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Algebraic families of p-adic tori

Werner Lütkebohmert and Ferdinand Vogel

Mathematisches Institut der Universität Münster, Einsteinstr. 62, W-4400 Münster, Federal Republic of Germany

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A p-adic torus is the rigid analytic quotient T/M of an affine torus T over a p-adic field K by a lattice M of rank equal to the dimension of T. If such a torus admits Riemann period relations, it is an abelian variety with torus reduction over the valuation ring R of K. Conversely, an abelian variety over K whose Néron model over R has torus reduction is a p-adic torus; cf. [R1] or [BL2]. Let us call such abelian varieties toric in the following.

In this paper we study abelian schemes $X \to S$ where S is an affinoid space or a scheme of finite type over K and where the fibres X_s are potentially toric; i.e., become toric after extending the residue field $k(s)$ of s, for the (closed) points s of S. We will show that such abelian schemes can be uniformized simultaneously after étale surjective base change, i.e., étale locally on S with respect to the rigid analytic topology there is a representation $\mathbb{G}^r_{m,S}/M$ where $M \cong \mathbb{Z}^r_S$ is an S-subgroup space of $\mathbb{G}^r_{m,S}$ which is constant.

As an application we will show a rigidity theorem. In the case where S is a connected K-scheme of finite type, (d-fold polarized) abelian schemes $X \to S$ with potentially toric fibres can occur only in quasi-isotrivial form; i.e., the associated map from S to the coarse moduli space of (d-fold polarized) abelian varieties is constant. The latter means that the fibres X_s become isomorphic to each other over an algebraic closure of K. Since the Jacobians of Mumford curves are toric, the rigidity result implies via the theorem of Torelli that an algebraic smooth family of Mumford curves over a K-scheme of finite type is quasi-isotrivial; cf. [L1].

Global rigidity problems for families of curves resp. of abelian varieties were considered by Arakelov [A] resp. by Faltings [F]; see also the lecture of Mumford [M2, Lecture II].

1 Uniformization

In the following let R be a complete discrete valuation ring with field of fractions K and residue field k, and let π be a uniformizing parameter of R. The purpose of this section is to study the simultaneous uniformization of abelian schemes over an affinoid space Sp(A); i.e., the analytification of an abelian scheme over Spec(A). We will assume that the rigid fibres of the given abelian scheme are potentially toric. The main result of this section is the following theorem.

Theorem 1 *Let S be a normal affinoid space and let $n \geqq 3$ be an integer prime to the residue characteristic of K. Let $X \to S$ be an abelian scheme such that all rigid fibres are potentially toric. Moreover assume that X and its dual X' admit level-n-structures. Then, locally on S with respect to the Grothendieck topology, X can be uniformized: i.e., $X \times_S S_i \cong \mathbb{G}^d_{m,S_i}/M$ where $M \cong \mathbb{Z}^d_{S_i}$ is a lattice in $\mathbb{G}^d_{m,S_i}$ for a finite covering $\{S_1, \ldots, S_r\}$ by open affinoid subdomains S_i of S.*

Proof. The abelian scheme $X \to S$ has generically a polarization. Since the base scheme is normal, the polarization extends to the whole S; cf. [R4, XI, 1.4]. Therefore we may assume that $X \to S$ is polarized.

Set $\bar{S} = \operatorname{Spec}(A)$ where A is a flat R-algebra of topologically finite type such that $S = \operatorname{Sp}(A \otimes K)$. Due to [AC, IX, § 4, Exercice 22] such a ring A is a Nagata ring in the sense of Bourbaki [AC, IX, § 4, n° 2, Definition 2]. So its normalization $\bar{A}$ is finite over A and, hence, it is of topologically finite type over R. Since $A \otimes_R K$ is normal, we may choose $A = \bar{A}$ to be normal. We remind the reader that $\bar{A}$ consists of all elements f of $A \otimes_R K$ with spectral norm $|f| \leqq 1$. As a first step we will prove:

(1.1) There is an S-admissible blowing-up $\bar{S}' \to \bar{S}$ of $\bar{S}$ and a finite morphism $\bar{S}^* \to \bar{S}'$ which is an isomorphism over the rigid part such that X extends to a semi-abelian scheme $\bar{X}$ over $\bar{S}^*$.

