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Partial homotopy type of finite two-complexes

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Introduction

The purpose of this paper is twofold. First, we wish to present some of the unpublished results of Wesley Browning, and secondly, to generalize them. Some comments are in order.

Browning's results appeared in his thesis and in three pre-prints written at ETH in 1979. These papers are extremely difficult to read, because they are written in a very general setting. In addition, some of the results attributed to other authors either were not quite there, or else they did not clearly imply what Browning claimed. All in all, these very important results deserved a thorough revision and abridgement. In addition to this, we generalize the results from homotopy equivalence to partial homotopy equivalence, i.e. equivariant homology equivalence of finite covers. Some of the algebraic results of this paper have been done independantly by Linnell [Li].

Basically we work with pointed lattices over a finite group Q and with their localizations at the set u of primes dividing the order of Q. If K is a 2-complex with fundamental group G and $\theta\colon G \to Q$ is a surjection, we construct a group $cl_u(\bar{K})$ using the cover $\bar{K}$ of K associated to $\ker(\theta)$. We define a total obstruction to partial homotopy equivalence in $cl_u(\bar{K})$ which is defined for all 2-complexes L with fundamental group G and Euler characteristic $\chi(K)$. This obstruction depends only on the Q-lattices, $\Sigma(\bar{K}) \subset H_2(\bar{K})$ and $\Sigma_2(\bar{L}) \subset H_2(\bar{L})$, of the spherical elements together with a "reduced" k-invariant. For fixed K and θ, there are only finitely many possible lattices N which localize to $\Sigma(\bar{K})_u$ as pointed lattices. This means the obstruction lies in a finite subgroup of $cl_u(\bar{K})$.

Browning obtains a realization theorem that we do not reproduce, since he works with finite complexes of any dimension. That makes our definition of $cl_u(\bar{K})$ slightly different from his. At any rate we are able to locate the obstruction as a torsion element in a quotient of $K_1(ZQ, u)$, which is defined to be the quotient of $K_1(Z_u Q)$ by $K_1(ZQ)$.

There are two technicalities. The first condition is that $H_2(\ker\theta)$ must be a lattice. This does not apear to be a serious setback. For example, if G is a finite free product of finite abelian groups, its commutator subgroup has free

Schur multiplicators. The second condition is that Q satisfy Eichler's condition. Again, this is not a serious constraint, since "most" finite groups satisfy Eichler's condition.

Section 1

Let G be a fixed group. We consider 2-dimensional CW-complexes K with only one vertex $K^{(0)}$ (the zero skeleton). All connected finite complexes are simply homotopic to one such. In general, we look at pairs (K, ϕ), where ϕ is an isomorphism, $\phi\colon \pi_1(K, K^0) \to G$. Given any two such complexes K, L, there is a map $f\colon K \to L$ such that f_* is an isomorphism of fundamental groups making the diagram

$$\begin{array}{ccc} \pi_1(K, K^0) & \xrightarrow{f_*} & \pi_1(L, L^0) \\ & {\scriptstyle \phi}\searrow \quad \swarrow{\scriptstyle \phi} & \\ & G & \end{array}$$

commute. Thus, we can identify the fundamental groups of K and L via f_*.

Let $\theta\colon G \to Q$ be a surjection of groups with kernel N. Let $\bar{K}$ be the covering of K associated to N (strictly speaking to $\phi^{-1}(N)$). Given K and L and f as above, there is a unique lift $\bar{f}\colon \bar{K} \to \bar{L}$ of f sending the preferred base point of $\bar{K}$ to that of $\bar{L}$.

(1.1) Definition. K and L are said to be *partially homotopy equivalent with respect to* Q if $\bar{f}$ induces an isomorphism of integral homology.

(1.2) Remarks. 1) Among the pairs (K, ϕ) as above, this is an equivalence relation: If L is partially homotopy equivalent (with respect to Q) to M via a map $g\colon L \to M$, then the lift of $g \circ f$ is $\bar{g} \circ \bar{f}$, and it clearly induces an isomorphism $H_*(\bar{K}) \to H_*(\bar{M})$.

2) Clearly the isomorphism of homology in the definition is Q-equivariant.

3) Partial homotopy equivalence with respect to G is just homotopy equivalence, since in that case $\bar{K} = \tilde{K}$ is the universal cover. At the other extreme, partial homotopy equivalence with respect to 1 is homology equivalence (with isomorphism of π_1). Thus, partial homotopy equivalence is necessary for homotopy equivalence and sufficient for homology equivalence.

More generally, Browning ([Br], p. 56) considers (G, m)-complexes, that is CW-complexes (K, ϕ), not necessarily 2-dimensional but with $\pi_i(K) = 0$, $2 \leqq i \leqq m-1$, where m is the dimension of K. The definition and remarks above generalize to this case. Browning showed that if G is finite (with one mild extra-condition, see below) and if the Euler characteristic is not the minimum possible then any two (G, m)-complexes with same Euler characteristic are homotopy equivalent. In the finite abelian case $(G, 2)$-complexes have been classified in all cases [Br 3], [M], [Si], and Section 9 below.

Section 2

In most of this paper we will consider $(G, 2)$-complexes only. Given K, L, f as above, it is possible to assume that $K^{(1)} = L^{(1)}$ and that $f | K^{(1)}$ is the identity.

This can be done by altering the spaces by simple homotopy equivalences. In that case f is a homotopy equivalence if and only if either vertical arrow in the diagram of G-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\tilde{K}) & \longrightarrow & C_2(\tilde{K}) & \longrightarrow & B_1(\tilde{K}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H_2(\tilde{L}) & \longrightarrow & C_2(\tilde{L}) & \longrightarrow & B_1(\tilde{L}) & \longrightarrow & 0
\end{array}$$

is an isomorphism. The equality of B_1 follows from our assumptions. We can reduce the problem to an easier necessary condition as follows: Assume that $\theta\colon G \to Q$ is a surjection with kernel N. Then the complexes $Z \otimes_N C_*(\tilde{K})$ and $Z \otimes_N C_*(\tilde{L})$ are just $C_*(\bar{K})$ and $C_*(\bar{L})$, where $\bar{K}$ is the cover associated to N, as in § 1. We obtain a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma_2(\bar{K}) & \longrightarrow & C_2(\bar{K}) & \longrightarrow & \Gamma_2(\bar{K}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Sigma_2(\bar{L}) & \longrightarrow & C_2(\bar{L}) & \longrightarrow & \Gamma_2(\bar{L}) & \longrightarrow & 0
\end{array} \tag{1}$$

where $\Sigma_2(\bar{K}) = Im(H_2(\tilde{K}) \to H_2(\bar{K}))$ and $\Gamma_2(\bar{K})$ is the quotient. The canonical isomorphism $\Gamma_2(\bar{K}) = \Gamma_2(\bar{L})$ is obtained by observing that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\bar{K}) & \longrightarrow & C_2(\bar{K}) & \longrightarrow & B_1(\bar{K}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H_2(\bar{L}) & \longrightarrow & C_2(\bar{L}) & \longrightarrow & B_1(\bar{L}) & \longrightarrow & 0
\end{array}$$

becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(N) & \longrightarrow & \Gamma_2(\bar{K}) & \longrightarrow & B_1(\bar{K}) & \longrightarrow & 0 \\
 & & \equiv & & \downarrow & & \| & & \\
0 & \longrightarrow & H_2(N) & \longrightarrow & \Gamma_2(\bar{L}) & \longrightarrow & B_1(\bar{L}) & \longrightarrow & 0
\end{array} \tag{1a}$$

upon division by Σ_2 on the left and middle terms of each line. Then f will be a partial homotopy equivalence with respect to Q if and only if either vertical arrow in (1) is an isomorphism of Q-modules.

For our purposes we shall assume that Q is finite. Let $R = ZQ$ and consider $\Gamma = \Gamma_2(\bar{K})$ as in diagram (1). Any R-free *presentation*

$$0 \to M \to F \to \Gamma \to 0$$

can be extended to a free resolution

$$\ldots \to F_3 \to F_2 \to F_1 \xrightarrow{\ \kappa\ } F \to \Gamma \to 0$$

where κ factors through an epimorphism which we will also call κ, $\kappa\colon F_1 \to M$. Thus κ determines a class $\kappa \in H^1(\mathrm{Hom}_R(F_i, M) = \mathrm{Ext}^1_R(\Gamma, M)$.

(2.1) Definition. The class $\kappa \in \mathrm{Ext}^1_R(\Gamma, M)$ is called the reduced κ-invariant of M. Notation: $\kappa = \kappa(M)$.

Convention. Whenever we use κ, we will mean $\kappa(M)$, consistently.

