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Linear degeneracy and shock waves

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1 Introduction

This paper is primarily concerned with certain jump discontinuities in weak solutions of hyperbolic systems of conservation laws, namely discontinuities associated with linearly degenerate modes for which the integral manifolds of the corresponding eigenspace bundle are compact. We prove that, besides the familiar contact discontinuities, there are also shocks associated with these modes, and that, in many cases, these shocks have viscous profiles. When applied to magnetohydrodynamics, this theory proves that for arbitrary dissipation certain shocks associated with the rotational *Alfvén* mode have viscous profiles. The abstract results are not restricted to simple eigenvalues. As a preliminary step, we establish basic notions and properties of general linearly degenerate modes corresponding to eigenvalues of arbitrary multiplicity.

The systems under consideration are of the form

$$\mathbf{u}_t(x,t)+(f(\mathbf{u}(x,t)))_x=0, \qquad (x,t)\in\mathbb{R}\times\mathbb{R}_+; \tag{1.1}$$

the *flux function* f maps the *state space* U, an open connected subset of $\mathbb{R}^n$, smoothly into $\mathbb{R}^n$, and hyperbolicity of (1.1) means that $Df(u)$ be $\mathbb{R}$-diagonalizable at any $u\in U$. We restrict attention to special weak solutions of (1.1) which have the form

$$\mathbf{u}(x,t)=\begin{cases}u^-, & x<st\\ u^+, & x>st\end{cases} \tag{1.2}$$

with appropriate $u^-, u^+\in U$, $s\in\mathbb{R}$; these frequently considered piecewise constant functions serve as prototypes for more general jump discontinuities. Given u^-, u^+, and s, (1.2) is a weak solution of (1.1) if and only if the *Rankine-Hugoniot* jump conditions

$$f(u^+)-f(u^-)=s(u^+-u^-) \tag{1.3}$$

are satisfied.

Now assume that $\lambda \in C(U, \mathbb{R})$ is an eigenvalue of Df of constant multiplicity.

Definition 1 λ is called *linearly degenerate* iff its gradient is everywhere orthogonal to the corresponding eigenspace bundle $R = \ker(Df - \lambda I)$.

Definition 2 A solution (1.2) of (1.1) is called a *contact discontinuity associated with* λ iff

$$\lambda(u^-) = s = \lambda(u^+). \tag{1.4}$$

Definition 3 A solution (1.2) of (1.1) is called a *shock associated with* λ iff

$$\lambda(u^-) > s > \lambda(u^+), \tag{1.5}$$

and $\lambda^- < s$ for any eigenvalue $\lambda^- < \lambda(u^-)$ of $Df(u^-)$ as well as $\lambda^+ > s$ for any eigenvalue $\lambda^+ > \lambda(u^+)$ of $Df(u^+)$.

Contact discontinuities and, especially, shocks which are associated with a linearly degenerate eigenvalue are the main objects of interest in this paper. The notions of linear degeneracy, contact discontinuity, and shock were introduced by Lax in [8] for the case of a simple eigenvalue. The above definitions give obvious generalizations to eigenvalues of arbitrary multiplicity. Whereas Definitions 1 and 2 seem to be appropriate in general, Definition 3 is made here only for convenience in the present context and is not intended to give a new general interpretation to the word shock. Although the case of a simple eigenvalue can be just the interesting one for applications – so for the one given in the Appendix, which has also been the original motivation for this study –, we keep the multiplicity arbitrary throughout this paper since actually all results do not depend thereupon.

The following five theorems contain the abstract results of the paper.

Theorem 1 *Assume $\lambda \in C(U, \mathbb{R})$ is an eigenvalue of Df of constant multiplicity l. Then the vectorspace bundle $R = \ker(Df - \lambda I)$ is integrable, and its integral manifolds χ constitute a foliation $\mathscr{F}$ of U. If l is greater than 1, λ must be linearly degenerate.*

Theorem 2 *Assume $\lambda \in C(U, \mathbb{R})$ is an eigenvalue of Df of constant multiplicity and linearly degenerate. If u^-, u^+ lie in the same contact leaf χ of the corresponding foliation and s is the (constant) value that λ attains on χ, the triple (u^-, u^+, s) defines a contact discontinuity associated with λ. For any $(u^-, u^+, s) \in \mathscr{N}$, an appropriate open neighborhood of $\{(u, u, \lambda(u)) \mid u \in U\} \subset U \times U \times \mathbb{R}$, these are the only weak solutions* (1.2) *of* (1.1). *Especially, no $(u^-, u^+, s) \in \mathscr{N}$ defines a shock associated with λ.*

Theorem 3 *Assume $\lambda \in C(U, \mathbb{R})$ is an eigenvalue of Df of constant multiplicity and linearly degenerate. If a corresponding contact leaf χ is compact, then there exist triples (u^-, u^+, s) such that* (1.2) *is a shock associated with λ. Such triples exist arbitrarily near $\chi \times \chi \times \lambda(\chi)$ and are structurally stable.*

Theorem 4 *Assume $\lambda \in C(U, \mathbb{R})$ is an eigenvalue of Df of constant multiplicity and linearly degenerate. If a contact leaf $\chi \in \mathscr{F}$ is compact, then a whole neighborhood of $\chi \subset U$ is covered by contact leaves which are all compact.*

