma') \operatorname{grad}(\sigma)_{\gamma(t)}$$

where this derivative makes sense in any differentiable local extension of $\gamma'(t)$ to a vector field in a neighborhood of $\gamma(0)$. We shall show first that the condition $g_{\gamma(t)}(\operatorname{grad}(h)_{\gamma(t)}, v)\, \forall v \in N(\gamma(t), \gamma'(t))$ is enough to deduce the fact that $\gamma(t), |t| \leqq L$ is still a geodesic segment of $\bar{g}$. Indeed, this means that all the components of $\operatorname{grad}(h)_{\gamma(t)}$ corresponding to the directions which are normal to $\gamma'(t) = e_0(t)$ are zero. This clearly holds as well for σ since $\operatorname{grad}(\sigma)_p = \dfrac{1}{2h(p)} \operatorname{grad}(h)_p$. This implies that

$$\begin{aligned}
\bar{\nabla}_{\gamma'(t)} \gamma'(t) &= 2 g(\operatorname{grad}(\sigma)_{\gamma(t)}, e_0(t))\, e_0(t) - g(\operatorname{grad}(\sigma)_{\gamma(t)}, e_0(t))\, e_0(t) \\
&= h(\gamma(t))\, g(\operatorname{grad}(\sigma)_{\gamma(t)}, e_0(t))\, e_0(t) \\
&= h(\gamma(t)) \frac{\partial}{\partial x_0}(\gamma(t))\, \gamma'(t).
\end{aligned}$$

But now, it is well known that if the covariant derivative of a curve is a multiple of its tangent vector at every point of the curve then it is a geodesic of the given metric, modulo reparametrization. And using the fact that $\operatorname{grad}(h(\gamma(t)) = 0 \forall |t| \leqq L$ we get

$$\bar{\nabla}_{\gamma'(t)} \gamma'(t) = 0.$$

$\forall |t| \leqq L$ which clearly concludes the proof of the Lemma. $\square$

Denote as ∂x_i: $V \subset M \to TM$ the coordinate vector field associated to the i^{th} coordinate of the system ϕ. Given two vector fields X, Y defined in V let $Q(X, Y)$ be the following linear operator defined in the set of C^∞ functions:

$$Q(X, Y)(h) = X(Y(h)) - (\nabla_X Y)(h) - X(h)\, Y(h)$$

where $X(h)=dh(X)$ is the derivative of $h: M \to R$ in the direction of X. Notice that Q is linear with respect to C^∞ functions, i.e., $Q(fX, Y)(h)=Q(X, fY)(h) = fQ(X, Y)(h)\ \forall f, h \in C^\infty(M, \mathbb{R})$ thus defining a tensor. Let $\mathscr{R}$ and $\bar{\mathscr{R}}$ be the curvature tensors associated to g and $\bar{g}$ respectively. Then we have

$$\begin{aligned} e^{-2\sigma}\bar{\mathscr{R}}(X, Y, Z, W) = \mathscr{R}(X, Y, Z, W) &+ [Q(Y, Z) + g(Y, Z)\|\operatorname{grad}(\sigma)\|^2]\, g(X, W) \\ &- [Q(X, Z) + g(X, Z)\|\operatorname{grad}(\sigma)\|^2]\, g(Y, W) \\ &+ g(Y, Z)\, Q(X, W) - g(X, Z)\, Q(Y, W). \end{aligned}$$

If we define $\mathscr{R}_{hijk}(p)$ by

$$\mathscr{R}_{hijk}(p) = g_p(\mathscr{R}(\partial x_h(p), \partial x_i(p))\, \partial x_j(p), \partial x_k(p))$$

and $g_{ij}(p) = g(\partial x_i(p), \partial x_j(p))$ then we get

$$\begin{aligned} e^{-2\sigma(p)}\bar{\mathscr{R}}_{0i0j}(p) = \mathscr{R}_{0i0j}(p) &+ [Q_{i0} + g_{i0}\,\|\operatorname{grad}(\sigma)\|^2]\, g_{j0} \\ &- [Q_{00} + g_{00}\,\|\operatorname{grad}(\sigma)\|^2]\, g_{ij} + g_{i0}\, Q_{0j} - g_{00}\, Q_{ij} \end{aligned}$$

where $\bar{\mathscr{R}}_{hijk}(p) = \bar{g}_p(\bar{\mathscr{R}}_h(\partial x_h(p), \partial x_i(p))\, \partial x_j(p), \partial x_k(p))$ and $Q_{ij} = Q(\partial x_i, \partial x_j)$. These formulae are calculated in [5] with detail (see also [6, 10]).