Let us first assume that $X \to S$ is principally polarized. Let $A_{g,n} \to \operatorname{Spec} \mathbb{Z}[1/n]$ be the moduli space of principally polarized abelian varieties with level-n-structure. Let $\bar{A}_{g,n}$ be a projective toroidal compactification of $A_{g,n}$ over $\mathbb{Z}[1/n]$; cf. the book of Chai and Faltings [CF, Chapt. 5]. Our abelian scheme $X \to S$ corresponds to a morphism $\Phi: S \to A_{g,n}$. Let $\Gamma \subset S \times_{\mathbb{Z}} A_{g,n}$ be the graph of Φ. Then consider the schematic closure $\bar{\Gamma} \subset \bar{S} \times_{\mathbb{Z}} \bar{A}_{g,n}$ of Γ. Since $\bar{A}_{g,n}$ is proper over $\mathbb{Z}[1/n]$, the projection $\bar{\Gamma} \to \bar{S}$ is proper. Due to the flattening technique [R2, 5.7.12], there exists an S-admissible blowing-up $\bar{S}' \to \bar{\Gamma}$ such that the composition $\bar{S}' \to \bar{\Gamma} \to \bar{S}$ is an S-admissible blowing-up of $\bar{S}$. The pull-back of the universal semi-abelian scheme over $\bar{A}_{g,n}$ yields a semi-abelian scheme $\bar{X} \to \bar{S}'$ extending the given abelian scheme $X \to S$.

If $X \to S$ is not principally polarized, we use Zarhin's trick to define a principal polarization on $P = (X \times_S X')^4$. By what we have proved so far there exists an S-admissible blowing-up $\bar{S}' \to \bar{S}$ such that the product $P = (X \times_S X')^4$ extends to a semi-abelian scheme $\bar{P}' \to \bar{S}'$. We can write X as a quotient $X = P/N$. Due to [R2, 5.2.2], there exists an S-admissible blowing-up $\bar{S}'' \to \bar{S}'$ such that N extends to a flat $\bar{S}''$-subgroup scheme $\bar{N}''$ of $\bar{P}'' = \bar{P}' \times_{\bar{S}'} \bar{S}''$. The representability of $(\bar{P}''/\bar{N}'')$ as an algebraic space follows from [A1, 6.3]. The representability by a scheme follows from Poincare's complete reducibility theorem [M1, § 19, Theorem 1] and extension properties of morphisms; cf. [D, 4.10]. For the latter one needs $\bar{S}''$ to be normal; so one has to replace $\bar{S}''$ by its normalization $\bar{S}^* \to \bar{S}''$. The morphism $\bar{S}^* \to \bar{S}''$ is finite, since the coordinate rings of open affine pieces of $\bar{S}''$ are Nagata rings. Since the rigid space S is assumed to be normal, $\bar{S}^* \otimes K \to \bar{S}'' \otimes K$ is an isomorphism of rigid spaces. The quotient is semi-abelian, since $\bar{P}''$ is semi-abelian; cf. [BLR, 7.4/2]. Since a composition of S-admissible blowing-ups is an S-admissible blowing-up [R2, 5.1.4] also, we have shown (1.1).

For the proof of the theorem, we may assume $\bar{S}^* = \bar{S}$. Indeed, let $\mathscr{S}$ be formal scheme obtained as the π-adic completion of S^*. It can be viewed as a formal structure on the rigid space S. Working locally on $\mathscr{S}$ means in the rigid context to perform base change from S to an admissible open covering; cf. [R3] or [L2]. Now the formal completion $\mathscr{X} \to \mathscr{S}$ of $\bar{X} \to \bar{S}$ with respect to the special fibre yields a formally smooth semi-abelian groups scheme which gives rise to an open rigid S-subgroup $\mathscr{X} \otimes K \to S$ of $X \to S$. The special fibre $\mathscr{X}_k$ is isomorphic to the special fibre $\bar{X}_k$ of $\bar{X}$; the notion "special fibre" is related to the situation over the base ring R. Let $R(s)$ be the valuation ring of the residue field $K(s)$ of a closed point s of S. Since $\bar{X} \to \bar{S}$ is a semi-abelian group scheme, $\bar{X} \times_{\bar{S}} \operatorname{Spec} R(s)$ is the identity component of the Néron model of the fibre $X \times_S \operatorname{Spec} K(s)$. Due to the assumption that all rigid fibres are analytic tori, the closed fibres of the $\bar{S}_k$-group scheme $\bar{X}_k$ are tori. Due to [SGA 3_{II}, X, 8.2] $\bar{X}_k$ is a torus over $\bar{S}_k$ and then by the lifting of tori [SGA 3_{II}, IX, 3.6] we see that the formal completion $\mathscr{X}$ of $\bar{X}$ is a formal torus over $\mathscr{S}$. Due to [SGA 3_{II}, X, 4.5] there exists an étale surjective affine base change $\mathscr{S}_k^* \to \mathscr{S}_k$ such that $\mathscr{X}_k^* = \mathscr{X} \times_{\mathscr{S}} \mathscr{S}_k^*$ is split. The morphism $\mathscr{S}_k^* \to \mathscr{S}_k$ lifts to a formally étale morphism $\mathscr{S}^* \to \mathscr{S}$ which is again surjective on the generic fibre. Due to the lifting of tori $\mathscr{X}^* = \mathscr{X} \times_{\mathscr{S}} \mathscr{S}^*$ is a split formal torus over $\mathscr{S}^*$. Thus we have shown:

(1.2) The formal completion $\mathscr{X} \to \mathscr{S}$ of $\bar{X} \to \bar{S}$ with respect to the special fibre is a formal torus. Furthermore there exists a formally étale surjective base change $\mathscr{S}^* \to \mathscr{S}$ such that $\mathscr{X}^* = \mathscr{X} \times_{\mathscr{S}} \mathscr{S}^* \to \mathscr{S}^*$ is a split formal torus.

In the case where $X \to S$ is principally polarized, we can obtain the uniformization of X over $S^* = \mathscr{S}^* \otimes K$ directly from the result of Chai and Faltings [CF, Chapt. 3]. Namely they have shown that the category of semi-abelian group schemes over S^* whose special fibres are extensions of abelian schemes by split tori is equivalent to the category of degeneration data for Mumford's construction. In our case when the special fibre is a split torus, this result implies the uniformization of X as claimed in the theorem. We want to work also with not necessarily principal polarizations. So we have to follow the program of Raynaud [R1] which can now easily be deduced from the result in the principally polarized case by using Zarhin's trick:

(1.3) In the following denote by $\bar{X}$ the rigid space $\mathscr{X} \otimes K$ associate to $\mathscr{X}$ viewed as an open S-subgroup of X. Then the restriction map induces a bijection of the set of rigid group morphisms

$$\operatorname{Hom}(\mathbb{G}_{m,S}, X) \xrightarrow{\sim} \operatorname{Hom}(\bar{\mathbb{G}}_{m,S}, \bar{X})$$

where $\bar{\mathbb{G}}_{m,S}$ is the open subtorus of units of $\mathbb{G}_{m,S}$. The latter is the formal completion of $\mathbb{G}_{m,S}$ with respect to its special fibre.

(1.4) Assume that $\bar{X} \to S$ is a split formal torus $\bar{T}$. Let $T \to S$ be the affine torus which contains $\bar{T}$ as an open analytic subgroup (as torus of units). The canonical map $\bar{T} \to X$ extends to a surjective group homomorphism $\rho: T \to X$. Let M be the kernel of ρ. Then M is a lattice in T and, hence, X is a quotient T/M.

Let $X' \to S$ be the dual of X and assume that X' admits a semi-abelian extension over $\bar{S}$. Denote by $\bar{X}' \to \bar{S}$ the formal completion of this model with respect to its special fibre; it is a formal torus also. If $\bar{X}' \to S$ is split over S, then M is a split lattice; i.e., $M \cong \mathbb{Z}_S^d$ is a lattice in the sense of [BL2, 3.3]. The lattice is split, since it is isomorphic to the character group of the formal torus $\bar{X}' \to S$ which is assumed to be split.

For the rigidity theorem in Sect. 2 we need to know that the uniformization is actually given locally on S and not only after some formal étale surjective base change which in general will not be finite. The key point which is left to be shown is the splitting of the formal S-tori $\bar{X}$ and $\bar{X}'$ locally over S.

(1.5) The splitting of the formal rigid torus $\bar{X}$ in (1.2) and the splitting of the lattice in (1.4) is already satisfied locally over S (with respect to the Grothendieck topology).