(2.2) Proposition. *Given any other presentation $0 \to M' \to F' \to \Gamma \to 0$, then a module map μ: $M \to M'$ extends to a map $\bar{\mu}$: $F \to F'$ making the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & \Gamma & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{\mu}} & & \| & & \\
0 & \longrightarrow & M' & \longrightarrow & F' & \longrightarrow & \Gamma & \longrightarrow & 0
\end{array}$$

commutative, if and only if the induced map μ_*: $\mathrm{Ext}^1_R(\Gamma, M) \to \mathrm{Ext}^1_R(\Gamma, M')$ sends $\kappa(M)$ to $\kappa(M')$.

Proof. If μ extends to a map $\bar{\mu}$ then, by the freeness of F_i, we have a chain map

$$\begin{array}{ccccccccccc}
\ldots F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \searrow M \nearrow & F & \longrightarrow & \Gamma & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow & & \| & & \\
\ldots F_3' & \longrightarrow & F_2' & \longrightarrow & F_1' & \nearrow M \searrow & F' & \longrightarrow & \Gamma' & \longrightarrow & 0.
\end{array}$$

It is clear from the commutivity of the above diagram that μ_* takes κ to κ'. Now suppose μ_* takes k to k'. We know that there exists a chain equivalence ψ_*,

$$\begin{array}{ccccccccccc}
\ldots F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \xrightarrow{\partial_1} & F & \longrightarrow & \Gamma & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} & & \| & & \\
\ldots F_3' & \longrightarrow & F_2' & \longrightarrow & F_1' & \longrightarrow & F' & \longrightarrow & \Gamma & \longrightarrow & 0.
\end{array}$$

We will modify this chain equivalence to a chain equivalence $\bar{\psi}$ such that $\bar{\psi}_0 | M = \mu$. Let i: $M \to F$ be the inclusion map and similarly with i': $M' \to F'$. Let κ: $F_1' \to M'$ and κ': $F_1' \to M'$ be the respective representative maps of the k-invariant. Since μ_* sends k-invariant to k-invariant, the image of κ will differ from κ' by a coboundary, and so there exists a map h: $F \to M'$ such that $\mu \circ \kappa - \kappa' \circ \psi_1 = h \circ \partial_1$. Since F is free and κ' is onto, we can lift h to a map λ: $F \to F_1'$ so that $\kappa' \circ \lambda = h$. Now define $\bar{\psi}_0 = \psi_0 + i' \circ h (i'$*: $M' \hookrightarrow F'$) and* $\bar{\psi}_1 = \psi_1 + \lambda \circ \partial_1$. Otherwise let $\bar{\psi}_i = \psi_i$. The $\bar{\psi}_i$'s will be a new chain map and the restriction of $\bar{\psi}_0$ to M will be μ. □

Such maps can be realized geometrically:

(2.3) Proposition. *Given a chain map μ_**

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma_2(\tilde{K}) & \longrightarrow & C_2(\tilde{K}) & \longrightarrow & \Gamma_2(\tilde{K}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{\mu}} & & \| & & \\
0 & \longrightarrow & \Sigma_2(\tilde{L}) & \longrightarrow & C_2(\tilde{L}) & \longrightarrow & \Gamma_2(\tilde{L}) & \longrightarrow & 0,
\end{array}$$

then there exists a map g: $K \to L$ such that the induced chain map is μ_*.

Proof. Given a map f: $K \to L$ such that $f_* = \psi$: $\pi_1(K) \to \pi_1(L)$ is the identity, then the induced chain map will differ from μ_* by a module map δ: $C_2(\tilde{K}) \to \Sigma_2(\tilde{L})$. By composing with the projection π: $C_2(\tilde{\tilde{K}}) \to C_2(\tilde{K})$ and lifting to $\pi_2(L)$. (We can do this since $C_2(\tilde{\tilde{K}})$ is free and $\pi_2(L) \to \Sigma_2(\tilde{L})$ is onto), we get $\tilde{\delta}$: $C_2(\tilde{\tilde{K}}) \to \pi_2(L)$ which "projects" to δ. So μ_* is induced by a map on the chain complex of the universal covers of K and L which differs from $\tilde{f}_*$ by the map $\tilde{\delta}$. By a Puppe modification (See [L 1], Lemma 1.4, p. 656) we may realize the map on the universal cover as a geometric map $g: K \to L$ with $g_* = f_*$. □

Section 3

Our complexes K are finite so that $\Gamma_2(\tilde{K})$ is a finitely generated Q-module. Thus we may assume F is a finitely generated free Q-module and, since Q is finite, M is a finitely generated free abelian group. In other words, we may assume M is a Q-lattice. We will do so hereafter.

Let ${}_Q\mathrm{Lat}$ be the category of Q-lattices and Ab the category of abelian groups. Let $\mathscr{F}: {}_Q\mathrm{Lat} \to Ab$ be an additive functor with $\mathscr{F}(ZQ)=0$.

(3.1) Definition. An $\mathscr{F}$-pointed lattice is a pair (M, k) where M is a Q-lattice and $k \in \mathscr{F}(M)$.

Example. $\mathscr{F}(M) = \mathrm{Ext}^1_R(\Gamma; M)$ and $k = \kappa(M)$ as in (2.1). $\mathscr{F}(ZQ)=0$ by [C–E], 8.2 a, p. 198.

(3.2) Convention. Hereafter Q is of finite order $|Q|=n$ and u is the set of primes in the decomposition of n.

(3.3) Lemma. *If $\mathscr{F}$ is as in the definition, then $n \cdot \mathscr{F}(M)=0$, for all M.*

Proof. Let M be a lattice, then there exists a finitely generated free module F and a surjection $\pi: F \to M$. Since M is a free abelian group, there exists a group splitting $s: M \to F$, such that $\pi s = id_M$. Define $t: M \to F$ as

$$t(x) = \sum_{q \in Q} q s(q^{-1} x).$$

Now t is a module map and $\pi t(x) = |Q| \cdot x$. So

$$\mathscr{F}(M) \xrightarrow{\mathscr{F}(t)} \mathscr{F}(F) \xrightarrow{\mathscr{F}(\pi)} |Q| \cdot \mathscr{F}(M).$$

But $\mathscr{F}(F) = \mathscr{F}(ZQ^k) = 0$, so $\mathscr{F}(\pi t) = 0$ and $|Q| \cdot \mathscr{F}(M) = 0$. □

(3.4) Definition. A Q-lattice map $f: M \to M'$ is a local equivalence if it induces an isomorphism $f_u: M_u \to M'_u$, where $M_u = Z_u \otimes_Z M$ and Z_u is the ring of integers localized at u.

Observe that, in particular, $Z_u Q = (ZQ)_u$.

A sequence $\mathbf{P}_*: 0 \to P_m \to \dots \to P_0 \to Z \to O$, exact except at P_m, P_i projective, is called a truncated Q-resolution of length m over Z. Given $\mathbf{P}_*$, consider the lattice (M, k), where $M = H_m(\mathbf{P}_*)$ and $k \in H^{m+1}(Q; M)$ is the (Postnikov) invariant defined in (2.1).

Quoting Bass cancellation, Wes Browning showed that any two pointed lattices arising from trunated projective resolution of the same length and with the same Euler characteristic will be "locally-equivalent" [Br 2]. Secondly, Browning showed that if the given Euler characteristic was above the minimum, then any two such pointed lattices were equivalent. In other words, any two such truncated projective resolutions were chain equivalent. Thirdly, Browning was able to show that at the minimum Euler characteristic level, the different classes of equivalent pointed lattices could be equated to a particular finite abelian group [Br 2], [Br 3].

We shall reproduce Browning's results in our context. Many of the proofs are the same or similar, but we reproduce them anyway for clarity and the fact that Browning's work is not published.

(3.5) Definition. Two $\mathscr{F}$-pointed Q-lattices (M, k) and (M', k') are stably equivalent if there exist finitely generated free Q-modules F_1 and F_2, and an isomorphism $\psi: M \oplus F_1 \to M' \oplus F_2$ such that $\mathscr{F}(\psi)(k)=k'$. If $F_1 = F_2$, we say that the pointed lattices are strongly stably equivalent.

Observe that, since $\mathscr{F}(M \oplus F_1) = \mathscr{F}(M)$, the statement $\mathscr{F}(\psi)(k)=k'$ is meaningful. We say that ψ is point-preserving.

