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On the rank of harmonic maps

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1 Introduction

One of the classical properties enjoyed by harmonic functions and more generally by harmonic maps is the *unique continuation property*, that is, if u maps an open subset to one point then it maps everything to that same point (see [A], [AKS], [GL], [K] for details). One possible generalization is the following natural question (see [EL], [K]). Let u: $M^m \to N^n$ be a harmonic map and say u maps an open subset of M to a submanifold of lower dimension. Does this imply that the whole image of u is contained in a submanifold of that lower dimension? More precisely, say V be an open subset of M^m such that $\operatorname{rank}(du) \leq k$ on V, where $k \geqq 0$ is an integer. Then does u have $\operatorname{rank}(du) \leqq k$ on all M^m? In the classical case where the dimension $k=0$ the answer is "yes".

The next progress on this question was in 1978 by Sampson [S] in the case $k=1$. He proved that if u: $M^m \to N^n$ is a harmonic map with $\operatorname{rank}(du)=1$ on some open set, then u maps all of M^m into a geodesic in N^n; also if u maps some open set of M^m into a complete totally geodesic submanifold $\mathscr{S}$ then $u(M^m) \in \mathscr{S}$. Sampson's idea was to reduce the problem to the unique continuation properties of the solutions of elliptic systems. The key property he used was that a harmonic map of rank one is necessarily a geodesic, and that there is only one way to extend a geodesic.

Of course, the answer to the more general question for larger values of k is "yes" if M^m and N^n are both real analytic, since in this case, the harmonic map is real analytic and so is the differential of the map. Therefore by this analyticity if on some open set the determinant of some p-submatrix of the differential is equal to zero, then (in a given coordinate patch) the determinant of the same p-submatrix of the differential is zero everywhere on the coordinate patch – and hence by analyticity on all of M^m.

Except this case and Sampson's result, nothing was known for the case where the rank is larger than one. In this short note, we construct an example which shows the answer is negative in general.

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Theorem. *There is a metric g on R^3 and a harmonic map u: $(R^3, g) \to (R^3, g_0)$ such that for some open sets V_1, V_2 in R^3 we have*

$$\operatorname{rank}(du) = 2 \text{ on } V_1 \quad \text{and} \quad \operatorname{rank}(du) = 3 \text{ on } V_2,$$

where g_0 is the standard Euclidean metric on R^3.

In our example, using standard coordinates (x, y, z) or R^3 the set V_1 will be where $z < 0$ while the set V_2 will be where $z > 0$.

2 The example

We begin with some background to fix the notation we will be using. Let (M^m, g) and (N^n, h) be Riemannian manifolds. A map u: $M^m \to N^n$ is called *harmonic* if u is a critical point of the energy functional

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dx$$

where, in local coordinates $u = (u^1, \ldots, u^n)$ using summation convention,

$$|\nabla u|^2 = g^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u(x)).$$

Here $g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ and $h = h_{ij}\, dy^i \otimes dy^j$ are the metrics on M and N, respectively. Also $g^{\alpha\beta}$ is the inverse of g and dx is the element of volume on (M^m, g). If $u \in C^2(M^m, N^n)$, then in local coordinates u satisfies the standard quasilinear elliptic system

$$\Delta_g u^k + g^{\alpha\beta}(x)\, \Gamma^k_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} = 0,$$

where Δ_g is the Laplacian operator on M and the Γ^k_{ij} are the Christoffel symbols of the metric h.

Notice that if (N^n, h) is (R^3, g_0), where g_0 is the standard flat metric on R^3, then in cartesian coordinates the Christoffel symbols are zero so u is harmonic if and only if each coordinate component of u is a harmonic function on (M^m, g).

We seek a harmonic map u: $(R^3, g) \to (R^3, g_0)$, so we need to find both the metric g and the map u. For convenience we use coordinates x, y, z on (R^3, g) and seek g^{-1}, the inverse of the metric g, in the special form

$$g^{-1} = \begin{pmatrix} 1 & a(z) & 0 \\ a(z) & 1 & 0 \\ 0 & 0 & b(z) \end{pmatrix},$$

where $b(z) = 1/(1 - a(z)^2)$ so that $\det g = 1$ and we specify that $|a(z)| < 1$ to insure that g is positive definite. To be more specific, say $a(z) = 0$ for $z < 0$ and $0 < a(z) < \frac{1}{2}$ for $z > 0$. Since the eigenvalues of g^{-1} are $\{1 - a,\ 1 + a, b\}$, this insures that

g and g_0, the standard metric on R^3, are quasi-isometric; in fact $\frac{1}{2}g_0 < g < 2g_0$ so, for instance, g is complete. Pick $a(z) \in C^\infty$ too.

Since g_0 is the standard metric on R^3, a map $u=(u^1, u^2, u^3)$ is harmonic if and only if

$$\Delta_g u^k = 0 \quad \text{on } R^3 \quad \text{for } k=1,2,3,$$

where for any scalar function f

$$\Delta_g f = f_{xx} + f_{yy} + 2a(z) f_{xy} + (b(z) f_z)_z.$$

We seek a harmonic map in the special form

$$u(x, y, z) = (x, y, -xy + \varphi(z)). \tag{1}$$

Then $\Delta_g x = 0$ and $\Delta_g y = 0$ while we must pick φ so that

$$0 = \Delta_g(-xy + \varphi(z)) = -2a(z) + (b(z)\varphi'(z))'. \tag{2}$$

In addition, we will arrange that $\varphi(z)=0$ for $z<0$ and $\varphi'(z)>0$ for $z>0$. This will show that du has rank 2 for $z<0$ and rank 3 for $z>0$. To find φ we simply integrate (2) using the initial conditions $\varphi(0)=0$ and $\varphi'(0)=0$. This yields

$$\varphi'(z) = \frac{2}{b(z)} \int_0^z a(s)\, ds$$

and

$$\varphi(z) = 2 \int_0^z \frac{1}{b(t)} \left(\int_0^t a(s)\, ds \right) dt.$$

The construction of our example is now completed.

Remark. In our example, we chose the set V_1 carefully, and our choice depended on the metric; for other choices of V_1 there might not be corresponding examples.

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