Following the notation in Corollary 1.2 we have that the operators $K(t): N_{\gamma(t)} \to N_{\gamma(t)}$ defined along $\gamma(t)$ have entries $K_{ik}(t) = \mathscr{R}_{0i0k}(\gamma(t))$. Let $\bar{K}(t)$ be a family of self-adjoint linear operators whose expressions in local coordinates are

$$\bar{K}_{ik}(t) = \bar{\mathscr{R}}_{0i0k}(\gamma(t)).$$

Notice that if g and $\bar{g}$ coincide along $\gamma(t)$, then $[\bar{K}_{ik}(t)]$ would be a matrix representation of the sectional curvature operator of $\bar{g}$ in $\gamma(t)$ in a Fermi coordinate system for $(M, \bar{g})$.

Lemma 2.2 *Let h be as in Lemma 2.1. If* $\dfrac{\partial h}{\partial x_0}(\gamma(t)) = 0$, $\dfrac{\partial^2 h}{\partial x_i\, \partial y_k}(\gamma(t)) = 0$ *for every $i, k > 0$, $i \neq k$, and $|t| < L$ then*

(a) $\bar{K}_{ik}(\gamma(t)) = K_{ik}(\gamma(t))\ \forall i, k > 0, i \neq k$.

(b) $\bar{K}_{ii}(\gamma(t)) = K_{ii}(\gamma(t)) - \dfrac{\partial^2 \sigma}{\partial x_i^2}(\gamma(t))$

where $\sigma = \frac{1}{2}\log(h)$ and $|t| < L$.

Proof. Since – from the hypotheses of Lemma 2.1 – h has the property that $\operatorname{grad}(h)_{\gamma(t)} = 0$ we have that

$$\frac{\partial \sigma}{\partial x_i}(\gamma(t)) = 0.$$

$\forall i$ and $|t| < L$. Moreover, from the hypotheses in the statement of Lemma 3.2 we also have that

$$\frac{\partial^2 \sigma}{\partial x_i\, \partial x_j}(\gamma(t)) = 0$$

$\forall i \neq j$, $|t|<L$. From the construction of the coordinate system it is easy to get that $g_{ij}(\gamma(t))=\delta_{ij}\,\forall |t|<L$. These remarks imply that

$$(*)\qquad \begin{aligned} Q(e_i,e_j)(\sigma)|_{\gamma(t)} &= \{e_i(e_j(\sigma))-(\nabla_{e_i} e_j)(\sigma)-e_i(\sigma)\,e_j(\sigma)\}|_{\gamma(t)} \\ &= \frac{\partial^2\sigma}{\partial x_i\,\partial x_j}(\gamma(t))-d\sigma_{\gamma(t)}(\nabla_{e_i} e_j)-\frac{\partial\sigma}{\partial x_i}(\gamma(t))\frac{\partial\sigma}{\partial x_j}(\gamma(t)) \\ &= \frac{\partial^2\sigma}{\partial x_i\,\partial x_j}(\gamma(t))-g(\mathrm{grad}(\sigma),\nabla_{e_i} e_j)_{\gamma(t)} \\ &= \frac{\partial^2\sigma}{\partial x_i\,d x_j}(\gamma(t)). \end{aligned}$$

So the conformal curvature formula and $(*)$ imply

$$\begin{aligned} e^{-2\sigma(\gamma(t))}\bar{K}_{ij}(\gamma(t)) &= K_{ij}(\gamma(t))+g_{0j}\frac{\partial^2\sigma}{\partial x_i\,\partial x_0}(\gamma(t))-g_{ij}\frac{\partial^2\sigma}{\partial x_0^2}(\gamma(t)) \\ &\quad -g_{0i}\frac{\partial^2\sigma}{\partial x_j\,\partial x_0}(\gamma(t))-g_{00}\frac{\partial^2\sigma}{\partial x_i\,\partial x_j}(\gamma(t)) \\ &= \begin{cases} K_{ij}(\gamma(t)) & \text{if } i,\ j>0\ i\neq j \\ K_{ii}(\gamma(t))-\dfrac{\partial^2\sigma}{\partial x_i^2}(\gamma(t)) & \text{if } i=j>0 \end{cases} \end{aligned}$$

where, in the last equality, we used the fact that $\frac{\partial^2\sigma}{\partial x_0^2}(\gamma(t))=0\,\forall |t|\leqq L$ which comes from the hypotheses on h. From the hypotheses we get also that $\sigma(\gamma(t))=0\,\forall |t|\leqq L$ so the statement of the lemma follows from the last formula. □