Theorem 5 *Assume $\lambda \in C(U, \mathbb{R})$ is an eigenvalue of Df of constant multiplicity l and linearly degenerate with compact contact leaves. Consider, in addition to the* *hyperbolic system, any strictly stable viscosity associated with it. Assume that $l=1$ or that the system of* o.d.e. *describing the traveling wave solutions of the associated parabolic system is gradient-like. Then there exist shocks associated with λ which have a viscous profile with respect to the given viscosity. More precisely, such shocks exist arbitrarily near any compact contact leaf χ in the sense that the corresponding triple (u^-, u^+, s) comes arbitrarily close to $\chi \times \chi \times \lambda(\chi)$, and each of these shocks has actually a whole $(l-1)$-parameter family of profiles associated with it. In all cases, the profiles are given by structurally stable heteroclinic orbits.*

Theorems 1 and 2 will be proved in Sect. 2, Theorems 3 and 4 in Sect. 3, and Theorem 5 in Sect. 4.

Similar results were already established in [3] for the case that the flux function is equivariant under $\mathbf{O}(m)$ (for some $m \leqq n$) as acting in a standard way on the state space. In that case the linearly degenerate mode is a rotational mode induced by the symmetry, and the compact integral manifolds are given by the orbits of the group action: spheres. Actually, for that specific case, more then is proved here was shown in [3]. The assumptions made in the present paper being more general, its results can be applied to less specialized cases, so to systems that have a rotational symmetry which is, however, not given by a standard representation of the orthogonal group. This property is shared e.g. by the system governing plane magnetohydrodynamic waves. In the Appendix, the theory is modified so as to establish, for arbitrary ratios of the commonly used four dissipation coefficients, the existence of viscous profiles for certain magnetohydrodynamic shocks associated with the linearly degenerate *Alfvén* mode, see Theorem A.1. Physically speaking, Theorem A.1 implies that in the presence of arbitrary dissipation – so especially for physically realistic values of the dissipation coefficients – some of the so-called intermediate shocks, which are unstable in the framework of ideal dynamics, can exist, see Theorem A.2. For these phenomena, the perception of which is suggested by previous observations in [A10] and [A8, A1, A10], respectively, no mathematical proofs seem to have been given prior to this work.

Theorems A.1 and A.2 and proofs thereof were presented during the IMA Workshop on Multidimensional Hyperbolic Problems in April 1989, see [4]. Most arguments used in the proofs of Theorems 1 and 3 had already been introduced in [2].

2 The geometry of linear degeneracy

In this section, we prove Theorems 1 and 2, which establish properties of arbitrary linearly degenerate modes. The proof of Theorem 1 is elementary, the existence of the foliation being a consequence of Frobenius' integrability theorem; Theorem 2 follows easily from a geometric consideration.

Proof of Theorem 1 In the case of a simple eigenvalue, the integrability of R is trivial: the leaves of $\mathscr{F}$ are given by the obvious integral curves of any vector field $r \in R$. Assume now that the eigenvalue $\lambda \in C(U, \mathbb{R})$ of Df has constant multiplicity greater than 1. According to Frobenius' Theorem (see e.g. [10]), a vector space bundle R is integrable if and only if for any two vector fields $r_1, r_2 \in R$

also their Lie-bracket $[r_1, r_2] = (r_1 \cdot \nabla) r_2 - (r_2 \cdot \nabla) r_1$ lies in R. In our case r_1, $r_2 \in R$ means

$$(Df - \lambda \mathrm{I}) r_i = 0, \qquad i = 1, 2.$$

This implies

$$\begin{aligned} 0 &= (r_1 \cdot \nabla)((Df - \lambda \mathrm{I}) r_2) - (r_2 \cdot \nabla)((Df - \lambda \mathrm{I}) r_1) \\ &= (D^2 f(r_2, r_1) - D^2 f(r_1, r_2)) - ((r_1 \cdot \nabla \lambda) r_2 - (r_2 \cdot \nabla \lambda) r_1) + (Df - \lambda \mathrm{I})[r_1, r_2]. \end{aligned}$$

Here, the first term in the last line vanishes because of the equality of mixed partial derivatives. Thus, since $R \cap (Df - \lambda \mathrm{I}) \mathbb{R}^n = \{0\}$, the second and third term must also both vanish. The evanescence of the third term means $[r_1, r_2] \in R$. By Frobenius' Theorem the integral manifolds χ of R exist and constitute a foliation $\mathscr{F}$ of U. Furthermore, at any point where r_1, r_2 are linearly independent, the evanescence of the second term means $r_i \cdot \nabla \lambda = 0$ for $i = 1, 2$. Since r_1, r_2 were arbitrary, this yields $r \cdot \nabla \lambda = 0$ for all $r \in R$, which means linear degeneracy.