(3.6) Theorem. *Let M and M' be Q-lattices. If*

$$0 \to M \to F \to \Gamma \to 0, \quad \textit{and}$$
$$0 \to M' \to F' \to \Gamma \to 0,$$

are presentations of Γ, then (M, κ) and (M', κ') are stably equivalent. If ψ is the isomorphism above, it extends to an isomorphism $F \oplus F_1 \to F' \oplus F_2$ which commutes in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \oplus F_1 & \longrightarrow & F \oplus F_1 & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & M' \oplus F_2 & \longrightarrow & F \oplus F_2 & \longrightarrow & \Gamma & \longrightarrow & 0. \end{array}$$

Proof. Given the surjective maps $F \to \Gamma \to 0$, and $F' \to \Gamma \to 0$, there exists some chain map ϕ extending the identity on Γ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & {\scriptstyle\phi_2}\downarrow & & {\scriptstyle\phi_1}\downarrow & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & F' & \longrightarrow & \Gamma & \longrightarrow & 0. \end{array}$$

Construct the mapping cone of ϕ.

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ C_2 & & F \\ \downarrow & & \partial_0 \downarrow \searrow \psi_1 \\ C_1 & & \Gamma \oplus F' \\ \downarrow & & \searrow \downarrow \partial'_0 \\ C_0 & & \Gamma \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

This mapping cone gives us an exact sequence in homology:

(2) $$0 \to H_2(C_*) \to M \xrightarrow{\phi_2} M' \to H_1(C_*) \to 0.$$

Now $C_1 = \Gamma \oplus F'$ is a finitely generated free abelian group, so $H_1(C_*)$ is a finitely generated abelian group and hence a finitely generated module. Consequently, there exists a finitely generated free module F_1 and a surjective map $F_1 \to H_1(C_*)$. Since F_1 is free this map will lift to a map λ: $F_1 \to M'$. The induced map $\phi_2 \oplus \lambda$: $M \oplus F_1 \to M'$ will be (a) surjective and (b) point preserving:

(a) $\phi_2 \oplus \lambda$ is surjective since the sequence (2) is exact and λ commutes with the map $M' \to H_1(C_*)$.

(b) $\phi_2 \oplus \lambda$ is point preserving. Consider the commutative diagram:

(3) $$\begin{array}{ccccccccc} 0 & \longrightarrow & M \oplus F_1 & \longrightarrow & F \oplus F_1 & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_2 \oplus \lambda} & & \downarrow{\scriptstyle \psi_1 \circ i'} & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & F' & \longrightarrow & \Gamma & \longrightarrow & 0. \end{array}$$

Now re-do the above construction with diagram (3). Let $\bar{M} = M \oplus F_1$, $\bar{F} = F \oplus F_1$, etc. Since $\bar{\phi}_2$ is onto we have that $H_1(\bar{C}_*) = 0$. The map $C_1 \to C_0$ is split by the identity on Γ. Since $H_1(\bar{C}_*) = 0$, we have an exact sequence $0 \to H_2(\bar{C}_*) \to \bar{C}_2 \to F' \to 0$, which splits. So, in particular, $H_2(\bar{C}_*)$ is stably free. Let ρ: $\bar{C}_2 \to H_2(\bar{C}_*)$ be the splitting map, and consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(\bar{C}_*) & \longrightarrow & \bar{M} & \longrightarrow & M' & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \rho} & & \downarrow{\scriptstyle j} & & & & \\ 0 & \longrightarrow & \bar{C}_2 & \underset{\alpha}{\cong} & \bar{F} & \longrightarrow & 0 & & \end{array}$$

The map $\rho \cdot \alpha^{-1} \cdot j$ is a splitting for the map $H_2(\bar{C}_*) \to \bar{M}$. So that $\bar{M}$, and therefore M, is stably isomorphic to M'. It is similar to the above that the isomorphism $\bar{M} \cong M' \oplus H_2(\bar{C}_*)$ is point preserving. □

(3.7) Corollary. *In the theorem, if $F = F'$ then (M, κ) and (M, κ') are strongly stably equivalent.*

Proof. With the notation of (3.6) since $F_1 \oplus F \cong F_2 \oplus F$ as Q-modules, and Q is finite, the Z-ranks of the F_i are equal and so are their Q-ranks.

Tietze's theorem as presented in [Wh] also implies the following corollary to (3.6):

(3.8) Proposition. *If K and L are finite 2-complexes with isomorphic fundamental group then $K \vee nS^2$ and $L \vee nS^2$ are homotopy equivalent for some n. (Here nS^2 is the n-fold wedge of 2-spheres).*

(3.9) Lemma. *$Z_v Q$ is a semi-local ring for any finite set of primes v.*

Proof. By [Sw 2], Lemma 9.2, there are only a finite number of maximal ideals, so Z_v is semi-local. Also, $Z_v Q$ is finitely generated since Q is finite. Therefore $Z_v Q$ is semi-local. □

(3.10) Definition. For a pointed lattice (M, k) define $(M, k)_v = (M_v, k)$.

Observe that k lies in $\mathscr{F}(M)$ (in general $\mathscr{F}(M_v)$ makes no sense as M_v is not finitely generated as a group). Now, let $v=u\equiv$ the primes in the order of Q, we are in the situation of Definition (3.5). Let ϕ: $M_u \to N_u$ be an isomorphism of localizations. Let r be an integer prime to $|Q|$ and such that $r\phi(M)\subset N$ (this is possible) and let s satisfy $rs\equiv 1 \bmod |Q|$. Then the formula $s\mathscr{F}(r\phi)$ gives a well-defined map $\mathscr{F}(M)\to\mathscr{F}(N)$.

(3.11) Lemma. (Semi-local cancellation for pointed modules.) If $[(M,k)\oplus ZQ]_v \cong [(N,l)\oplus ZQ]_v$ then $(M,k)_v\cong(N,l)_v$, for any finite set of primes $v\supset u$.

Proof. Browning ([Br 2] (a), p. 12) refers to work of [Wi] to prove this. We shall prove it directly by modifying the unpointed proof in [Sw 1], Lemma 11.7, p. 176. We merely need to check that the defined maps send points to points.

Note first that since $\mathscr{F}(ZQ)=0$, the notation $[(M,k)\oplus ZQ]_v$ is consistent, i.e. $k\in\mathscr{F}(M\oplus ZQ)=\mathscr{F}(M)$. Now suppose ϕ: $[M,k)\oplus ZQ]_v\cong[(N,l)\oplus ZQ]_v$. Let us consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N,l)_v & \xrightarrow{i} & [(N,l)\oplus ZQ]_v & \xrightarrow{j} & Z_v Q & \longrightarrow & 0 \\ & & & \searrow^{\phi^{-1}i} & \uparrow{\scriptstyle\phi} & & & & \\ & & & & [(M,k)\oplus ZQ]_v. & & & & \end{array}$$

Since i and ϕ are point preserving, so is $\phi^{-1}i$ and we have the exact sequence:

$$0\to(N,l)_v\xrightarrow{\phi^{-1}i}[(M,k)\oplus ZQ]_v\xrightarrow{j\phi}Z_vQ\to 0.$$

By composing with the appropriate injections $j\phi$ is the sum of two maps ψ: $M_v\to Z_vQ$ and $\bar{r}$: $Z_vQ\to Z_vQ$ where $\bar{r}$ is multiplication by $r\in Z_vQ$. By a lemma of Bass ([Sw 1], Lemma 11.8, p. 176), there is an $m\in M_v$ such hat $r+\psi(m)=\mu$ is a unit in Z_vQ. Now consider the diagram:

$$\begin{array}{ccccc} Z_vQ & \xrightarrow{\text{id}} & Z_vQ & \xrightarrow{\bar{\mu}} & Z_vQ \\ \oplus & {\scriptstyle -\bar{m}}\searrow & \oplus & \nearrow{\scriptstyle\psi} & \oplus \\ M_v & \xrightarrow[\text{id}]{} & m_v & \xrightarrow[\text{id}]{} & M_v, \end{array}$$

where $-\bar{m}(r)=rm$. Notice that the composition h of the two maps described above is a module isomorphism which, since it restricts to the identity on M, is point preserving. Therefore the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N,l)_v & \longrightarrow & [(M,k)\oplus ZQ]_v & \longrightarrow & Z_vQ & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle\text{id}} & & \\ 0 & \longrightarrow & (M,k)_v & \longrightarrow & [(M,k)\oplus ZQ]_v & \longrightarrow & Z_vQ & \longrightarrow & 0. \end{array}$$

where the induced map λ is point preserving. By the Five lemma, λ is an isomorphism. □

(3.12) Corollary. *Given any two ZQ-lattices M and M′ with* $0 \to M \to F \to \Gamma \to 0$, *and* $0 \to M' \to F \to \Gamma \to 0$, *then the pointed lattices* (M, κ) *and* (M', κ') *are locally equivalent with respect to any finite set of primes* $v \supset u$. *(Note that the free module in each short exact sequence is the same F.)*

Proof. By Corollary 3.7, (M, κ) and (M', κ') are strongly stably equivalent, so they are strongly stably equivalent when localized. By Lemma 3.11, they are locally equivalent. □

Section 4

In this section, u is the set of primes of the order n of Q (cf. (3.3)). Let v be any set primes $v \supset u$.