To conclude this section we shall point out some of the consequences of Lemma 2.2. Observe that for every $(p,v)\in TM$ the subspace $N(p,v)=\{w\in T_p M, g(v,w)=0\}$ is the same for every conformal change of g. So from statement (a) we deduce that the curvature operators of the family of conformal deformations of the metric g obtained by $\bar{g}_p=h(p)\,g_p$, where $h\in C^\infty(M,\mathbb{R})$ is as in the statement of this lemma, generate a family of operators

$$\begin{aligned} &\bar{D}(t)\colon\ N_{\gamma(t)}\to N_{\gamma(t)} \\ &\bar{D}(t)=\bar{K}(t)-K(t) \end{aligned}$$

which is a curve of diagonal operators, i.e., $\bar{D}_{ij}(t)=0\,\forall i\neq j$, where $N_{\gamma(t)}=N(\gamma(t),\gamma'(t))$. Moreover, from Lemma 2.2 (b)

$$\bar{D}_{ii}(t)=-\frac{\partial^2\sigma}{\partial x_i^2}(\gamma(t)).$$

3 Metric perturbations with prescribed curvature

Proposition 3.1 *Let $L>0$ be such that every g-geodesic segment of length $2L$ is an imbedding. Then, for every C^2 neighborhood U of g in the set of Riemannian* *metrics there exist* $\varepsilon_0>0$, $\delta_0=\delta_0(\varepsilon_0)>0$, *such that given a geodesic* $\gamma: R\to M$ *of* g *and every* $0<\varepsilon\leqq\varepsilon_0$ *there exist one parameter families of metrics* $\{g_\delta\}\subset U$, $\delta\in(0,\delta_0]$, *and self-adjoint operators with respect to* g

$$K_\delta(t):\ N_{\gamma(t)}\to N_{\gamma(t)}$$

defined for every $t\in R$ *and every* $\delta\in[0,\delta_0]$ *with the following properties*:

(a) g_δ *and* g *are conformal for every* δ, $g_\delta=g$ *along* $\gamma(t)$, $|t|<L$ *and outside of a* δ*-tubular neighborhood of* $\{\gamma(t),\ |t|<L\}$, *and there exists* $C>0$ *such that* $\|g-g_\delta\|_{C^1}<C\delta\,\forall\,\delta\in(0,\delta_0]$. *Moreover*, g *coincides with* g_δ *along* $\gamma(t)$ *up to the first jet, and if* $\gamma_\delta(t)$ *is the geodesic of* g_δ *defined by* $\gamma_\delta(0)=\gamma(0)$, $\gamma'_\delta(0)=\gamma'(0)$ *then* $\gamma_\delta(t)=\gamma(t)\,\forall\,|t|\leqq L$.

(b) *The operators* $K_\delta(t)$ *are the sectional curvature operators of the metrics* g_δ, i.e., *if* $\mathscr{R}_{g_\delta}$ *is the curvature tensor associated to* g_δ *then*

$$(K_\delta)_{ij}(t)=(\mathscr{R}_{g_\delta})_{0i0j}(\gamma(t))$$

$\forall\,\delta\in(0,\delta_0]$, $\forall\,t\in R$ *and*

$$g((K_\delta(t)-K(t))\,w,w)\geqq 0$$

$\forall\,|t|\leqq L$, $w\in N_{\gamma(t)}$.

(c) *The operators* $K_0(t)$ *are defined by*

(i) $K_0(t)=K_\delta(t)$ *for every* $|t|\leqq L$ *and every* $\delta\in(0,\delta_0]$.

(ii) $K_0(t)=K(t)$ *for every* $|t|>L$

and they satisfy

(iii) $g((K_0(t)-K(t))\,w,w)>\varepsilon\,\|w\|^2\,\forall\,|t|<\varepsilon$, $w\in N_{\gamma(t)}$.

Let $\{e_{\delta i}(t)\}$ $i=0,1,\dots,n-1$ be the orthonormal frame defined along $\gamma_\delta(t)$ by $e_{\delta i}(0)=e_i(0)$. Then a matrix representation of $K_\delta(t)$ according to Proposition 3.1 (b) is given by

$$(K_\delta)_{ij}(t)=g_\delta(\mathscr{R}_{g_\delta}(\gamma'_\delta(t),e_{\delta i}(t))\,\gamma'(t),e_{\delta j}(t)).$$

Remark that $e_{\delta i}(t)=e_i(t)\,\forall\,i=0,1,\dots n-1$, $\forall\,|t|\leqq L$. This is because the metrics g and g_δ coincide along $\gamma(t)|t|\leqq L$ up to the first jet (Proposition 3.1 (a)).

Let $C>0$ be a fixed constant. We proceed to construct a family of C^∞ functions $h_\delta: M\to R$ satisfying the hypotheses of Lemmas 2.1 and 2.2 and such that the matrices $[D_{ii}(t)]$ defined at the end of Sect. 2 are just $C\cdot I^{(n-1)^2}$ for every $|t|<L$, where $I^{(n-1)^2}$ is the identity matrix on R^{n-1}. This will conclude the proof of Proposition 3.1.

Let $f: R\to R$ be a C^∞ bump function satisfying the following properties:

(a) $f(x)=1\,\forall\,|x|\leqq\frac{1}{2}$

(b) $f(x)=0\,\forall\,|x|>1$

(c) $0\leqq f(x)\leqq 1\,\forall\,\frac{1}{2}\leqq|x|\leqq 1$.