Proof of Theorem 2 Since grad λ is orthogonal to R everywhere, the restriction $\lambda | \chi$ is a constant for any $\chi \in \mathscr{F}$, which will be denoted s_χ in the sequel. Consider a $\chi \in \mathscr{F}$ and two arbitrary states u^-, $u^+ \in \chi$. Then (1.4) holds with $s = s_\chi$. Integrating along any curve u: $[0, 1] \to \chi$ which joins u^- to u^+ yields

$$f(u^+) - f(u^-) = \int_0^1 Df(u(\tau))\, u'(\tau)\, d\tau = \int_0^1 \lambda(u(\tau))\, u'(\tau)\, d\tau = s_\chi (u^+ - u^-)$$

because of $u'(\tau) \in R(u(\tau))$. We have thus shown that, for any χ, all pairs $(u^-, u^+) \in \chi \times \chi$ define contact discontinuities. In order to prove the rest of the theorem, fix $\chi \in \mathscr{F}$, $u^- \in \chi$, and a sufficiently small number $\delta > 0$. Consider now any $(u^+, s) \in U \times \mathbb{R}$ with

$$f(u^+) - f(u^-) = s(u^+ - u^-)$$

and

$$|u^+ - u^-| < \delta, \qquad |s - s_\chi| < \delta.$$

We will prove that actually $u^+ \in \chi$, $s = s_\chi$. For any point $u \in \chi$, we have

$$f(u^+) - f(u^-) = f(u^+) - f(u) + s_\chi (u - u^-),$$

which yields

$$f(u^+) - f(u) - s(u^+ - u) = (s - s_\chi)(u - u^-) =: c.$$

Let $\hat{R}$ denote the $Df(u^-)$-invariant complement of $R(u^-)$ in $\mathbb{R}^n$ and choose u to be the locally unique point in χ with $u^+ - u \in \hat{R}$.

Now, on one hand

$$\begin{aligned} c &= f(u^+) - f(u) - s(u^+ - u) \\ &= \left(\int_0^1 Df(\tau u^+ + (1 - \tau) u)\, d\tau - s\mathrm{I} \right) (u^+ - u) \\ &= (Df(u^-) - s_\chi \mathrm{I} + O(\delta))(u^+ - u). \end{aligned}$$

Since $(Df(u^-)-s_\chi \mathrm{I})|\hat{R}$ is non-singular, this implies that

$$c\in(\mathrm{I}+O(\delta))\hat{R}$$

and

$$c=0 \quad \text{only if } u=u^+.$$

On the other hand,

$$c=(s-s_\chi)(u-u^-)\in(\mathrm{I}+O(\delta))\,R(u^-).$$

For sufficiently small δ,

$$(\mathrm{I}+O(\delta))\,R(u^-)\cap(\mathrm{I}+O(\delta))\,\hat{R}=\{0\},$$

which implies $u=u^+$, i.e. actually $u^+\in\chi$ and $s=s_\chi$. The proof is complete.

3 The geometry of compact linear degeneracy

In this section, Theorems 3 and 4 will be proved. They deal with the special case that the foliation associated with a linearly degenerate mode contains a compact leaf. Theorem 3 proves an important consequence of this geometric property, whereas Theorem 4 establishes its structural stability. Theorem 3 is proved by applying a topological argument to the *Rankine-Hugoniot* conditions (cp. [2]), Theorem 4 by viewing the leaf as an invariant manifold of an appropriate flow.

Proof of Theorem 3 Define sets $\Lambda, \Lambda^-, \Lambda^+\subset U\times\mathbb{R}$, and a map $F\colon U\times\mathbb{R}\to\mathbb{R}^{n+1}$ by

$$\Lambda^-=\{(u,s)\,|\,s<\lambda(u) \text{ and } [s,\lambda(u))\cap\sigma(Df(u))=\varnothing\},$$

$$\Lambda^+=\{(u,s)\,|\,s>\lambda(u) \text{ and } (\lambda(u),s]\cap\sigma(Df(u))=\varnothing\}.$$

$$\Lambda=\{(u,s)\,|\,s=\lambda(u)\}=\partial\Lambda^-\cap\partial\Lambda^+,$$

$$F(u,s)=(f(u)-su,\,s).$$

Consider a compact contact leaf $\chi\in\mathscr{F}$ and let

$$C=\chi\times\{s_\chi\}\subset\Lambda.$$

As consequences of Theorem 2, F maps all of C into one point $q_0\in\mathbb{R}^{n+1}$ and C is isolated in $F^{-1}(q_0)$, i.e. there exists an open neighborhood $P_0\subset\subset U$ of C such that

$$F^{-1}(q_0)\cap\bar{P}_0=C.$$

Moreover, P_0 can be chosen such that also

$$\bar{P}_0\subset\Lambda^-\cup\Lambda\cup\Lambda^+.$$

Since $q_0\notin F(\partial P_0)$, there exists a connected open neighborhood $Q_0\subset\mathbb{R}^{n+1}$ of q_0 such that $F(\partial P_0)\cap Q_0=\varnothing$. For $Q_0'=Q_0\setminus F(\Lambda\cap\bar{P}_0)$ we claim

(i) $$Q_0'\subset F(P_0\cap\Lambda^-)\cap F(P_0\cap\Lambda^+).$$

Assuming (i) for a moment, we choose any $q \in Q'_0$ and find $p^- \in P_0 \cap \Lambda^-$, $p^+ \in P_0 \cap \Lambda^+$ with

$$F(p^-) = F(p^+) = q.$$

By the definition of F, $p^- = (u^-, s)$, $p^+ = (u^+, s)$ for a certain triple $(u^-, u^+, s) \in U \times U \times \mathbb{R}$ which also satisfies the Rankine-Hugoniot conditions (1.3). Since $p^\pm \in \Lambda^\pm$, (1.2) as given by this triple defines a shock associated with λ. Moreover, since P_0 can be chosen to be an arbitrarily small neighborhood of C, such triples can be found arbitrarily near $\chi \times \chi \times \{s_\chi\}$. These are structurally stable since Q'_0 is open. Thus Theorem 3 will be proved once (i) is shown to be true. We show that $Q'_0 \subset F(P_0 \cap \Lambda^+)$. This is a consequence of the following three facts:

(ii) $F(P_0 \cap \Lambda^+) \cap Q'_0 \neq \emptyset$,
(iii) $\partial(F(P_0 \cap \Lambda^+)) \cap Q'_0 = \emptyset$,
(iv) Q'_0 is connected.