Let M and M' be Q-lattices, and suppose $f\colon M \to M'$ induces a local isomorphism $f_v\colon M_v \to M'_v$. Since $f = f_v | M$, f is monic. Consequently, we have an exact sequence $0 \to M \to M' \to X \to 0$, where $X_v = 0$. We say that such an f v-localizes to an isomorphism, and that M and M' are v-locally equivalent.

(4.1) Definition. Let $A_v(M)$ be the set of Q-modules X which are epimorphic images of M and which satisfy $X_v = 0$.

Notice that X is the cokernel of some map $f\colon N \to M$ of Q-lattices which v-localizes to an isomorphism.

(4.2) Definition. Let $G_v(M)$ be the Groethendiek group of $A_v(M)$, that is, the free abelian group in the isomorphism classes $[X]$ of modules X in $A_v(M)$ modulo the relations $[Y] = [X] + [Z]$ whenever $0 \to X \to Y \to Z \to 0$ is exact.

(4.3) Lemma [Br 1]: *$G_v(M)$ is a free abelian group with basis* $\{[S] \mid S \in A_v(M),$ *S simple*$\}$.

Proof. Use Swan's rearrangement theorem, [Sw 2], Lemma 9.3.

(4.4) Definition. Let $\mathcal{M}_v(M)$ be the category whose objects are the lattices N which are locally equivalent to M and whose morphisms $f\colon N \to N'$ are module maps which localize to an isomorphism.

(4.5) Lemma [Br 1]. *For any N in* $\mathcal{M}_v(M)$, $G_v(N) = G_v(M)$.

Proof. Let $\phi\colon M_v \to N_v$ be an isomorphism. ϕ can be thought of as a Z_v-module isomorpism. Since M and N are finitely generated free abelian groups, M_v and N_v will also be finitely generated Z_v-modules generated by the free Z-generators of M and N. Given these generators, the matrix representative of ϕ will have rational numbers as entries. Let s be the least common denominator of these fractions. So $s\phi$ will map M into N and will still be an isomorphism on M_v, $s \cdot \phi\colon M_v \cong N_v$. So we can assume, by changing ϕ to $s \cdot \phi$ if necessary, that $\phi(M) \subset N$. Now let $\pi\colon M \to S$ be onto with S simple and $S_v = 0$. Since ϕ is $1-1$ and $N \supset \phi(M) \cong M$, then $N/\phi(\ker \pi)$ will contain an isomorphic copy of S. And therefore S must appear in a composition series for N. Therefore $G_v(N)$ and $G_v(M)$ have the same free generators. □

The above proof provides us with an explicit identification $G_v M = G_v N$ which is independent of ϕ. We need $\mathcal{F}$-pointed versions of the above. The most important case is where $\mathscr{F}(M)$ is a finite abelian group. For example, if $\mathscr{F}(M)=\mathrm{Ext}^1_Q(\Gamma, M)$ and $k=\kappa(M)$, this is satisfied.

(4.6) Definition. Let $\mathscr{M}_v(M, k)$ be the category whose objects are the pointed lattices (N, l) which are v-locally equivalent to (M, k), with respect to v and whose morphisms are maps $f: (N, l)\to(N', l')$ which v-localize to isomorphisms.

(4.7) Lemma [Br 1]. *$\mathscr{M}_v(M, k)$ has finitely many isomorphism classes.*

Proof. By the Jordan-Zassenhaus theorem for ZQ([Sw 2] p, 43), there are only finitely many isomorphism types of ZQ-lattices of a given Z-rank. The Z-rank of the modules locally isomorphic to M is the same as that of M. Since $\mathscr{F}(N)$ is finite by (3.2) for any Q-lattice N, there can only be a finite number of such pointed modules locally equivalent to (M, k). □

(4.8) Definition. Let (N, l) and (N', l') be pointed lattices v-locally equivalent to (M, k). Let $f: (N, l)\to(N', l')$ be any map that v-localizes to an isomorphism, then define $[f]_v$ as $[\mathrm{cok}(f)]\in G_v(M)$.

(4.9) Proposition. a) *$[f]_v$ is well defined.*
b) $[f\circ g]_v=[f]_v+[g]_v$.

Proof. a) since $f: (N, l)\to(N', l')$ v-localizes to an isomorphism, the cokernel of f will localize to zero, and so $\mathrm{cok}(f)$ does represent an element of $G_v(N)$ which equals $G_v(M)$ by (4.5).

b) Any map that localizes to an isomorphism must be one-to-one. Consequently, given monomorphisms
$g: (N, l)\to(N', l')$ and $f: (N', l')\to(N'', l'')$ then

$$0\to\mathrm{Cok}(g)\to\mathrm{Cok}(f\circ g)\to\mathrm{Cok}(f)\to 0$$

So by the definition (4.2) of the Grothendieck group $G_v(M)$, $[\mathrm{cok}(f\circ g)]_v =[\mathrm{cok}(f)]_v+[\mathrm{cok}(g)]_v$. □

(4.10) Definition. Let $\mathrm{Aut}(M, k)_v$ be the subgroup of $G_v(M)$ of the $[f]_v$, where $f: (M, k)\to(M, k)$ v-localizes to an isomorphism. Define $cl_v(M, k)$ as the quotient $G_v M/\mathrm{Aut}(M, k)_v$.

Normally we omit the reference to k and write $cl_v M$ instead.

(4.11) Lemma. *If (N, l) is v-locally equivalent to (M, k) then $cl_v N=cl_v M$.*

Proof. By (4.5) and symmetry, it suffices to show $\mathrm{Aut}(M, k)_v\subset\mathrm{Aut}(N, l)_v$. If ϕ: $(M, k)\to(N, l)$ v-localizes to an isomorphism, there is a map ψ: $N\to M$ so that $\phi\psi$ and $\psi\phi$ v-localize to point preserving automorphisms. Thus in $G_v M=G_v N$, $[\phi f\psi]_v\in\mathrm{Aut}(N, l)_v$, for any f, as in (4.10). By (4.9), $[\phi f\psi]_v=[\phi]+[f]_v+[\psi]_v=[f]_v+[\phi\psi]_v$, so that $[f]_v=[\phi f\psi]_v-[\phi\psi]_v$ is obviously in $\mathrm{Aut}(N, l)_v$. □

The above result allows the following:

(4.12) Definition. If (N, l) is as in (4.11) and ζ: $(N, l)\to(N', l')$ v-localizes to an isomorphism, define $\langle\zeta\rangle_v=[\mathrm{cok}(\zeta)]_v$ in $cl_v M$.

Our main goal is to extend this definition for isomorphisms ζ: $(N, l)_v\to(N', l')_v$. Since $N\subset N_v$ it is meaningful to talk about $\zeta(N)$. Unfortunately $\zeta(N)$ need not be contained in N'. However, since N and N' are finitely generated, there exists $r\in Z_v^*\cap Z$ such that $r\zeta(N)\subset N'$.

(4.13) Lemma ([Br 2]). *If r is mutually prime with all the elements of v (notation $(r, v)=1$), then there exists an r' with $r|r'$, $(r', v)=1$ and $r'\equiv 1 \bmod n$.*

Proof. Let s be the number such that $rs\equiv 1 \bmod n$. By the Chinese Remainder Theorem there is an $s'\equiv s \bmod n$ with $s'\equiv 1 \bmod p$ for all $p\in u$. Let $r'=rs'$. □

We may now assume that $r\zeta$ will (i) map N to N', (ii) v-localize to an isomorphism and (iii) be point preserving, the later since $r=1+kn$ and, by (3.2) $(1+kn)\,l'=l'$, since $n\mathscr{F}(M)\equiv 0$(3.3).

(4.14) Definition. If ζ: $(N, l)_v \to (N', l')$ is a v-local isomorphism, define $\langle\zeta\rangle_v$ to be $\langle r\zeta\rangle_v$, where $r\zeta(N)\subset N$, $r\in Z_v^*\cap Z$ and $r\equiv 1 \bmod n$.

The above definition makes sense, for if r and s both satisfy the conditions above then multiplication by r is a v-self-equivalence of (N', l'). So by the definition of $cl_v(M)$, (4.10), $\langle r\cdot s\cdot\zeta\rangle_v\equiv\langle s\cdot\zeta\rangle_v$. Similarly, $\langle s\cdot r\cdot\zeta\rangle_v\equiv\langle s\cdot\zeta\rangle_v$.

(4.15) Definition. Let (N, l) and (N', l') be pointed lattices v-locally equivalent to (M, k), then define the *Browning Obstruction* $\langle(N, l), (N', l')\rangle_v\equiv\langle\phi\rangle_v\in cl_v(M)$ for some local isomorphism ϕ: $(N, l)_v\to(N', l')_v$.