For a given $C>0$ let $p(x)=\dfrac{C}{2}\cdot x^2(x^2-1)$. Define $Q(x)=f(x)\cdot p(x)$. For a real number $\delta>0$ recall that $R_\delta=\bigcup_{|t|\leqq L} B_\delta(\gamma(t))$, where $B_r(q)$ is the closed ball of radius r with center at q, and define $\eta_\delta: R\to R$ by

$$\eta_\delta(t)=\delta^2 Q\left(\frac{t}{\delta}\right).$$

The function $\eta_\delta(t)$ satisfies the following properties: for a given function $\beta: R\to R$ let $\operatorname{supp}(\beta)=\{t\in R\backslash\beta(t)\neq 0\}$ be the support of β. Then

Lemma 3.1 (a) $\operatorname{supp}(\eta_\delta)=\operatorname{supp}(\eta'_\delta)=\operatorname{supp}(\eta''_\delta)=[-\delta,\delta]$.

(b) *There exists a constant $A>0$ such that*

(b.1) $\|\eta_\delta\|_\infty < CA\delta^2$

(b.2) $\|\eta_\delta\|_{C^1} < CA\delta$

(b.3) $\|\eta_\delta\|_{C^2} < CA$

(c) $\eta_\delta(0)=\eta'_\delta(0)=0$, *and* $\eta''_\delta(0)=-C$.

Lemma 3.1 follows from elementary calculations. Define $\lambda_\delta: M\to R$ as

$$\lambda_\delta(q)=\lambda_\delta(\phi^{-1}(x_0,\ldots,x_{n-1}))=f\left(\frac{x_0}{L}\right)\eta_{b\delta}\left[\left(\sum_{i=1}^{n-1}x_i^2\right)^{\frac{1}{2}}\right]$$

where $b>0$ is defined in the remarks preceeding Lemma 2.1. From the very definition of b we have that the support of λ_δ and all its derivatives is included in R_δ, and subsequently we give a series of properties of λ_δ:

Lemma 3.2 (a) $\lambda_\delta\in C^\infty(M,R)\,\forall\, 0<\delta<\delta_0$

(b) $\lambda_\delta(\gamma(t))=0\,\forall\,|t|\leqq L$, $\lambda_\delta(q)=0\,\forall\, q\in M-R_\delta$

(c) $\dfrac{\partial\lambda_\delta}{\partial x_i}(\gamma(t))=0\,\forall\, i,\forall\,|t|\leqq L$

(d) $\dfrac{\partial^2\lambda_\delta}{\partial x_i\,\partial x_j}(\gamma(t))=0\,\forall\, i\neq j,\forall\,|t|\leqq L$

(e) $\dfrac{\partial^2\lambda_\delta}{\partial x_i^2}(\gamma(t))=-Cf\left(\dfrac{t}{L}\right)\leqq 0,\forall\, i\neq 0,\forall\,|t|\leqq L$

$\dfrac{\partial^2\lambda_\delta}{\partial x_i^2}(\gamma(t))=-C\,\forall\, i\neq 0,\forall\,|t|\leqq\dfrac{L}{2}$

$\dfrac{\partial^2\lambda_\delta}{\partial x_0^2}(\gamma(t))=0\,\forall\,|t|\leqq L$

Notice that the second derivatives of λ_δ at the points $\gamma(t)$ do not depend on δ.

(f) *There exists $B>0$ such that $\|\lambda_\delta\|_{C^1}<BC\delta$, and $\|\lambda_\delta\|_{C^2}<BC$.*

Remark that the function λ_δ is concave when restricted to small normal sections of $\gamma(t)|\,|t|<L$, i.e., for each fixed $t\in(-L,L)$ the Hessian of the function

$$\lambda_\delta:\ V_{\gamma(t)}(\rho)\cap\underset{\gamma(t)}{\exp}(N_{\gamma(t)})\to R$$

is a negative definite matrix for $\rho=\rho(\delta)$ suitably small, where recall that $N_g(\gamma(t),\gamma'(t))$ is the subspace of $T_{\gamma(t)}M$ which is normal to $\gamma'(t)$, $V_{\gamma(t)}(\rho)$ is a ball of radius ρ centered at $\gamma(t)$ and $\underset{p}{\exp}$ is the exponential map. Now, define the family of C^∞ functions $h_\delta: M\to R$ by $h_\delta(q)=1+\lambda_\delta(q)$, and consider the family of conformal metrics

$$(g_\delta)_q=h_\delta(q)\,g_q.$$

Observe that $h_\delta(\gamma(t))=1\,\forall\,|t|\leqq L$ and $h_\delta(q)=1\,\forall\,q\in M-R_\delta$. Statement (a) of Proposition 3.1 follows from Lemmas 3.1 and 3.2.