Here, (ii) follows obviously from the local behavior of F near any point in C; (iii) follows from

$$\begin{aligned} \partial(F(P_0 \cap \Lambda^+)) \cap Q'_0 &\subset (F(\partial P_0) \cap F(\Lambda \cap \bar{P}_0)) \cap Q'_0 \\ &\subset F(\partial P_0) \cap Q_0; \end{aligned}$$

(iv) follows from the fact that the codimension of $F(\Lambda)$ is at least 2, which in turn is a consequence of

$$T\Lambda = (\ker DF) | \Lambda.$$

Thus $Q'_0 \subset F(P_0 \cap \Lambda^+)$; by analogy $Q'_0 \subset F(P_0 \cap \Lambda^-)$. These yield (i) and the proof is complete.

The proof of Theorem 4 is based on the following fact, which is an obvious consequence of the fundamental theorem on normally hyperbolic invariant manifolds (see [6]):

Lemma. *Let $v \in C^\infty(U, \mathbb{R}^n)$ be a smooth vector field on an open set $U \subset \mathbb{R}^n$ and $M \subset U$ a compact smooth submanifold which is stationary for the (local) flow of v: $v | M \equiv 0$. Assume that at any $u \in M$ the eigenvalue 0 of $Dv(u)$ has algebraic multiplicity equal to* $\dim M$ *and there is no other purely imaginary eigenvalue of $Dv(u)$. Then for any vector field $\tilde{v}$ sufficiently close to v there is a unique smooth submanifold $\tilde{M}$ close to M which is invariant under the (local) flow of $\tilde{v}$.*

Proof of the Lemma. Note first that it is unimportant whether we have flows or only local flows, since we are only interested in their behavior on a compact neighborhood of M in U. Assuming tacitly that, whenever necessary, vector fields have been redefined outside such a neighborhood so as to make their local flows global in time, we will only speak of flows in the sequel. If Φ: $U \times \mathbb{R} \to U$ is the flow of v, we have to consider the advance maps $\Phi_t = \Phi(\cdot, t)$: $U \to U$, which are governed by

$$\frac{d}{dt} \Phi_t = v \circ \Phi_t, \quad \Phi_0 = \mathrm{id}$$

(cp. [1]). We need to show that, for $t>0$, M is normally hyperbolic invariant under Φ_t, i.e. (cp. [6]) the tangent bundle $T_M U$ of U along M decomposes as

$$T_M U = V_- \oplus TM \oplus V_+,$$

where the subbundles V_-, TM, V_+ are invariant under the differential $D\Phi_t$ of Φ_t and $V_-[V_+]$ is contracted [expanded] more sharply than TM. Since $\Phi_t | M = \mathrm{id}_M$ for all $t \in \mathbb{R}$, $D\Phi_t$ satisfies

$$\frac{d}{dt} D\Phi_t = D v D\Phi_t, \qquad D\Phi_0 = \mathrm{I} \quad \text{at } M$$

and thus

$$D\Phi_t = e^{tDv} \quad \text{at } M.$$

Thus, the Dv-invariant subbundles V_-, TM, V_+ of $T_M U$ corresponding to eigenvalues of Dv with negative, zero, and positive real part are also invariant under $D\Phi_t$. Obviously, for any fixed $t>0$, the restriction of $D\Phi_t$ to TM is identity, and its restriction to $V_-[V_+]$ has only eigenvalues of modulus less [greater] than 1. From this it follows easily (cp. e.g. Lemma 4 on p. 147 of [1]) that, with respect to a suitable metric, $D\Phi_t$ contracts V_-[expands V_+] more sharply than TM.

Proof of Theorem 4 For any $\chi \in \mathscr{F}$ denote the constant value of $f - s_\chi \mathrm{id}_U$ on χ by c_χ. Theorem 4 follows from applying the Lemma with $M = \chi \in \mathscr{F}$ being any compact contact leaf and

$$v = f - s_\chi \mathrm{id}_U - c_\chi.$$

Obviously, along χ, $v=0$ and $Dv = Df - \lambda \mathrm{I}$ so that

$$\ker Dv = R = T\chi.$$

By hyperbolicity, the algebraic multiplicity of the zero eigenvalue equals $\dim R$, and the other eigenvalues are real; so the assumptions of the Lemma are satisfied. It yields the existence of open neighborhoods $U_0 \subset U$ of χ and $Q_0 \subset \mathbb{R}^{n+1}$ of (c_χ, s_χ) such that for any vector field

$$\tilde{v} = f - s\,\mathrm{id}_U - c$$

with $(c, s) \in Q_0$ there exists a unique compact submanifold $\tilde{M} \subset U_0$ which is the maximal invariant set in U_0 for the flow of $\tilde{v}$. On the other hand, there exists a neighborhood $U_1 \subset U$ of χ such that $(c_{\tilde{\chi}}, s_{\tilde{\chi}}) \in Q_0$ for any $\tilde{\chi} \in \mathscr{F}$ with $\tilde{\chi} \cap U_1 \neq \varnothing$. Now, choosing any such $\tilde{\chi}$ and setting

$$(c, s) = (c_{\tilde{\chi}}, s_{\tilde{\chi}}),$$

we find that the corresponding $\tilde{M}$ must coincide with $\tilde{\chi}$, which implies that $\tilde{\chi}$ is compact.