Note. The Browning torsion of a pair of 2-complexes will live in a subgroup of $cl_u(M)$, where M is an "appropriate" 2nd homology group and u is the set of prime divisors of $|Q|$. However, we will need the other groups $cl_v(M)$ as well. Eventually (in Sect. 6), we will need to show that for the right M, $cl_v(M)\cong cl_u(M)$ for finite sets of primes $v\supset u$.

Section 5

Hereafter Q is a finite group, and R_0 is the *rational* group algebra of Q. As usual, u is the set of primes of $|Q|$. We denote the rationals as $\mathbf{Q}$.

(5.1) Definition. A ZQ lattice M satisfies Eichler's Condition if for every simple left ideal S of R_0 such that End S is a totally definite quaternion algebra, then either S is *not* a direct sumand of $\mathbf{Q}M$ or $S\oplus S$ is a direct summand of $\mathbf{Q}M$.

(5.2) Remark. According to ([Sw 2], p. 177 or [Br 1], p. 14) examples of groups for which every lattice satisfies Eichler's condition include all finite abelian groups, all simple groups, and all finite groups whose order is not divisible by 4.

(5.3) Eichler's Theorem. ([Sw 2], p. 177 or [BR 1], p. 14): *Let M be a ZG-lattice and suppose M satisfies Eichler's condition, then there exists a finite set of primes $v\supset u$ such that, given any two epimorphisms f: $M\to S$ and g: $M\to S$, with S simple and $S_v=0$, then there exists automorphisms ε: $S\to S$ and η: $M\to M$ such that*

$$\begin{array}{ccc} M & \xrightarrow{\ \eta\ } & M \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{\ \varepsilon\ } & S \end{array}$$

commutes, and $\eta\equiv \mathrm{id}_M \bmod |G|\cdot M$.

(5.4) Definition. A ZG-lattice M is said to satisfy $ET(v)$ if any module locally equivalent to M at v satisfies Eichler's theorem for the given v.

(5.5) Proposition. *For G finite, any ZG-lattice satisfies $ET(v)$ for some finite set of primes $v \supset u$.*

Proof. Let M be a ZG-lattice such that $u' \supset u$ is the finite set of primes of Eichler's thorem. Then any $v \supset u'$ will also satisfy Eichler's theorem. Now there are only a finite number of lattices locally equivalent to M at u up to isomorphism. Let v be the union of the finite set of primes that "works" in Eichler's theorem for each lattice locally equivalent to M. Now M will satisfy $ET(v)$, since any module locally equivalent to M at v is in the set of modules locally equivalent at u, and all these modules satisfy Eichler's theorem for v. □

(5.6) Theorem [Br 1]. *Suppose M satisfies $ET(v)$. Let (N', l), (N, l) be lcoally equivalent to (M, k) at v, then $(N', l') \cong (N, l)$ if and only if $\langle (N', l'), (N, l) \rangle_v = 0$ in $cl_v(M)$.*

The following proof is taken directly out of [Br 1], *Theor. 2, p. 15 with only minor changes.*

Proof. If $(N', l') \cong (N, l)$ then from construction $\langle (N', l'), (N, l) \rangle_v = 0$ in $cl_v(M)$. So suppose $\langle (N, l'), (N, l) \rangle_v = 0$ in $cl_v(M)$. Let f be a map f: $(N', l') \to (N', l)$ which localizes to an isomorphism. From the hypothesis $\langle f \rangle_v = 0$, there exists a map g: $(N, l) \to (N, l)$, which localizes to an equivalence, so that $[f]_v = [g]_v$ in $G_v(M)$. Now we have maps f: $(N', l') \to (N, l)$ and g: $(N, l) \to (N, l)$ which localize to isomorphisms, and $\langle \text{cok} f \rangle = \langle \text{cok}\, g \rangle$ in $G_v(N)$. Let $\text{cok} f = X$ and $\text{cok}\, g = Y$.

$$0 \to N' \xrightarrow{f} N \xrightarrow{\phi} X \to 0$$

$$0 \to N \xrightarrow{g} N \xrightarrow{\psi} Y \to 0.$$

By 4.3, X and Y have the same composition series. That is, there are composition series

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots \supset X_r = 0 \quad \text{and}$$
$$Y = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \supset Y_r = 0,$$

such that $X_{i-1}/X_i = Y_{i-1}/Y_i = S_i$ is simple for all i.

Note that X and Y are both quotient modules of N. Let ϕ: $N \to X$, and ψ: $N \to Y$ be the quotient maps. Let $N_i = \phi^{-1}(X_i)$ and $M_i = \psi^{-1}(Y_i)$. Let α_i: $N_i \to N$ and β_i: $M_i \to N$ be the inclusion maps. α_i and β_i are module maps that localize to isomorphisms, since $(X_i)_v = (Y_i)_v \equiv 0$, for each i. Therefore the maps $\text{Ext}^1(\alpha_i)$ and $\text{Ext}^1(\beta_i)$ are also isomorphisms. Let $x_i = [\text{Ext}^1(\alpha_i)]^{-1}(l)$ and $y_i = [\text{Ext}^1(\beta_i)]^{-1}(l)$ be the "points" associated to N_i and M_i. So (N_i, x_i) and (M_i, y_i) are pointed modules locally equivalent, in the pointed category, to (N, l). Since N satisfies $\text{ET}(v)$, by hypothesis, then by the definition of $\text{ET}(v)$, so do the N_i and the M_i.

Note that $(N_0, x_0)=(M_0, y_0)=(N, l)$ and $(N', l)\cong(N_r, x_r)$, $(N, l)\cong(M_r, y_r)$. So it is sufficient to show that $(N_i, x_i)\cong(M_i, y_i)$ for each i. We will do this by induction. To start the induction, $(M_0, y_0)=(N_0, x_0)=(N, l)$.

Now suppose that $h_{i-1}: (N_{i-1}, x_{i-1})\to(M_{i-1}, y_{i-1})$ is an isomorphism. Consider the epimorphism ϕ_i: $N_{i-1}\to S_i$ given by $N_{i-1}\to X_{i-1}\to X_{i-1}/X_i$, and the epimorphism ψ_i: $M_{i-1}\to S_i$ given $M_{i-1}\to Y_{i-1}\to Y_{i-1}/Y_i$ followed by any isomorphism of S_i. Now we have two epimorphisms from N_{i-1} to S_i. That is, ϕ_i and $\phi_i\circ h_{i-1}$. So by ET(v) there are automorphisms ε and η such that

$$\begin{array}{ccccc} N_{i-1} & & \xrightarrow{\phi_i} & & S_i \\ \downarrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle \varepsilon} \\ N_{i-1} & \xrightarrow{h_{i-1}} & M_{i-1} & \xrightarrow{\psi_i} & S_i \end{array}$$

commutes, and $\eta\equiv\mathrm{id} \bmod |Q|\cdot N_{i-1}$. Note that $h_{i-1}\circ\eta$ is an isomorphism. So we have exact sequences

$$0\to N_i\to N_{i-1}\to S_i\to 0 \quad \text{and}$$
$$0\to[h_{i-1}\circ\eta]^{-1}(M_i)\to N_{i-1}\to S_i\to 0.$$

So $h_{i-1}\circ\eta$ restricts to an isomorphism from N_i to M_i. We merely need to show that $h_i\equiv h_{i-1}\circ\eta|_{N_i}$ is point preserving.

Finally, η is point preserving since by Eichler's Theorem $\eta\equiv\mathrm{id} \bmod |Q|\cdot N_{i-1}$. Since h_{i-1} is point preserving by induction, h_i is point preserving. □

(5.7) Corollary. *The group $cl_u(M)$ is a finite abelian group.*

Proof. By the theorm, $(N, l)\mapsto\langle(N, l), (M, k)\rangle_u$ is an onto map, $\mathcal{M}_u(M, k)\to cl_u(M, k)$. Since the former set is finite, (4.7), then so is the latter.

Section 6

For $\mathcal{F}$-pointed lattices we shall assume that $\mathcal{F}(M)=\mathrm{Ext}^1_Q(\Gamma, M)_u$ where u is the set of primes which divide $|Q|$. Hereafter, k and l will be in $\mathrm{Ext}^1_Q(\Gamma, M)_u$ and will usually be $\kappa(M)$ and $\kappa(N)$ respectively, cf. (2.1).

(6.1) Theorem [Br 2]. *Let (M, k) and (N, l) be two pointed lattices. Then if $v\supset u$, then $(M, k)_u\cong(N, l)_u$ if and only if $(M, k)_v\cong(N, l)_v$ for any finite set of primes v containing u.*

To prove the above theorem we will need the following lemma:

(6.2) Lemma (Browning, [Br 2]). *Let M and N be ZQ-lattices, and let v be any finite set of primes containing the prime divisors of $|Q|$. If $G_v(M)=G_v(N)$, then $M\oplus N$ satisfies* ET(v).