To prove (b) and (c) in Proposition 3.1, remark that the function $\sigma_\delta(q)=\frac{1}{2}\log h_\delta(q)$ satisfies

$$\sigma_\delta(\gamma(t))=\frac{\partial\sigma_\delta}{\partial x_i}(\gamma(t))=\frac{\partial^2\sigma_\delta}{\partial x_0^2}(\gamma(t))=0$$

for every $|t|\leqq L$, $i=0,1,\ldots,n-1$ and for all δ, which follows easily from Lemma 3.2. Besides, all the mixed derivatives of σ_δ are zero along $\gamma(t)$ from (d) in the same lemma. So we conclude that the function h_δ satisfies the hypotheses of Lemmas 2.1 and 2.2. In particular, the curve $\{\gamma(t),|t|\leqq L\}$ is a segment of a unit geodesic of g_δ for all δ and the frame $\{e_i(t)\}$ is an orthonormal frame for g_δ. Thus, using the formulae of Lemma 2.2 and the remarks above we have that the sectional curvature operators K_δ associated to g_δ at the points $\gamma(t)$, $|t|\leqq L$, are given by:

$$(K_\delta)_{ik}(t)=K_{ik}(t)$$

for every $i\neq k$ and

$$(K_\delta)_{ii}(t)=K_{ii}(t)-\frac{\partial^2\sigma_\delta}{\partial x_i^2}(\gamma(t)).$$

But now,

$$\frac{\partial^2\sigma_\delta}{\partial x_i^2}(\gamma(t))=\frac{1}{2h_\delta(\gamma(t))}\frac{\partial^2 h_\delta}{\partial x_i^2}(\gamma(t))\leqq 0$$

for every $|t|\leqq L$ by the properties of the function h_δ given in Lemma 3.2. Therefore, statement (b) holds from the last two inequalities. It is also true that

$$\frac{\partial^2\sigma_\delta}{\partial x_i^2}(\gamma(t))=\frac{1}{2h_\delta(\gamma(t))}(-C)=-\frac{C}{2}$$

$\forall\,|t|\leqq\frac{L}{2}$, $\forall\,i\neq 0$ and $\forall\,0<\delta<\delta_0$, with δ_0 suitably small, which implies that the operators $D_\delta(t)=(K_\delta-K)(t)$ are diagonal and

$$(D_\delta)_{ii}(t)=(K_\delta)_{ii}(t)-K_{ii}(t)\geqq\frac{C}{2}$$

$\forall\,|t|<\frac{L}{2}$. Taking $C=2\varepsilon$ we obtain statement (c).

Corollary 3.1 *If $\delta\to 0$, then $g_\delta\to g$ in the C^1 topology.*

Proof. This is immediate from Lemma 3.2 (f).

Corollary 3.2 *There exists $D>0$ such that*

$$\sup_{0<\delta\leqq\delta_0}\|g_\delta-g\|_{C^2}<D\varepsilon$$

and in particular

$$\sup_{0<\delta\leqq\delta_0}\|\mathscr{R}_{g_\delta}-\mathscr{R}_g\|_{C^0}<D\varepsilon.$$

Proof. This follows from Lemma 3.2 (f) and the formula for the curvature of a conformal metric.

Let (x_k, y_k) $K=0, \ldots, n-1$ be the canonical coordinate system of TM induced by the system $\phi=(x_0, x_1, \ldots, x_{n-1})$. The geodesic flow of g in this coordinate system is the set of integral curves of the following vector field of M:

$$Z(t)=\begin{cases} \dfrac{\partial}{\partial t}(x_k)=y_k \\ \dfrac{\partial}{\partial t}(y_k)=-\sum\limits_{ij}\Gamma_{ij}^k\, y_i\, y_i \end{cases}$$

where Γ_{ij}^k are the so-called Cristoffel symbols of the metric g with respect to the coordinate system ϕ. Let G and G_δ be functions of R^{2n} into itself such that the differential equations

$$X'(t)=G(X(t))$$
$$X'(t)=G_\delta(X(t))$$

define the geodesic flows of g and g_δ respectively in the coordinate system ϕ. Then, it is clear that by looking at the formulae of Cristoffel symbols we get that

Corollary 3.3 *$G_\delta \to G$ in the C^1 topology. Moreover, there exists $E>0$ such that*

$$\|G_\delta - G\|_{C^1} < E\delta.$$

Corollary 3.4 *Let $T>0$. Then $\gamma_\delta|_{[-T,T]}$ converges uniformly to $\gamma|_{[-T,T]}$ as $\delta \to 0$.*

This lemma follows from of Proposition 3.1 (a), Corollary 3.3 and the following basic fact of the theory of ordinary differential equations:

Lemma 3.3 *Let $x'(t)=f(x(t))$ be a differential equation defined in $U \subset R^n$, where $f\colon U \to R^n$ is of class C^1. Let $f_m\colon R^n \to R^n$ be a sequence of C^1 maps satisfying*

(a) $\lim\limits_{m\to\infty} \|f_m - f\|_\infty = 0$ *in every open set $V \subset U$.*

(b) $\sup\limits_{m\in N} \|f_m'\|_\infty < C$, *where C is a constant.*

Then, if a sequence $\{a_i\} \subset U$ satisfies $\lim\limits_{i\to\infty} a_i = a_0 \in U$, and $x_i\colon [-b, b] \to R^n$ is a sequence of solutions of $x'=f_m(x)$ with initial conditions $x_i(0)=a_i$, we have that

$$\lim_{i\to\infty} x_i = x_0$$

uniformly in $[-b, b]$, where x_0 is the solution of $x'=f(x)$ with initial condition $x_0(0)=a_0$.