4 Viscous profiles for shocks associated with compact linear degeneracy

Theorem 5 establishes the existence of viscous profiles for (some of) the shocks found in Theorem 3. Here we prove Theorem 5 by applying the same idea as in the proof of Theorem 4 to a different flow.

Proof of Theorem 5 The viscous counterpart of (1.1) with respect to a given viscosity, a smooth matrix field B: $U \to \mathbb{R}^{n \times n}$ on the state space, is the system

(4.1) $$\mathbf{w}_t(x, t) + (f(\mathbf{w}(x, t)))_x = (B(\mathbf{w}(x, t))\, \mathbf{w}_x(x, t))_x$$

of partial differential equations. A solution of the form

(4.2) $$\mathbf{w}(x, t) = w(x - s t)$$

is called a *traveling wave* of speed s for (4.1). If $\mathbf{w}$ tends to limits u^-, u^+ for $x \to -\infty$, $x \to +\infty$, respectively, $\mathbf{w}$ is called a viscous profile of the corresponding discontinuous solution (1.2) of (1.1). In this case, w is a heteroclinic orbit of the system

(4.3) $$B(w)\, w' = f(w) - s w - c \quad (c = f(u^\pm) - s u^\pm)$$

of ordinary differential equations that runs from u^- to u^+:

(4.4) $$w(-\infty) = u^-, \qquad w(+\infty) = u^+.$$

We assume that the viscosity B is strictly stable. This very natural requirement was introduced in [9], where also its meaning and implications were investigated thoroughly. We will make use of an algebraic property only that has been demonstrated in [9]: for all $u \in U$ there exists a $\delta_0 > 0$ such that for all $\eta \in \mathbb{R}$

(4.5) $$\kappa \in \sigma(-i \eta D f(u) - \eta^2 B(u)) \Rightarrow \operatorname{Re} \kappa \leqq -\delta_0 \eta^2.$$

We consider a compact contact leaf $\chi \in \mathscr{F}$, and let – as before – $s_\chi \in \mathbb{R}$, $c_\chi \in \mathbb{R}^n$ denote the (constant) values of λ and $f - s_\chi \operatorname{id}_U$ on χ. We show that the vector field

(4.6) $$v_B = B^{-1}(f - s_\chi \operatorname{id}_U - c_\chi)$$

satisfies the assumptions of the Lemma along $M = \chi$. Obviously, v_B vanishes on χ, and

(4.7) $$D v_B = B^{-1}(Df - \lambda \mathrm{I}) \quad \text{on } \chi,$$

so that

(4.8) $$\ker D v_B = \ker(Df - \lambda \mathrm{I}) = R = T\chi \quad \text{on } \chi.$$

Next we prove

(4.9) $$\operatorname{image} D v_B \cap \ker D v_B = \{0\} \text{ along } \chi,$$

which together with (4.8) implies that the eigenvalue 0 of $D v_B$ has algebraic multiplicity equal to $\dim \chi$.

Consider an $r_0 \in \operatorname{image} D v_B(u_0) \cap \ker D v_B(u_0)$, i.e.

$$r_0 = B_0^{-1} A_0 a \quad \text{with some } a \tag{4.10}$$

and

$$B_0^{-1} A_0 r_0 = 0, \tag{4.11}$$

where A_0, B_0 abbreviate $Df(u_0) - \lambda(u_0)\mathrm{I}$, $B(u_0)$. (4.10), (4.11) are equivalent to

$$A_0 r_0 = 0 \tag{4.12}$$

and

$$m_0 B_0 r_0 = 0 \quad \text{for any } m_0 \text{ with } m_0 A_0 = 0. \tag{4.13}$$

Assuming for a moment that 0 is a simple eigenvalue of A_0, i.e. $l = 1$, we find, for $\eta \in \mathbb{R}$, η near 0, an (eigenvalue, eigenvector)-pair (μ, m) of $-\mathrm{i} A_0 - \eta B_0$:

$$m(\eta)(-i A_0 - \eta B_0 - \mu(\eta)\mathrm{I}) = 0 \tag{4.14}$$

which depends smoothly on η and satisfies

$$\mu(0) = 0, \qquad m(0) = m_0 \neq 0, \qquad m_0 A_0 = 0. \tag{4.15}$$

Differentiating (4.14) and evaluating at $\eta = 0$, we get

$$m_0(B_0 + \mu'(0)\mathrm{I}) + m'(0)\, i A_0 = 0. \tag{4.16}$$

Multiplication by r_0 yields

$$\mu'(0)\, m_0 r_0 = -m_0 B_0 r_0. \tag{4.17}$$

By (4.13), (4.15), the right hand side of (4.17) vanishes, and (4.5) implies $\mu'(0) \neq 0$. Thus (4.17) yields

$$m_0 r_0 = 0. \tag{4.18}$$

By the biorthogonality of left and right eigenvectors, this implies $r_0 = 0$, which proves (4.9) for $l = 1$. For general l, 0 is still a semi-simple eigenvalue of A_0 in the sense of [7], and as a consequence of Theorem 2.3 in Chapt. 2 of [7] we can still apply the same argument with an appropriately *chosen* family (μ, m): (4.9) is considered proved. Finally, assume that, at a point $u_0 \in \chi$, $D v_B(u_0)$ has a non-vanishing purely imaginary eigenvalue $i\alpha \neq 0$, i.e.