This follows from Proposition 3 below, since the n-torsion ${}_n\bar{X}$ of $\bar{X}$ and the n-torsion ${}_n\bar{X}'$ of $\bar{X}'$ are also rational as we will show now. Indeed, let S be connected and let $\sigma: S \to {}_nX$ be an n-torsion point such that for some point $s \in S$ the image $\sigma(s)$ lies in $\bar{X}$. Then we have to show that σ maps S to $\bar{X}$. It suffices to show that $\sigma^{-1}(X - \bar{X})$ is empty. Of course we may assume that S is affinoid. Assume first that there is a covering map $\varphi: T \to X$ from a split torus $T = \mathbb{G}_{m,S}^d$ to X and that the kernel M of φ is split; i.e., $M \cong \mathbb{Z}_S^d$. For an element $\alpha \in \sqrt{|K^*|}$ with $\alpha \leqq 1$, we set

$$T_\alpha = \{t = (t_1, \ldots, t_d) \in T;\ \alpha \leqq |t_i| \leqq \alpha^{-1} \text{ for } i = 1, \ldots, d\}.$$

Since M is a closed subvariety of T and since $M \cap T_1 = \{1\}$, there exists an element $\alpha \in \sqrt{|K^*|}$ with $\alpha < 1$ such that $M \cap T_\alpha = \{1\}$, due to the maximum modulus principle. In other terms, this means that the canonical map $\rho: T_\varepsilon \to X$ of (1.4) is injective for any $\varepsilon \in \sqrt{|K^*|}$ with $\alpha < \varepsilon^2 < 1$. Denote by X_ε the image of T_ε under ρ. Setting $\delta = \sqrt[n]{\varepsilon}$ there is no point $t \in T_\delta - \bar{T}$ which is mapped to ${}_nX$. Returning to our original problem, we see $\sigma^{-1}(X - X_\delta) = \sigma^{-1}(X - \bar{X})$. Therefore $\sigma^{-1}(X - \bar{X})$ is an admissible open subvariety of S and, hence, it is empty due to the connectedness of S. The general case is reduced to the special case treated just before by base change using (1.2) and (1.4).

Thus the proof of Theorem 1 is clear. □

In view of Sect. 2, let us state the following remark.

Corollary 2 *Let $X \to S$ be an abelian scheme over a normal K-scheme of finite type such that all closed fibres are potentially toric. Then there exists an algebraic étale finite surjective map $S' \to S$ such that $X \times_S S'$ can be uniformized locally (with respect to the rigid Grothendieck topology) over S'.*

A different proof which relies on the Lemma of Gabber [D] and which follows strictly the program of Raynaud is given in [V]. Finally it remains to supply the splitting of the formal torus.

Proposition 3 *Let $\mathcal{T} \to \mathcal{S}$ be a formal morphism of admissible formal schemes where $\mathcal{T}$ is a formal torus over $\mathcal{S}$. Let $\bar{T} \to S$ be the associate rigid morphism. Assume that S is normal and that the n-torsion ${}_n\bar{T}$ is rational over S for some integer $n \geqq 3$ which is prime to* char(K). *Then there exists an admissible blowing-up $\mathcal{S}' \to \mathcal{S}$ such that $\mathcal{T}' = \mathcal{T} \times_{\mathcal{S}} \mathcal{S}'$ is locally trivial over $\mathcal{S}'$.*

Proof. Due to [SGA 3_{II}, X, 4.5] there exists an étale surjective, affine base change $\mathcal{S}_k^* \to \mathcal{S}_k$ such that $\mathcal{T}_k^* = \mathcal{T} \times_{\mathcal{S}} \mathcal{S}_k^*$ splits. The morphism $\mathcal{S}_k^* \to \mathcal{S}_k$ lifts to a formally étale morphism $\mathcal{S}^* \to \mathcal{S}$ which is again surjective on the generic fibre. Due to the lifting of tori [SGA 3_{II}, IX, 3.6] the formal torus $\mathcal{T}^* = \mathcal{T} \times_{\mathcal{S}} \mathcal{S}^*$ is split over $\mathcal{S}^*$. We may assume that $\mathcal{S}$ is affine and, hence, that $\mathcal{T}$ is affine, since $\mathcal{T} \to \mathcal{S}$ is formal affine; cf. [SGA 3_{II}, X, 4.9]. Denote by $\bar{T} \to S$ and by $S^* \to S$ the associate rigid morphism; these are morphisms of affinoid spaces, and let $\bar{T}^*$ be $\bar{T} \times_S S^*$ which is the generic fibre of $\mathcal{T}^*$. Now, using the fact that the n-torsion of $\bar{T}$ is rational, we will show that $\bar{T}$ is actually split, locally over S.