4 Proof of Theorem B

We begin this section by stating a characterization in terms of Jacobi fields of metrics with no conjugate points which are not Anosov metrics. This property plays a key role in understanding the boundary behavior of those metrics in the set of metrics with no conjugate points.

Lemma 4.1 *Let g be a metric of M with no conjugate points which is not an Anosov metric. Then, there exists a geodesic $\gamma: R \to M$ such that for every $\lambda > 0$ there exist perpendicular Jacobi vector fields $J_1(t)$ and $J_2(t)$ defined along $\gamma(t)$ with the following properties*:

(a) $J_1(0) = J_2(0)$, $\|J_1(0)\| = \|J_2(0)\| = 1$.
(b) *There exists* $a > 0$ *such that* $J_2(-a) = 0$, $J_1(a) = 0$.
(c) $\|J_1'(0) - J_2'(0)\| < \lambda$.
(d) *There exists* $\alpha > 0$ *such that* $\inf_{|t| < \alpha} \{\|J_1(t)\|, \|J_2(t)\|\} > \frac{1}{2}$.

Proof. By the results of Eberlein [4], there exist a geodesic $\gamma(t)$, a constant $D > 0$, and a perpendicular Jacobi field $J(t)$ in $\gamma(t)$ such that $\|J(t)\| < D \; \forall t \in R$. It is clear that $J(t) \neq 0 \; \forall t \in R$, otherwise we would have that $\limsup_{t \to +\infty} \|J(t)\| = +\infty$ by elementary properties of manifolds without conjugate points. Indeed, from [4] Proposition 2.9 we have

Sublemma. Let $\gamma(t)$ be a unit geodesic with curvature $> -k^2$. For a given $A > 0$ there exists $T = T(A, \gamma) > 0$ such that

$$\|X(s)\| > A \, \|X'(0)\|$$

for every Jacobi field $X(t)$ with $X(0) = 0$, $X'(0) \neq 0$. Given $T \in R$, denote by $J_T(t)$ the Jacobi field on $\gamma(t)$ determined by the initial conditions $J_T(0) = J(0)$ and $J_T(T) = 0$. From the sublemma and the fact that $\|J(T) - J_T(T)\| = \|J(T)\| \leqq D$ we also get that

$$\lim_{|T| \to \infty} \|J'(0) - J_T'(0)\| = 0.$$

Since $J(t)$ is continuous and $\|J(0)\| = 1$ there exists $\alpha > 0$ such that $\|J(t)\| > \frac{3}{4}$ for every $|t| < \alpha$. Now set $J_1(t) = J_a(t)$ and $J_2(t) = J_{-a}(t)$ for $a > 0$ big enough and then the lemma follows. □

Now we proceed to the proof of Theorem B. Suppose that g is a metric with no conjugate points which is not Anosov. Let $L > 0$ be the same of Proposition 2.1 and let $\gamma(t)$ be given as in Lemma 2.1. For a given C^2 neighborhood U of g let $\varepsilon_0 > 0$, $\delta_0 > 0$, be as in Proposition 2.1 and let g_δ be the family of metrics of Proposition 2.1 constructed along $\gamma(t)$. Take $0 < \varepsilon_1 < \frac{1}{2}\varepsilon_0^2$. Let $J_1(t)$, $J_2(t)$, α and $a > 0$ be defined as in Lemma 2.1 for $\lambda = \varepsilon_1$. Without loss of generality we can suppose that $\alpha > \varepsilon_0$. Fix an orthonormal, parallel frame $\{e_i(t)\}$, $i = 0, 1, \ldots, n-1$ along $\gamma(t)$ with $e_0(t) = \gamma'(t)$. Consider the operators $K(t)$ in these coordinates. By means of $J_1(t)$ and $J_2(t)$ we find a continuous, piecewise C^2

solution $X(t)$ of $x''(t)+K(t)\,x(t)=0$ in the interval $[-a, a]$ satisfying the following properties:

(a) $\|X(t)\|>\frac{1}{2}\forall t\in[-\alpha, \alpha]$.
(b) $X(-a)=X(a)=0$.
(c) $\|X(0)\|=1$.
(d) $X(t)$ is C^2 for every $t\neq 0$ and $\|X'^{+}(0)-X'^{-}(0)\|<\varepsilon_1$.