$$B^{-1}(u_0)(Df(u_0) - s_\chi \mathrm{I})\, r = i\alpha r \qquad \text{for an } r \neq 0, \tag{4.19}$$

or, equivalently,

$$(-i\alpha Df(u_0) - \alpha^2 B(u_0))\, r = -i\alpha s_\chi r. \tag{4.20}$$

With $\kappa = -i\alpha s_\chi$ this means

$$\kappa \in \sigma(-i\alpha Df(u_0) - \alpha^2 B(u_0)) \quad \text{and} \quad \operatorname{Re}\kappa = 0, \tag{4.21}$$

which contradicts (4.5) for $\eta=\alpha$. It follows that $D v_B(u_0)$ cannot have any non-vanishing purely imaginary eigenvalue for any $u_0 \in \chi$. We have thus shown that v_B satisfies the assumptions of the Lemma along $M=\chi$.

Now, let again U_0, P_0, Q_0 denote sufficiently small neighborhoods of χ, $\chi \times \{s_\chi\}$ and (c_χ, s_χ). Again, for any $(c, s) \in Q_0$ we find a unique smooth l-dimensional submanifold $\tilde{M} \subset U_0$ which is the maximal invariant set in U_0 for the flow of the vector field

$$\tilde{v}_B = B^{-1}(f - s\,\mathrm{id}_U - c). \tag{4.22}$$

Now assume that $(c, s) \notin F(\Lambda)$. Then any rest point u_0 of $\tilde{v}_B$ which lies on $\tilde{M}$ is necessarily non-degenerate: in the notation used in the proof of Theorem 3, either $(u_0, s) \in \Lambda^-$ or $(u_0, s) \in \Lambda^+$. Since $\tilde{M}$ is invariant under the flow of $\tilde{v}_B$, $\tilde{v}_B | \tilde{M}$ is tangential to $\tilde{M}$, of course. As a rest point of $\tilde{v}_B | \tilde{M}$, any such u_0 is a node which is stable or unstable according to whether $(u_0, s) \in \Lambda^-$ or $(u_0, s) \in \Lambda^+$. Since the rest points of $\tilde{v}_B$ are the same as the rest points of $\tilde{v} = \tilde{v}_{\mathrm{I}}$, Theorem 3 guarantees the existence of unstable and stable nodes, which have been called u^-, u^+ there. Choose now an arbitrary unstable node u^- of $\tilde{v}_B | \tilde{M}$, and consider any orbit w in its unstable manifold. If $l=1$, $\tilde{M}$ must be a regular closed non-selfintersecting curve. It is obvious that w cannot run all around $\tilde{M}$, but must terminate in another rest point u^+, a stable node. Now consider the case of arbitrary l, on assuming, however, that $\tilde{v}_B$ is gradient-like, i.e. there exists a (Lyapunov) function $\varphi \in C^\infty(U, \mathbb{R})$ that increases along any orbit of $\tilde{v}_B$. Since φ is increasing and at the same time bounded on w, it is again obvious that w connects the unstable node u^- to a stable node u^+.

It is obvious that in all cases the established structure persists under small perturbations of (c, s).

Appendix. Application to magnetohydrodynamic shock waves

Plane magnetohydrodynamic waves are governed by the equations

$$\begin{gathered}
\rho_t + (\rho\,\alpha)_x = 0 \\
(\rho\,\alpha)_t + (\rho\,\alpha^2 + p + \tfrac{1}{2}|b|^2)_x = \nu\,\alpha_{xx} \\
(\rho\,a)_t + (\rho\,\alpha\,a - b)_x = \mu\,a_{xx} \\
b_t + (\alpha\,b - a)_x = \eta\,b_{xx} \\
E_t + (\alpha(E + p + \tfrac{1}{2}|b|^2) - a \cdot b)_x = \nu(\alpha\,\alpha_x)_x + \mu(a \cdot a_x)_x + \eta(b \cdot b_x)\,x + \kappa\,\Theta_{xx}
\end{gathered} \tag{A.1}$$

where ρ, $\Theta > 0$ denote density and temperature, $\alpha \in \mathbb{R}$ the longitudinal velocity and $a \in \mathbb{R}^2$ the transverse velocity (i.e. the components of the velocity vector parallel resp. orthogonal to the direction of wave propagation), $b \in \mathbb{R}^2$ the transverse magnetic field (since the magnetic field is divergence-free, its longitudinal component is a constant, which has been chosen to be 1 here by normalization); $p = p(\rho, \Theta)$ is the pressure which we assume to satisfy Weyl's conditions, and $E = \rho(e + \frac{1}{2}(\alpha^2 + |a|^2)) + \frac{1}{2}|b|^2$ is the total energy with $e = e(\rho, \Theta)$ being the internal energy, $e_\Theta > 0$. The coefficients ν and μ are convex combinations of the two fluid viscosities, η is electrical resistivity, κ the heat conductivity. For "real" magnetohydrodynamics, these dissipation coefficients are all positive; setting them to zero makes (A.1) the hyperbolic system of conservation laws describing "ideal" magnetohydrodynamics. (See e.g. [A2] for a derivation of the equations). The hyperbolic system has the well-known rotational *Alfvén* mode, and our primary interest here is in viewing certain shocks as associated with this mode and applying the previously described ideas to them; the result is