Let us first discuss the special case where $\mathcal{S}^* = \mathrm{Spf}(A^*) \to \mathcal{S} = \mathrm{Spf}(A)$ is finite. In this case $A \to A^*$ is faithfully flat and finite. Let T^* be a split affine A^*-torus whose formal completion is isomorphic to $\mathcal{T}^*$. Since a formal morphism of formal tori is algebraic as given by the map between their character groups, we get an (algebraic) descent datum on T^* with respect to $A \to A^*$. Hence, since T^* is affine over A^*, the descent is effective. So it descends to an affine A-torus T whose formal completion is isomorphic to $\mathcal{T}$. The n-torsion of $T \otimes K$, which coincides with the n-torsion of $\bar{T}$, is rational. So, S being normal, $T \otimes K$ is split. Namely, it is split over the generic points of S as follows from [SGA 7_{I}, IX, 4.7.1] by a galois argument, and due to the Néron property the splitting extends to an open subscheme of S which contains all points of codimension 1, and then the assertion follows from Weil's extension argument of group morphisms [BLR, 4.4/1]. Thus we see that $\bar{T}$ is split over S.

The general case will be reduced to the above special case by an approximation argument. There is an open affine subscheme $\mathcal{S}_k^1$ of $\mathcal{S}_k$ which contains all the generic points of $\mathcal{S}_k$ such that the restriction of $\mathcal{S}^* \to \mathcal{S}$ to $\mathcal{S}_k^1$ is finite, since an étale morphism of finite type is generically finite. Continuing this way we get a finite partition

$$\mathcal{S}_k = \mathcal{S}_k^1 \cup \mathcal{S}_k^2 \cup \ldots \cup \mathcal{S}_k^r$$

of $\mathcal{S}_k$ by (locally closed) affine subschemes such that the restriction of $\mathcal{S}^* \to \mathcal{S}$ to $\mathcal{S}_k^i$ is finite for $i = 1, \ldots, r$. Furthermore we can choose the subschemes $\mathcal{S}_k^i$ of the following special type:

$$\mathcal{S}_k^1 = \{s \in \mathcal{S}_k ; f_k^1(s) \neq 0\},$$
$$\mathcal{S}_k^i = \{s \in \mathcal{S}_k ; f_k^j(s) = 0 \quad \text{for } j = 1, \ldots, i-1 \quad \text{and} \quad f_k^i(s) \neq 0\},$$

where $f^i \in \mathcal{O}(\mathcal{S})$ and where f_k^i denotes the restriction of f^i to $\mathcal{S}_k$. Furthermore, for $\varepsilon_1, \ldots, \varepsilon_r \in \sqrt{|K^*|}$ with $\varepsilon_\rho < 1$ which will be specified later, we define open affinoid subvarieties

$$S(1) = \{s \in S ; |f^1(s)| \geqq 1\}$$
$$S(i) = \{s \in S ; |f^j(s)| \leqq \varepsilon_j \quad \text{for } j = 1, \ldots, i-1 \quad \text{and} \quad |f^i(s)| \geqq 1\}.$$
$$S(1)' = \{s \in S ; |f^1(s)| \geqq \varepsilon_1\}$$
$$S(i)' = \{s \in S ; |f^j(s)| \leqq \varepsilon_j \quad \text{for } j = 1, \ldots, j-1 \quad \text{and} \quad |f^i(s)| \geqq \varepsilon_i\}.$$

Then $\{S(1)', \ldots, S(r)'\}$ is an open affinoid covering of S. Now $S(i)$ is contained in $S(i)'$ and $\mathcal{O}(S(i)') \to \mathcal{O}(S(i))$ is dense for $i = 1, \ldots, r$. We will show that we can choose $\varepsilon_1, \ldots, \varepsilon_r$ such that $\bar{T}$ is split over $S(i)'$ for $i = 1, \ldots, r$. It is clear that this implies the assertion of the proposition. First choose some $\varepsilon_1 < 1$ which will be modified later on. Then

$$\bar{T}(1) = \bar{T} \times_S S(1) \subset \bar{T}(1)' = \bar{T} \times_S S(1)'$$

are affinoid and $\mathcal{O}(\bar{T}(1)') \to \mathcal{O}(T(1))$ is dense also. Similarly,

$$S(1)^* = S^* \times_S S(1) \subset S(1)^{*\prime} = S^* \times_S S(1)'$$

are affinoid and $\mathcal{O}(S(1)^{*\prime}) \to \mathcal{O}(S(1)^*)$ is dense. Furthermore set

$$\bar{T}(1)^* = \bar{T}(1) \times_S S^* \subset \bar{T}(1)^{*\prime} = \bar{T}(1)^* \times_S S(1)' \subset \bar{T}^* = \bar{T} \times_S S^*.$$