This solution is defined as:

$$X(t)=\begin{cases} J_1(t)\;\forall t\in[0, a] \\ J_2(t)\;\forall t\in[-a, 0]. \end{cases}$$

Consider the curve $c(t)=(g(t), K(t))$. From the index formula we get that

$$\begin{aligned} |I_{C,[-a,a]}(X, X)| &= |g(X'^{+}(0)-X'^{-}(0), X(0))| \\ &\leqq \|X'^{+}(0)-X'^{-}(0)\|\,\|X(0)\| \\ &\leqq \varepsilon_1 . \end{aligned}$$

Now, let $K_0(t)$ be as in Proposition 2.1 and define $c_0(t)=(g(\gamma(t)), K_0(t))$. It follows from the properties of g_δ that $K_0(t)$ is of class C^∞ and that it is C^0-close to $K(t)$.

Claim. The equation

$$x''(t)+K_0(t)\,x(t)=0$$

has conjugate points in the interval $[-a, a]$.

Indeed, let us estimate $I_{C_0,[-a,a]}(X, X)$:

$$\begin{aligned} I_{C_0,[-a,a]}(X, X) &= I_{C,[-a,a]}(X, X)-\int_{-a}^{a} g((K_0(t)-K(t))(X(t)), X(t))\,dt \\ &\leqq I_{C,[-a,a]}(X, X)-\int_{-\varepsilon_0}^{\varepsilon_0} g((K_0(t)-K(t))(X(t)), X(t))\,dt. \end{aligned}$$

But from assertions (b) and (c) of Proposition 3.1 we get

$$\begin{aligned} &< I_{C,[-a,a]}(X, X)-\int_{-\varepsilon_0}^{\varepsilon_0} \varepsilon_0\,\|X(t)\|\,dt \\ &< \varepsilon_1-\tfrac{1}{2}\varepsilon_0^2<0 \end{aligned}$$

by the choice of ε_1. Therefore, by Theorem 1.1 the equation above has conjugate points.

The next step toward the proof of Theorem B is to show the connection between conjugate points of the equation $x''(t)+K_0(t)\,x(t)=0$ and conjugate points in the manifold (M, g_δ). Remark that if γ possesses an *isolated point* p, i.e., there is an open ball B with center at p and radius $\beta>0$ such that $\gamma\cap B$ consists in only one connected segment σ of γ, then by taking $L=2$ length(σ) in the above argument one can easily verify that the existence of conjugate points for the equation is equivalent to the existence of conjugate points along $\gamma(t)$ in the metric g_δ (remark that γ is still a geodesic of g_δ). If $\gamma(t)$ is either periodic or non-recurrent the last assertion holds for instance. And in both cases we are also able to prove the statement of Theorem B in every C^k topology with $k \geqq 2$. This fact is straightforward from the construction of the g_δ.

However, recurrent orbits of the geodesic flow form a set of total Lebesgue measure by Poincare's lemma for measure preserving flows, so the above cases are certainly far away from the general case. The problem here is that Proposition 3.1 assures that $K_0(t)$ coincides with a curve of curvature operators of a geodesic of some Riemannian metric of M only in some imbedded segment of the given geodesic γ. At this point it becomes more appropriate to look at curves of symmetric operators from the point of view of Corollary 1.1 instead of looking at curvature operators coming from Riemannian metrics. We shall assume also that the geodesic γ has no isolated points.

Lemma 4.2 *Let $L>0$ be as in Proposition* 3.1, *and take $b>L$. Then $\forall |t|>b$ we have*

$$\lim_{\delta \to 0} \| K_\delta(t) - K(t) \|_\infty = 0.$$

Proof. Given a metric ω in M and $(p,v) \in T_1 M$ define $N_\omega(p,v) = \{w \in T_p, M, \omega(w,v)=0\}$. Let $K_\omega(p,v): N_\omega(p,v) \to N_\omega(p,v)$ be the associated sectional curvature operator. Now let g, g_δ be the metrics in Proposition 3.1. Since they are all conformal to $g N_g(p,v) = N_{g_\delta}(p,v) \, \forall \delta$ so we shall denote all those subspaces by $N(p,v)$. Then, by the definition of the operators $K_\delta(t)$ and $K(t)$ we have

$$\begin{aligned} \| K_\delta(t) - K(t) \|_\infty &= \| K_{g_\delta}(\gamma_\delta(t), \gamma'_\delta(t)) - K_g(\gamma(t), \gamma'(t)) \|_\infty \\ &= \| K_{g_\delta}(\gamma_\delta(t), \gamma'_\delta(t)) - K_{g_\delta}(\gamma(t), \gamma'(t)) \|_\infty \\ &\quad + \| K_{g_\delta}(\gamma(t), \gamma'(t)) - K_g(\gamma(t), \gamma'(t)) \|_\infty . \end{aligned}$$