Theorem A.1 *For any given value of* $(\nu, \mu, \eta, \kappa) \in (\mathbb{R}_+)^4$, *certain magnetohydrodynamic shocks associated with the rotational Alfvén mode possess viscous profiles. The heteroclinic orbits (of* (A.2) *below) that represent these profiles are structurally stable.*

Proof. Restricting attention without loss of generality to stationary discontinuities (i.e. $s=0$ in (1.2), $\partial/\partial t=0$ in (A.1)), we find that traveling waves of (A.1) are described by

(A.2)
$$\begin{pmatrix} \mu I_2 & 0 & 0 & 0 \\ 0 & \eta I_2 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \kappa \end{pmatrix} \begin{pmatrix} a \\ b \\ \alpha \\ \Theta \end{pmatrix}' = \begin{pmatrix} m a - b - c_0 \\ \alpha b - a - c_1 \\ m\alpha + p + \frac{1}{2}|b|^2 - c_2 \\ me - \frac{1}{2}m(\alpha^2 + |a|^2) - \frac{1}{2}\alpha |b|^2 + a\cdot b + a\cdot c_0 + b\cdot c_1 + \alpha c_2 - c_3 \end{pmatrix}$$

where $m \in \mathbb{R}$, c_0, $c_1 \in \mathbb{R}^2$, c_2, $c_3 \in \mathbb{R}$ are constants of integration and p, e are evaluated at $\rho = \frac{m}{\alpha}$. Note that the first equation in (A.1) does not give rise to a differential equation for the traveling waves, but to a constraint (namely constancy of the mass flux $m = \rho\alpha$). This occurs since the viscosity matrix of (A.1) as it stands is obviously singular. For the same reason, Theorem 5 cannot be applied literally to (A.1), (A.2) since, of course, a singular viscosity cannot be strictly stable. However, we can still view (A.2) as

(A.3) $$B w' = v(w)$$

where $w = (a, b, \alpha, \Theta)$ attains values in $U = \mathbb{R}^4 \times (\mathbb{R}_+)^2$ and the meaning of v: $U \to \mathbb{R}^6$, $B \in \mathbb{R}^{6\times 6}$ is obvious. Assume in the following that $c_0 = 0$ (which because of Galilean invariance means no loss of generality) and $m \neq 0$ (since we are not interested in entropy waves here, which are characterized by vanishing mass flux). Now, if also $c_1 = 0$ and c_2, c_3 are sufficiently large, the fixed point set $v^{-1}(0)$ obviously contains a circle

(A.4) $$\chi = \{(a, b, \alpha, \Theta) \in U \mid a = m^{-1} b, |b| = \beta_0, \alpha = m^{-1}, \Theta = \Theta_0\}, \beta_0, \Theta_0 > 0.$$

(χ corresponds to a contact leaf associated with the linearly degenerate *Alfvén* mode; any two elements of it give rise to a contact discontinuity traveling at *Alfvén* speed, cp. [A2].) Theorem A.1 follows from applying the lemma stated in Sect. 3 to the invariant manifold χ, in complete analogy to the proof of Theorem 5. Obviously, it suffices to check the following

Auxiliary Lemma *At any* $u \in \chi$, *the eigenvalue* 0 *of* $B^{-1} D v(u)$ *is simple and there is no other purely imaginary eigenvalue of* $B^{-1} D v(u)$.

To see this, we compute that, at $u=(a, b, \alpha, \Theta)\in\chi$,

$$\text{(A.5)} \qquad D\,v(u)=\begin{pmatrix} m\mathrm{I}_2 & -\mathrm{I}_2 & 0 & 0 \\ -\mathrm{I}_2 & m^{-1}\mathrm{I}_2 & b & 0 \\ 0 & b^T & m+\dfrac{dp}{d\alpha} & p_\Theta \\ 0 & 0 & 0 & me_\Theta \end{pmatrix}$$

and write this as

$$\text{(A.6)} \qquad A=\begin{pmatrix} A_{55} & A_{51} \\ A_{15} & A_{11} \end{pmatrix}$$

with $(j\times k)$-matrices A_{jk}, $j, k\in\{1, 5\}$. Now,

$$\text{(A.7)} \qquad B^{-1}D\,v(u)=B^{-1/2}\tilde{A}B^{1/2} \quad \text{with} \quad \tilde{A}=B^{-1/2}AB^{-1/2}$$

so that it suffices to show that $\tilde{A}$ has the properties claimed for $B^{-1}D\,v(u)$ in the assertion of the auxiliary lemma. We decompose

$$\text{(A.8)} \qquad \tilde{A}=\begin{pmatrix} \tilde{A}_{55} & \tilde{A}_{51} \\ \tilde{A}_{15} & \tilde{A}_{11} \end{pmatrix}$$

is analogy to (A.6). It is obvious that $\tilde{A}_{55}$ is symmetric and $\tilde{A}_{15}$ vanishes, properties inherited from analogous properties of A.