Proof. [Sw 2], Proposition 9.5. □

Proof of (6.1): The key fact to prove is that two pointed lattices are locally equivalent with respect to any finite set of primes $v\supset u$ if and only if the two pointed lattices are strongly stably equivalent.

One direction follows from lemmas (3.9) and (3.11), for if v is any finite set of primes then $Z_v Q$ is semi-local, by lemma 3.9. By (3.11) strongly stably equivalent modules are locally equivalent.

Now suppose $(M, k)_v \cong (N, l)_v$, we will show that they are strongly stably equivalent, and therefore, by the previous paragraph, they will be v-locally equivalent. To prove that (M, k) and (N, l) are strongly stably equivalent we will follow second half of the proof of Theorem 4.11 of [Br 2], p. 16.

Let ϕ: $(M, k)_v \cong (N, l)_v$ be a local isomorphism. Since $u \subset v$, we may choose r as in (4.12) so that $r\phi$ maps (M, k) to (N, l) and multiplication by r is point preserving. So assume we started with ϕ: $(M, k) \to (N, l)$. Since ϕ localizes to an isomorphism, we know that ϕ is $1-1$. Let $X = \operatorname{cok}(\phi)$. Note that $X_v = 0$. Since $cl_v(M)$ is finite by (5.7), $[X]_v$ has finite order. So there is a Y(e.g. $X \oplus X \oplus X \ldots \oplus X$) such that $[Y]_v \equiv -[X]_v \bmod \{\operatorname{Aut}(M, k)_v\}$. Rim's Theorem, ([R], Thm 4.12, p. 705) tells us that for any Y which localizes to zero (specifically, Y is cohomologically trivial) the Y has projective dimension $\leqq 1$. That is, there is a short exact sequence

$$0 \to Q \xrightarrow{\psi} P \to X \to 0,$$

with P and Q projective. We may assume that $P = (ZQ)^k$, for some $k \geqq 2$.

Now consider the map $\phi \oplus \psi$: $M \oplus Q \to N \oplus P$. Then $\operatorname{cok}(\phi \oplus \psi) = X \oplus Y$, so $[\operatorname{cok}(\phi \oplus \psi)]_v = [X]_v + [Y]_v \equiv 0 \bmod \{\operatorname{Aut}(M, k)_v\}$. So there is a v-local self-equivalence h of (M, κ) so that $[h]_v = [\operatorname{cok}(\phi \oplus \psi)]_v$. Now $0 = \langle h \oplus \mathrm{id}_p \rangle_v$ in $cl_v(M \oplus P)$. But $M \oplus P$ is $\mathrm{ET}(v)$, by (6.2). By letting $M \oplus ZQ$ be one factor and ZQ^{k-1} be the other factor, (6.2) implies that $M \oplus P$ satisfies $\mathrm{ET}(v)$. But $\langle (M \oplus P, k), (N \oplus P, l) \rangle = \langle h \oplus \mathrm{id}_p \rangle_v = 0$. So by (5.6) $M \oplus P \cong N \oplus P$, and we are done. □

(6.3) Lemma. *Let v be a finite set of primes containing u, and M a lattice, then $\langle X \rangle_v \mapsto \langle X \rangle_u$ defines an isomorphism $cl_v(M) \cong cl_u(M)$.*

Proof. The map is well-defined since $G_v(M) \subset G_u(M)$ and $\operatorname{Aut}(M, k)_v \subset \operatorname{Aut}(M, k)_u$. The map is onto by (6.1). We merely need to show that the map is $1-1$. To show this, we need to define $cl_u^2(M)$, in the following way: consider pointed modules of the form $(N_1 \oplus N_2, l_1 \oplus l_2)$, where N_1 and N_2 are each u-locally equivalent to M, and $l_1 \oplus l_2 \in T(M) \oplus T(M)$. A map f: $N_1 \oplus N_2 \to N_1' \oplus N_2'$ is point preserving if $T(f_i)$: $T(N_i) \to T(N_i')$, $i = 1, 2$, takes l_i to l_i'. We can construct $G_u^2(M)$ as the Grothendieck group of the cokernels of local equivalences of pairs as in (4.2) and define $cl_u^2(M)$ *as* $cl_u^2(M) \equiv G_u^2(M)/\{\operatorname{Aut}(M \oplus M, k \oplus k)_u\}$. Note that by (6.2) $M \oplus M$ satisfies $ET(v)$ for any $v \supset u$.

Now we can define the map j: $cl_u(M) \to cl_u^2(M)$, by merely sending $\langle X \rangle_u$ to $\langle X \rangle_u$, since simple quotients of $M \oplus M$ are the same as those of M, and the cokernel of a local automorphism of M can be realized as the cokernel of a local automorphism of $M \oplus M$. j is obviously an isomorphism.

Now suppose $\langle X \rangle_v \in cl_v(M)$ such that $\langle X \rangle_u = 0$ in $cl_u(M)$. Let f: $N \to M$ be a map with cokernel X, then $\langle N, M \rangle_u = \langle f \rangle_u = \langle X \rangle_u = 0$. Now $j(\langle X \rangle_u) = \langle N \oplus M, M \oplus M \rangle_u$. But this is zero, since j is an isomorphism. Since $M \oplus M$ satisfies $ET(u)$, then by (5.6), $N \oplus M \cong M \oplus M$. This implies that $\langle N \oplus M, M \oplus M \rangle_v = 0$, for any v. But $\langle N \oplus M, M \oplus M \rangle_v = j(\langle X \rangle_v)$. And since j is an isomorphism $\langle X \rangle_v = 0$ and $cl_v(M) \cong cl_u(M)$. □

Section 7

Notation

Let $G \xrightarrow{\theta} Q$ be a homomorphism onto a finite group. Let $\bar{K}$ be the cover of K corresponding to Q. Let $(\Sigma_2(\bar{K}), \kappa)$, be the pointed lattice with $\Sigma_2(\bar{K})$ the spherical elements of $H_2(\bar{K})$, and κ the reduced k-invariant corresponding to the short exact sequence $0 \to \Sigma_2(\bar{K}) \to C_2(\bar{K}) \to \Gamma_2(\bar{K}) \to 0$. Let u be the set of the prime divisors of $|Q|$. To insure that Γ_2 is a Q-lattice, we impose the condition that $H_2(\ker\theta)$ be a lattice (See Section 9, examples 2 and 3, and also (1a), Section 2).

(7.1) Definition. Let $cl_u(\Sigma_2(\bar{K}))$ be denoted as $cl_u(\bar{K})$ and $\kappa(\Sigma_2(\bar{K}))$ be denoted as $\kappa(\bar{K})$.

Recall from (4.10) that $cl_u(\bar{K})$ depends on the reduced k-invariant.

(7.2) Theorem. *Let Q be finite, and $H_2(\ker\theta)$ a free abelian group. Let (N, l) be any pointed lattice locally equivalent to $(\Sigma_2(\bar{K}), \kappa)$ with respect to u. Then $(\Sigma_2(\bar{K}), \kappa) \cong (N, l)$ if and only if $\langle(\Sigma_2(\bar{K}), \kappa), (N, l)\rangle_u = 0$ in $cl_u(\bar{K})$.*

Proof. If $(\Sigma_2(\bar{K}), \kappa) \cong (N, l)$ then $\langle(\Sigma_2(\bar{K}), \kappa), (N, l)\rangle_u = 0$, by construction. So suppose $\langle(\Sigma_2(\bar{K}), \kappa), (N, l)\rangle_u = 0$. By (5.5), $(\Sigma_2(\bar{K}), \kappa)$ satisfies $ET(v)$ for some v. By (6.3), $\langle(\Sigma_2(\bar{K}), \kappa)(N, l)\rangle_u = 0$, we are done by (5.6). □

(7.3) Corollary. *Let K and L be two finite 2-dimensional CW-complexes with $G = \pi_1(K) \overset{\phi}{\cong} \pi_1(L)$ and $\chi(K) = \chi(L)$. Let $\theta\colon \pi_1(K) \to Q$ be a homomorphism onto a finite group. Let u be the set of prime divisors of $|Q|$. Assume further that $H_2(\ker\theta)$ is free abelian and that $\bar{K}$ (resp. $\bar{L}$) is the cover of K (resp. L) corresponding to $\ker(\theta)$ (resp. $\phi(\ker(\theta))$. Then K and L are partially homotopy equivalent with respect to Q if and only if*

$$\langle(\Sigma_2(\bar{K}), \kappa(\bar{K})), (\Sigma_2(\bar{L}), \kappa(\bar{L}))\rangle_u = 0 \quad in \quad cl_u(\bar{K}).$$

Proof. If $\chi(K) = \chi(L)$ then in diagram (1), section 2, $C_2(\bar{K})$ and $C_2(\bar{L})$ have the same rank. Consequently, Theorem (3.6) applies with $M = \Sigma_2(\bar{K})$, $M' = \Sigma_2(\bar{L})$ and $F = F' = C_2(\bar{K})$. So $(\Sigma_2(\bar{K}), \kappa(\bar{K}))$ and $(\Sigma_2(\bar{L}), \kappa(\bar{L}))$ are strongly stably equivalent, and by (3.11) they are v-locally equivalent for some v with $u \subset v$. By (5.6) our conclusion follows with u replaced by v, which in turn implies the final conclusion by (6.2). □

Section 8

Now we will use the exact sequence of Bass ([Ba], p. 494):

$$0 \to K_1(ZQ) \xrightarrow{i_*} K_1(Z_u Q) \xrightarrow{\omega} G_u \xrightarrow{\gamma} \tilde{K}_0(ZQ) \to 0,$$

where G_u is defined as in (4.2), but generated by all simple ZQ-modules. In particular $G_u(M) \subset G_u$ for any module M. The map i_* is induced by inclusion, and ω and γ will be described shortly.