Since $S(1)^* \to S(1)$ is finite, it follows by the special case discussed above that $\bar{T}(1)$ is split over $S(1)$. Let $\underline{f}$ be a basis of the cocharacters on $\bar{T}(1)$. Now we can approximate $\underline{f}$ by a system $\underline{f}'$ of functions on $\bar{T}(1)'$ as close as we want. Now we know that $\bar{T}^*$ is split over S^*, so we can choose a basis $\underline{F}^*$ of its group of cocharacters. Then we obtain

$$\underline{f} = M \cdot \underline{F}^* \qquad \text{over } \bar{T}(1)^* = \bar{T}^* \times_S S(1) = \bar{T} \times_S S(1)^*$$

where M is a matrix with entries being locally constant sections of $\mathbb{Z}$; i.e., $M \in \mathrm{GL}(d, \mathbb{Z})(\bar{T}(1)^*)$. By an approximation argument it is clear that we can choose $\varepsilon_1 < 1$ so large that any connected component of $\bar{T}(1)^*$ is a restriction of a connected component of $\bar{T}(1)^{*\prime}$. Thus we can view M as a matrix of $\mathrm{GL}(d, \mathbb{Z})$ defined over $\bar{T}(1)^{*\prime}$. Then we can write

$$\underline{f}' = M \cdot \underline{F}^* + \underline{\Delta}$$

where $\underline{\Delta}$ is a vector of functions on $\bar{T}(1)^{*\prime}$. Since $\underline{f}'$ approximates $\underline{f}$, we may assume that

$$\Delta|_{\bar{T}(1)^*} \in \pi^2 \mathcal{O}(\mathcal{T}(1)^*)^d$$

where $\mathcal{T}(1)^* = \mathcal{T}^* \times_{\mathcal{S}} \mathcal{S}(1)$ and $\mathcal{S}(1)$ is the open subscheme of $\mathcal{S}$ corresponding to $S(1)$. Then we can choose $\varepsilon_1 < 1$ so large that

$$\underline{\Delta}|_{\bar{T}(1)^{*\prime}} \in \pi \mathcal{O}(\mathcal{T}(1)^{*\prime})^d$$

where $\mathcal{S}' \to \mathcal{S}$ is an admissible formal blowing-up such that $S(1)'$ is induced by a formally open subscheme $\mathcal{S}(1)'$ of $\mathcal{S}'$ and where $\mathcal{T}(1)^{*\prime}$ is induced by base change of $\mathcal{T} \to \mathcal{S}$ with $\mathcal{S}(1)' \to \mathcal{S}$. It is clear that $\underline{f}'$ gives rise to a trivialization of the torus $\mathcal{T}(1)'_k$ over $\mathcal{S}(1)'_k$. By the lifting of tori [SGA 3_{II}, IX, 3.6] this trivialization gives rise to a trivialization of $\bar{T}(1)'$ over $S(1)'$. If, in addition, we choose $\mathcal{S}' \to \mathcal{S}$ in such a way that $S(2)$ is induced by an open subscheme $\mathcal{S}(2)'$ of $\mathcal{S}'$, then $\mathcal{S}(2)'_k$ is mapped to $\mathcal{S}^2_k$ set-theoretically. Therefore, the induced map $\mathcal{S}^* \to \mathcal{S}$ becomes finite over $\mathcal{S}(2)'$ after the base change $\mathcal{S}(2)' \to \mathcal{S}$. So we can do the same reasoning as before to define ε_2. Continuing this way we complete the proof by induction. □

2 Rigidity

Now we will apply the uniformization theory to derive a rigidity theorem for abelian schemes with potentially toric fibres.

Theorem 4 *Let S be a connected scheme of finite type over a p-adic field K. Let $X \to S$ be an abelian scheme such that all closed fibres are potentially toric. Then X is constant; i.e., the fibres $X_{\bar{s}}$ are isomorphic to each other where $\bar{s}$ runs over the geometric points of S.*

Proof. For the proof we may assume that S is a smooth connected affine curve and, due to Corollary 2, we may assume that X can be uniformized locally (with respect to the rigid Grothendieck topology) over S. Thus there exists an open admissible covering $\{S_i; i \in I\}$ of S such that $X|S_i \cong T_i/M_i$ where $M_i \cong \mathbb{Z}^d_{S_i}$ is a lattice in $T_i \cong \mathbb{G}^d_{m,S_i}$, for $i \in I$. Let $p: \hat{S} \to S$ be the universal covering (cf. Sect. 3) of S. We may assume that the inverse image of each S_i can be decomposed into a disjoint union