The map which assigns to each $p \in M$ the curvature tensor $\mathcal{R}_\delta$ of g_δ at p is a continuous function, so given $\varepsilon>0$ there exists $\mu>0$ such that for every $(p,v) \in TM$, $g(v,v)=1$, if $d((p,v),(q,w))<\mu$ then $\| K_{g_\delta}(p,v) - K_{g_\delta}(q,w) \|_\infty < \varepsilon$. Here $d(,)$ is the distance on TM induced by g. Also, from Corollary 3.4 there exists $\delta_1>0$ such that $d((\gamma(t),\gamma'(t)),(\gamma_\delta(t),\gamma'(t))<\mu$ for every $|t|<b$ and $0<\delta<\delta_1$. This implies that

$$\| K_{g_\delta}(\gamma_\delta(t), \gamma'_\delta(t)) - K_{g_\delta}(\gamma(t), \gamma'(t)) \|_\infty \leqq \varepsilon$$

for every $0<\delta<\delta_1, \forall |t| \leqq b$, so the first term of the right hand side of the above inequality goes to zero with δ. It remains to estimate the second term of the inequality. Recall first that the support of g_δ – i.e., the set of $p \in M$ such that $g_\delta(p) \neq g(p))$ is included in the set

$$R_\delta = \{p \in M, d_g(\gamma(t), p) < \delta \forall |t| < L\}$$

(see Proposition 3.1(a)). On the other hand, the metrics g and g_δ agree outside R_δ, so their corresponding curvatures agree as well. This means that for every

$\gamma(t)\notin R_\delta$ we have $K_{g_\delta}(\gamma(t),\gamma'(t))=K_g(\gamma(t),\gamma'(t))$, so the second term in the inequality is zero, thus concluding the proof of the lemma. □

Observe that $K_\delta(t)=K_0(t)$ for every δ and $|t|<L$, so assuming that $a>L$ we conclude the following from the last lemma:

Corollary 4.1 *Consider the curves* $c_\delta:[-a,a]\to A(n-1)$ defined by $c_\delta(t)=(g_\delta(\gamma_\delta(t)), K_\delta(t))$. *Let* $c_0:[-a,a]\to A(n-1)$ *be defined* by $c_0(t)=(g(\gamma(t)), K_0(t))$. *Then*

$$\lim_{\delta\to 0}\|c_\delta(t)-c_0(t)\|_\infty=0$$

where $c_\delta(t)-c_0(t)=(g_\delta(t)-g(t), K_\delta(t)-K(t))$ *and*

$$\|c_\delta(t)-c_0(t)\|_\infty=\sup_{|t|\leq a}\{\|g_\delta(\gamma_\delta(t))-g(\gamma(t))\|_\infty, \|K_\delta(t)-K_0(t)\|_\infty\}.$$

We have already proved that the equation $x''(t)+K_0(t)\,x(t)=0$ has conjugate points in the interval $[-a,a]$. So from Corollaries 1.1 and 4.1 we get that the equations

$$x''(t)+K_\delta(t)\,x(t)=0$$

have conjugate points for δ small enough. But now, the operators $K_\delta(t)$ are curvature operators of the metrics g_δ, which means that the Riemannian manifolds (M, g_δ) have conjugate points for δ close to zero. This concludes the proof of Theorem B.

References

1. Anosov, D.: Geodesic flow on closed Riemannian manifolds of negative curvature. Tr. Mat. Inst. Steklova **90** (1967)
2. Cheeger, J., Ebin, D.: Comparison Theorems in Riemannian Geometry. Amsterdam: North-Holland 1975
3. Eberlein, P.: Geodesic flows in certain manifolds without conjugate points. Trans. Am. Math. Soc. **167**, 151–162 (1972)
4. Eberlein, P.: When is a geodesic flow of Anosov type I. J. Differ. Geom. **8**, 437–463 (1973)
5. Eisenhart, L.: Riemannian Geometry. Princeton: Princeton University Press 1964
6. Gromoll, D., Klingenberg, W., Meyer, W.: Riemannsche Geometrie im Grossen. (Lect. Notes Math., vol. 55) Berlin Heidelberg New York: Springer 1968
7. Hopf, E.: Closed surfaces without conjugate points. Proc. Natl. Acad. Sci. USA **34**, 47–51 (1948)
8. Klingenberg, W.: Riemannian manifolds with geodesic flow of Anosov type. Ann. Math. **99**, 1–13 (1974)
9. Klingenberg, W.: Lectures on closed geodesics. Berlin Heidelberg New York: Springer 1978
10. Kulkarni, R.: Curvature structures and conformal transformations. J. Differ. Geom. **4**, 425–451 (1970)
11. Pugh, C.: An improved closing lemma and a general density theorem. Am. J. Math. **89**, 1010–1021 (1967)