Since $\tilde{A}_{55}$ is symmetric, it can be diagonalized by a rotation $O\in\mathbf{O}(5)$. Transforming $\tilde{A}$ by $O\times 1\in\mathbf{O}(6)$ yields

$$\text{(A.9)} \qquad \tilde{\tilde{A}}=\begin{pmatrix} \tilde{\tilde{A}}_{55} & \tilde{\tilde{A}}_{51} \\ \tilde{\tilde{A}}_{15} & \tilde{\tilde{A}}_{11} \end{pmatrix}=(O\times 1)\,\tilde{A}(O\times 1)^T=\begin{pmatrix} O\tilde{A}_{55}O^T & O\tilde{A}_{51} \\ \tilde{A}_{15}O^T & \tilde{A}_{11} \end{pmatrix}.$$

Since $\tilde{\tilde{A}}_{55}$ is diagonal and $\tilde{\tilde{A}}_{15}=\underset{\sim}{0}$, $\tilde{\tilde{A}}$ is upper triangular. The eigenvalues of A are the diagonal elements of $\tilde{\tilde{A}}$, i.e. the eigenvalues of $\tilde{A}_{55}$ and the value $\tilde{A}_{11}=\kappa^{-1}me_\Theta$. It is obvious that they are all real numbers. Since $e_\Theta\neq 0$, the existence and simplicity of the zero eigenvalue of $\tilde{A}$ follow from

$$\text{(A.10)} \qquad \operatorname{rank}\tilde{A}_{55}=4.$$

Keeping in mind that $b\neq 0$ on χ, this is easy to check. The proof of the auxiliary lemma and thus of Theorem A.1 is complete.

We conclude by a very brief discussion of the context around Theorem A.1, referring the reader to [A4] for a more detailed account. Viscous profiles for magnetohydrodynamic shock waves have been the subject of many previous investigations, so [A5, A3, A8]. It is well-known (since [A5]) that for $c_1\neq 0$ (A.2) has up to four rest points u_0, u_1, u_2, $u_3\in U$ (where the numbering is as usual chosen according to decreasing value of the α-coordinate). These rest points may or may not exist for a given value of c, and also u_0 and u_1 or u_2 and u_3 may coalesce, but whenever u_i exists and does not coalesce with any of the other zeros of v, it is a hyperbolic fixed point of (A.2) with stable manifold of dimension i. All discontinuities (1.2) with $u^+=u_i$, $u^-=u_j$, $i, j\in\{0, 1, 2, 3\}$ satisfy the *Rankine-Hugoniot* jump conditions. However, only those with $i<j$ can have viscous profiles at all. (This follows from the fact that (A.2) is gradient-like, see [A5]). So viscous profiles of six different types might exist, often denoted as

(A.11) $$u_0 \to u_1,\ u_2 \to u_3,\ u_1 \to u_2,\ u_0 \to u_2,\ u_0 \to u_3,\ u_1 \to u_3\,.$$

Of these, the first two are the so-called *fast* and *slow* shocks and the remaining ones are called *intermediate shocks.* The existence of viscous profiles $u_0 \to u_1$, $u_2 \to u_3$ has been proved several times (see [A5, A3, A7]). Also it is well-known that for any fixed intermediate shock there are choices of the dissipation coefficients ν, μ, η, $\kappa \in \mathbb{R}_+$ such that the shock has no viscous profile (with respect to the B defined by these coefficients) (see [A5, A3]). In contrast to this, results obtained in [A8, A1, A10] suggested strongly that intermediate shocks may possess viscous profiles. Notice that the shocks considered in Theorem A.1 are of type $u_1 \to u_2$ since they are associated with the eigenvalue given by the *Alfvén* speed, cp. [A5]. Thus we can restate Theorem A.1 as follows:

Theorem A.1′ *For all* $(\nu, \mu, \eta, \kappa) \in (\mathbb{R}_+)^4$, *there are intermediate shocks* $u_1 \to u_2$ *which have a viscous profile, given by a structurally stable heteroclinic orbit.*

Actually, in [A9] it was already shown that there exist values of c and (positive) values of ν, μ, η, κ such that the intermediate shock of type $u_1 \to u_2$ has a viscous profile. However, from the results in [A9] one cannot tell for which choices of the dissipation coefficients this happens. Nor does one know whether the corresponding orbit is structurally stable.

Note now further that for the shocks under consideration the magnetic field b does not vanish on either side since $|b^-| \approx |b^+| \approx \beta_0 > 0$. Note also that in the absence of dissipative effects intermediate shocks are dynamically unstable (cp. e.g. [A2]). Thus, another trivial consequence of Theorem A.1 is

Theorem A.2 *For an arbitrary choice of the four positive dissipation coefficients in the dissipative version* (A.1) *of magnetohydrodynamics certain magnetohydrodynamic shock waves with magnetic field* $b \neq 0$ *which are dynamically unstable in the ideal framework possess structure.*

Here, *structure* is just another frequently used word for viscous profile. Note that for $b=0$ the existence of a viscous profile $u_0 \to u_3$ follows trivially from [A6]. This can be used to derive Theorem A.2 much more easily than in the above way, see [5] and [A4]. In contrast, no easier way than the above of proving Theorem A.1 (A.1′) is known to the author at present.

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Note added in proof

As I learned recently, the fact that constant higher multiplicity implies linear degeneracy was proofed already in Boillat, G.: Chocs charactér tiques. C. R. Acad. Sci. Paris, Ser. A, **274**, 1018–1021 (1972). I think that this basic property should become known more widely than it has in the past.