(8.1) Definition. $K_1(ZQ, u) \equiv K_1(Z_u Q)/i_*[K_1(ZQ)]$.

We can write the Bass exact sequence as a short exact sequence:

$$0 \to K_1(ZQ, u) \xrightarrow{\omega_*} G_u \xrightarrow{\gamma} \tilde{K}_0(ZQ) \to 0.$$

Our goal here is to show that the map γ sends every element of the form $\langle(\Sigma_2(\bar{K}), \kappa(\bar{K})), (\Sigma_2(\bar{L}), k(\bar{L}))\rangle_u$ to zero. Consequently, when we are measuring partial homotopy type obstructions, the obstruction will lie in $K_1(ZQ, u)$.

So let us now define ω and γ. Given any $[X]_u \in G_u$, by Rim's theorem, ([R], Thm 4.12, p. 705) there exists projective modules P and P' such that

$$0 \to P' \to P \to X \to 0.$$

$\gamma([X]_u)$ is defined to be $\gamma([X]_u) \equiv [P] - [P']$. We may define ω in the following way. Let $[\phi] \in K_1(Z_u Q)$, with ϕ: $(Z_u Q)^k \to (Z_u Q)^k$. Then there is an integer r relatively prime to $|Q|$ such that $r \cdot \phi((ZQ)^k) \subset (ZQ)^k$ (See proof of (4.5)). Define $\omega([\phi]) \equiv [(ZQ)^k/r \cdot \phi((ZQ))^k]_u - [(ZQ)^k/r \cdot (ZQ)^k]_u$. Notice that if $r = 1$, ω will map the class of ϕ to the class of its cokernel. We will leave the proof that this is well-defined etc. to [Ba].

(8.2) Lemma. *Let ψ: $\pi_1(K) \to \pi_1(L)$ be an isomorphism. Let g: $K \to L$, with $g_* = \psi$. Let $\bar{g}$: $\bar{K} \to \bar{L}$ be the lift of g corresponding to $\ker(\theta)$, where θ: $\pi_1(K) \to Q$ is a homomorphism onto a finite group and let u be the prime divisors of $|Q|$. If $\bar{C}_*$ be the algebraic mapping cone of the cellular chain map of $\bar{g}$, then $H_2(\bar{C}_*)_u = 0$.*

Proof. We know from (2.2), (2.3), (3.4) and Lemma (3.9) that given any map f: $K \to L$ with $f_* = \psi$, *there is a map* g: $K \to L$ with $g_* = f_*$, so that $\bar{g}_2$: $H_2(\bar{K}) \to H_2(\bar{L})$ is a u-local isomorphism. Since $H_2(\bar{C}_*) = \operatorname{cok} \bar{g}_2$, $(H_2(\bar{C}_*))_u = 0$. □

(8.3) Lemma. *If $\sigma = \langle(\Sigma_2(\bar{K}), \kappa(\bar{K})), (\Sigma_2(\bar{L}), \kappa(\bar{L}))\rangle$, then $\gamma\sigma = 0$ in $\tilde{K}_0 ZQ$.*

Proof. By (1) and (1a) in section 2, $\sigma = \langle H_2(\bar{K}), \kappa(\bar{K})), (H_2(\bar{L}), \kappa(\bar{L}))\rangle = [\text{coker } \bar{g}_2] = [H_2(\bar{C}_*)]$, which, by (8.2), lies in G_u. We may assume that $\bar{g}_*$: $C_*(\bar{K}) \to C_*(\bar{L})$ is the identity on C_0 and C_1, so that $H_2(\bar{C})$ is the cokernel of g_2: $C_2(\bar{K}) \to C_2(\bar{L})$. But this is a free resolution of $H_2(\bar{C})$. Therefore, $\gamma([H_2(\bar{C}_*)]) = 0$. □

In view of (8.3) we can define the following:

(8.4) Definition. With the hypotheses of (8.2), $\tau(\bar{g})$ is defined to be the unique element of $K_1(ZQ, u)$ which satisfies $\omega_*(\tau(\bar{g})) = \text{coker}(\bar{g}_2)$. Note that this will coincide with the usual definition of Whitehead torsion [Co] due to the way ω_* has been defined.

Consider u, $\bar{K}$ and Q as usual with θ: $\pi_1 K \to Q$ onto, Q finite of order n, u the set of primes of n, $H_2(\ker(\theta))$ free abelian, and $\bar{K}$ is the cover of K associated to $\ker(\theta)$.

(8.5) Definition. Let $\bar{A}_u$ be the set of all maps g: $K \to K$, which induce an isomorphism on $\pi_1(K)$, and induce a localized homology isomorphism on $\bar{K}$, i.e. g_u^*: $H_2(\bar{K}, Z_u Q) \cong H_2(\bar{K}, Z_u Q)$. (Note: The coefficients $Z_u Q$ are expressing the fact that the chain map $\bar{g}_*$: $C_*(\bar{K}) \to C_*(\bar{K})$ is equivariant with respect to the Q-action.)

(8.6) Theorem. *Let* θ: $\pi_1(K) \to Q$ *be a homomorphism onto a finite group. Let* f: $K \to L$ *be a map with* $f_* = \psi$: $\pi_1(K) \to \pi_1(L)$ *an isomorphism. Let* $\bar{f}$: $\bar{K} \to \bar{L}$, *be the lift of* f *corresponding to* $\ker(\theta)$, *be a local homology isomorphism, then* $\sigma = \langle(\Sigma_2(\bar{K}), k(\bar{K})), (\Sigma_2(\bar{L}), k(\bar{L}))\rangle_u = 0$ *in* $cl_u(\bar{K})$ *if and only if* $\tau(\bar{f}) \equiv 0 \bmod \bar{A}$ *in* $K_1(ZQ, u)$.

Proof. By (8.3) $\sigma = \omega_* \tau(\bar{f})$ and $\omega_* \tau \bar{A} = \mathrm{Aut}(\Sigma_2(\bar{K}), \kappa(\bar{K}))_u$. So the result follows. □

(8.7) Definition. Let θ: $\pi_1(K) \to Q$ be a homomorphism onto a finite group. Let ψ: $\pi_1(K) \to \pi_1(L)$ be an isomorphism and let f: $K \to L$ be a map with $f_* = \psi$. Let $\bar{f}$: $\bar{K} \to \bar{L}$, the lift of f corresponding to $\ker(\theta)$, be a local homology isomorphism w.r.t. u, then define $\tau(\psi) \equiv \tau(f) \bmod \bar{A}$ in $K_1(ZQ, u)/\bar{A}$.

(8.8) Corollary. *Given two finite 2-complexes* K *and* L *with* ψ: $\pi_1(K) \cong \pi_1(L)$. *Then for* θ: $\pi_1(K) \to Q$ *be a homorphism onto a finite group, then there exists a partial homotopy equivalence* f: $K \to L$ *with respect to* Q *with* $f_* = \psi$ *is and only if* $\tau(\psi) = 0$ *in* $K_1(ZQ, u)/\bar{A}$.

Section 9

In this section, we will discuss three examples. The first example will be a review of the results of Browning for finite abelian groups (where $Q = G$). The next two examples will involve finite quotients. Let ε: $RG \to R$ be the augmentation map.