$$p^{-1}(S_i) = \cup_j \hat{S}_{ij}$$

of connected open subvarieties $\hat{S}_{ij}$ isomorphic to S_i. Furthermore we may assume that all intersections $\hat{S}_{ij} \cap \hat{S}_{i'j'}$ are connected. The local uniformizations give rise to a cocycle

$$(\varphi_{ij}) \in Z^1(\{S_i, i \in I\}, \mathrm{GL}(d, \mathbb{Z}))$$

associated to the tori $\{T_i; i \in I\}$ and to a cocycle

$$(\mu_{ij}) \in Z^1(\{S_i, i \in I\}, \mathrm{GL}(d, \mathbb{Z}))$$

associated to the lattices $\{M_i; i \in I\}$. All these cocycle become trivial under pullback to $\hat{S}$. So we arrive at a global uniformization

$$X \times_S \hat{S} \cong \hat{T}/\hat{M}$$

where $\hat{M} \cong (\mathbb{Z}^d)_{\hat{S}}$ is a lattice in $\hat{T} \cong \mathbb{G}^d_{m,\hat{S}}$. It is clear that we may assume that X is polarized over S. So we have a polarization over $\hat{S}$ which gives rise to a positive definite (multiplicative) bilinear form (cf. [BL 2, 2.4])

$$B = (b_{ij}): (\mathbb{Z}^d)_{\hat{S}} \times (\mathbb{Z}^d)_{\hat{S}} \to \mathbb{G}_{m,K}.$$

The entries b_{ij} are global functions on $\hat{S}$. Rewriting this in the additive form, this means that, for any $s \in S$, the matrix

$$-\log|B| = (-\log|b_{ij}(s)|): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

is a positive definite bilinear form in the ordinary sense. Note that, for a function f on $\hat{S}$, the condition $-\log|f(s)|$ being positive implies f to be bounded. Due to Proposition 6 below, all bounded global functions on $\hat{S}$ are constant. Therefore, the positive definiteness of $-\log|B|$ implies that all entries b_{ij} are constant. Namely, it is clear for the entries in the diagonal, then it follows by induction that the entries in all subdiagonals are constant by the criterion of Hurwitz. Indeed, consider a positive definite (symmetric) matrix

$$A=\begin{pmatrix} a_{11}, & \dots, & a_{1n} \\ a_{j1}, & A', & a_{jn} \\ a_{n1}, & \dots, & a_{nn} \end{pmatrix}$$

For the determinant of A we obtain

$$\det A=(-1)\cdot\det A'\cdot(a_{1n})^2+\alpha\cdot a_{1n}+\beta$$

where α and β are independent of $a_{1n}=a_{n1}$. In our case where A is a submatrix of $-\log|B|$ which is symmetric with respect to the diagonal of $-\log|B|$, we know that $\det A'$, α, and β are constant, due to the induction hypothesis, and that $\det A$ and $\det A'$ are positive. Therefore, the absolute value $|a_{1n}|=|\log|b_{1n}||$ is bounded as $(-1)\cdot\det A'$ being negative. So b_{1n} is bounded and, hence, constant as a function of $s\in\hat{S}$. Thus we see that the matrix (b_{ij}) is constant as a function of $s\in\hat{S}$. This implies that the lattice is constant; i.e., the map

$$(\mathbb{Z}^d)_{\hat{S}} \xrightarrow{\sim} \hat{M} \hookrightarrow \mathbb{G}^d_{m,\hat{S}} \longrightarrow \mathbb{G}^d_{m,K}$$

is constant as follows from [BL 2, 3.2]. □

Corollary 5 *Let S be a connected scheme of finite type over K. Let $C\to S$ be a proper smooth curve such that all fibres are Mumford curves. Then C is constant; i.e., the fibres $C_{\bar{s}}$ are isomorphic to each other where $\bar{s}$ runs over the geometric points of S.*

Proof. The Jacobi variety of a Mumford curve is a p-adic torus. So the assertion follows from Theorem 3 by the theorem of Torelli. □

3 The universal covering of an algebraic curve

Let S be a smooth projective algebraic curve over K. After finite separable field extension there exists a semi-stable curve $\bar{S}$ over the valuation ring R with generic fibre S; cf. [AW] or [BL 1, 7.1]. Denote by $\mathcal{S}$ the formal completion of $\bar{S}$ with respect to the special fibre. After a suitable blowing-up of certain double points of $\bar{S}$, we may assume that the irreducible components of $\bar{S}_k$ have no self-intersection. Let Γ be the graph of coincidence of the irreducible components of $\bar{S}_k$; i.e., the vertices of Γ are the irreducible components of $\bar{S}_k$ and the edges of Γ joining two vertices are the double points lying on the corresponding irreducible components. Let $\hat{\Gamma}\to\Gamma$ be the universal covering of Γ in the sense of trees. Now use $\hat{\Gamma}$ as a rule of how to glue formal open parts of $\mathcal{S}$. In this way one obtains a formal scheme $\hat{\mathcal{S}}\to\mathcal{S}$ whose special fibre $\mathcal{S}_k$ has $\hat{\Gamma}$ as graph of coincidence. Denote by $\hat{S}$ the associate rigid space $\hat{\mathcal{S}}\otimes K$ and denote by $p\colon\hat{S}\to S$ the associate map $\hat{\mathcal{S}}\to\mathcal{S}$. The map $\hat{S}\to S$ serves as a universal covering in the sense of rigid spaces. For example it has the following property: For any open rigid analytic covering $\{S_i, i\in I\}$ of S there exists a refinement $\{S_j, j\in J\}$ such that $p^{-1}(S_j)$ decomposes into a disjoint union $\cup_k \hat{S}_{jk}$ such that

$p|\hat{S}_{jk}$ induces an isomorphism between $\hat{S}_{jk}$ and S_{jk} and such that all intersections $\hat{S}_{jk} \cap \hat{S}_{j'k'}$ are connected. This easily follows from the fact that one can refine the formal structure of S in such a way that the given covering $\{S_i, i \in I\}$ admits a refinement by a formal covering $\{S_j, j \in J\}$ where the reduction of any S_j is connected with at most one singular point which has to be an ordinary double point.

If S is an affine smooth curve, one has a universal covering $p: \hat{S} \to S$ also. Namely, choose a smooth compactification $\bar{S}$ of S which exists after extending the base field. Then take the universal covering $\bar{p}: \hat{\bar{S}} \to \bar{S}$ of $\bar{S}$ and define $p: \hat{S} \to S$ by removing the inverse images of the finitely many points of $\bar{S} - S$ in $\hat{\bar{S}}$. It is clear that $p: \hat{S} \to S$ has the property for rigid analytic coverings mentioned above. An important point in the proof of the rigidity theorem was the following proposition.

Proposition 6 *A bounded holomorphic function on $\hat{S}$ is constant.*

Proof. It suffices to prove the assertion only in the case where $\hat{S}$ is the universal covering of a smooth projective curve. Indeed, if $\bar{S}$ is a smooth compactification of S, we may view $\hat{S}$ as a open subvariety on the universal covering $\hat{\bar{S}}$ of $\bar{S}$ such that $\hat{\bar{S}} - \hat{S}$ consists of isolated points. So a bounded holomorphic function on $\hat{S}$ extends to $\hat{\bar{S}}$. Thus we see that we can reduce to the case where S is a smooth projective curve. Now let f be a holomorphic function on $\hat{S}$. Let $\hat{S}_i$ be the irreducible components of the reduction of $\hat{S}$, and let c_i be the norm of f on $\hat{S}_i$; i.e., the supnorm of f on a dense open part of $\hat{S}_i$. If f is not constant, there exists an index i_0 such the reduction of $f_{i_0} = f/c_{i_0}$ does not induce a constant function on $\hat{S}_{i_0}$. Thus we see that there exists a pole x on $\hat{S}_{i_0}$ for f_{i_0} which has to be a singular point in the reduction of $\hat{S}$. Now x lies also on a different component $\hat{S}_{i_1}$. The reduction of f_{i_1} induces a function on $\hat{S}_{i_1}$ which has a zero at x_0; cf. [BL 1, 3.1]. Due to [BL 1, 3.2] we have

$$c_{i_0} \leqq \alpha(x_0)^{-1} c_{i_1}$$

where $\alpha(x_0) < 1$ is the height of the annulus associated to x. Now f_{i_1} must have a pole x_1 on $\hat{S}_{i_1}$ which lies also on a component on $\hat{S}_{i_2}$ which is different from on $\hat{S}_{i_1}$. As above we obtain

$$c_{i_1} \leqq \alpha(x_1)^{-1} c_{i_2}.$$

Continuing this way we obtain

$$c_{i_0} \leqq \alpha(x_0)^{-1} \alpha(x_1)^{-1} \ldots \alpha(x_{n-1})^{-1} c_{i_n}.$$

Since only finitely many types of $\alpha(x_i) < 1$ occur, we see that f cannot be bounded. Thus we see that a bounded holomorphic function on $\hat{S}$ is constant. □

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