Example 1 Let G be a finite abelian group, $G \simeq Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_m}$, where $n_1 | n_2 \ldots | n_m$. Let $N_1, N_2, \ldots, N_m$ be the respective sums $N_i = \sum_{j=1}^{n_j - 1} a_i^j$, where a_i generates Z_{n_i}. Let $N = \sum_{g_i \in G} g_i$. The following may be easily proved by induction:

(9.1) Lemma. Let u be the set of prime divisors of G, then for any $\mu \in Z_u G$ with augmentation $\varepsilon(\mu) = 1$, μ may be written as a product, $\mu = \mu_1 \cdot \mu_2 \ldots \mu_m$, where $\mu_i \cdot N_i = N_i$.

(9.2) Proposition [Br 3]. *Let* K *be the standard complex of the presentation*

$$\{a_1, a_2, \ldots, a_m | a_1^{n_1}, a_2^{n_2}, \ldots, a_m^{n_m}, [a_i, a_j], i<j\},$$

and let $\bar{A}_u$ *be as in* (8.3), *then the sets* $\{1 + rN | r \in Z\}$ *and* $\{\mu \in (Z_u G)^* | \varepsilon(\mu) = 1\}$ *are contained in* $\tau(\bar{A}_u) \subset K_1(ZG, u)$.

Proof. To realize the elements that augment to 1, we merely note that Lemma 9.1 tells us that the matrix:

$$\begin{pmatrix} \mu_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \mu_2 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_m & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

will commute with the boundary map $C_2(\tilde{K}) \xrightarrow{\partial_2} C_1(\tilde{K})$. Consequently, using Puppe modifications on the identity map id: $K \to K$, we can construct a geometric map, f: $K \to K$, which induces the identity on the 1-skeleton and whose torsion is the torsion of the above matrix, which is μ by [L1].

Similarly, to realize $1+rN$, we merely need to point out that $N \cdot \partial(\tilde{S}_{i,j}) = 0$ where $S_{i,j}$ is the algebraic boundary in $C_1(\tilde{K})$ of the 2-cell corresponding to the commutator relator $[a_i, a_j]$. Therefore, if we start with the identity matrix on $C_2(\tilde{K})$, and replace the 1 corresponding to the $\tilde{S}_{i,j}$ entry with $1+kN$, we see that the new matrix again commutes with the boundary map. □

(9.3) Corollary. *The elements of $K_1(ZG, u)/\tau(\bar{A}_u)$ are representable by integers in $(Z/mZ)^*/\pm 1$, where $m = |G|$ is the order of the group.*

Proof. Since $Z_u G$ is semi-local, $K_1(Z_u G)$ is representable by units, [Sil], Section 6.5. So let $[\kappa] \in K_1(ZG, u)/\tau(\bar{A}_u)$, where $\kappa \in (Z_u G)^*$. Let $k = \varepsilon(\kappa) \in Z_u$. Since $\varepsilon\left(\kappa \cdot \frac{1}{k}\right) = 1$, it is in $\tau(\bar{A}_u)$. Consequently, κ is equivalent to $k \in (Z_u)^*$.

From the above argument $1+rN$ is equivalent to $1+rm \in Z$, where $m = |G|$. So any integer congruent to 1 mod m is equivalent to 1. In particular, $\left[b \cdot \frac{1}{b}\right] = [1]$, so $\left[\frac{1}{b}\right] = [b^{-1}]$, where b^{-1} refers to the multiplicative inverse of b in Z/mZ. Therefore, if $r = \frac{a}{b}$, then r is equivalent in $K_1(ZG, u)/\tau(\bar{A}_u)$ to ab^{-1}. □

(9.4) Theorem. *Let K be a finite 2-complex with minimal Euler characteristic and with $\pi_1(K) = G$, where G is finite and abelian, then K is homotopy equivalent to the standard complex of a presentation of G of the form:*

$$\{a_1, a_2, \dots a_m \mid a_1^{n_1}, a_2^{n_2}, \dots, a_m^{n_m}, a_1^r\, a_2\, a_1^{-r}\, a_2^{-1}, \dots, [a_i, a_j], i<j \neq 1, 2\}$$

where $(n_1, r) = 1$.

Proof. If K is the standard 2 complex of the usual presentation of G and L is the standard complex of the above presentation, we can construct a map f: $K \to L$ which is the identity on the 1-skeleton and whose torsion is $\left[\sum_{i=0}^{r-1} a_1^i\right] \in K_1(Z_u G)$, since $\sum_{i=0}^{r-1} a_1^i \cdot \partial \tilde{S}_{1,2} = \tilde{R}_{1,2}$, where $\tilde{R}_{1,2}$ is the 2-cell in $\tilde{L}$ corresponding to the relator $a_1^r\, a_2\, a_1^{-r}\, a_2^{-1}$, and $\tilde{S}_{1,2}$ is the 2-cell in $\tilde{K}$ corresponding to the relator $[a_1, a_2]$. Note that $\sum_{i=0}^{r-1} a_1^i$ is a unit in $Z_u G$. For if $rs = 1+qn$ for some s and q then $\sum_{i=0}^{r-1} a_1^i \left(\sum_{i=0}^{s-1} (a_1^r)^i - \frac{q}{r} \sum_{i=0}^{n-1} a_1^i\right) = 1$. Also notice that $\sum_{i=0}^{r-1} a_1^i$ augments to r. Consequently all of the elements of $K_1(ZG, u/\tau(\bar{A}_u)$ are realizable by obstructions to complex K being homotopy equivalent to a complex of the form in the hypothesis. Therefore all 2-complexes with minimal Euler characteristic and fundamental group G are homotopy equivalent to one of the type described. □

Note that Latiolais [L 2] has obtained results similar to (9.4) for iterated semi-direct products of finite cyclic groups. Sieradski [Si] used the above model to describe the homology classes of 2-complexes with finite abelian fundamental group. Also see [M].

Example 2 Now we will consider the case where $G=H*J$ where H and J are finite grops.

(9.5) Definition. For any finitely presentable group G, let $\min_\chi(G)$ be the lowest Euler characteristic possible for a 2-complex with fundamental group G.

(9.6) Proposition. *Let θ: $H*J\to Q$ be a surjective map onto a finite group with the restriction to H and J injective, then* $\ker\theta$ *is a free group.*

Proof. Let $N=\ker\theta$. Let K and L be finite 2-complexes with $\pi_1(K)=H$, $\pi_1(L)=J$, $\chi(K)=\min\chi(H)$ and $\chi(L)=\min\chi(L)$. $K\vee L$ will have fundamental group $H*J$. Let $\overline{k\vee L}$ be the finite lift of $K\vee L$ whose fundamental group is N, with covering map p: $\overline{K\vee L}\to K\vee L$. Since H and J inject into Q, then the covers of K and L in $\overline{K\vee L}$ will be universal covers, $p^{-1}(K)=\sqcup_i\tilde{K}$, $p^{-1}(L)=\sqcup_j\tilde{L}$, where $\sqcup$ is disjoint union and $\tilde{K}$ and $\tilde{L}$ are universal covers. Now replace the 'wedge', $\vee$, in $K\vee L$ by an arc connecting a point of K with a point of L, then $\overline{K\vee L}=p^{-1}(K)\cup p^{-1}(\text{arc})\cup p^{-1}(L)$. So off of the covers of K and L, $\overline{K\vee L}$ is 1-dimensional. Since the components of $p^{-1}(K)$ and $p^{-1}(L)$ are simply-connected, $\overline{K\vee L}$ is the fundamental group of a graph. Therefore $N=\pi_1(\overline{K\vee L})$ is free. □

(9.7) Corollary. *Let $H=Z_{m_1}\times Z_{n_1}$* and *$J=Z_{m_2}\times Z_{n_2}$, where n_i and m_i satisfy the above conditions, then there is only one partial homotopy type for 2-complexes K with $\pi_1(K)=H*J$ and Euler characteristic $\chi(K)>\min\chi(H*J)$.*

Proof. Let L be the 2-complex with minimal Euler characteristic and with $\pi_1(L)=H*J$. Then $\chi(K)=\chi(L\vee nS^2)$ for some $n>0$. Consequently, we may always construct a map f: $L\vee nS_2\to L\vee nS^2$ which realizes all the elements of $K_1(Z_u G)$, by merely mapping one of the S^2's to a unit multiple of itself in $C_2(\tilde{L}\vee nS^2)$. Therefore, $K_1(ZG, u)=\tau(\bar{A}_u)$. □

The above corollary is of particular interest, since there are examples where $\min\chi(H*J)<\min\chi(H)+\min\chi(J)$, [H-L-M]. This is the case if we let m_1, m_2, r_1, r_2, n_1, and n_2 be integers such that $r_i>1$, $r_i^{m_i}-1=n_i\cdot q_i$, $(q_1, q_2)=1$, $r_i\equiv 1 \bmod n_i$, and $(m_i, n_i)\neq 1$. For example, let $m_1=n_1=2$, $m_2=n_2=3$, $r=9$. The examples of [H-L-M] have been shown to have the same homology type (Metzler, private communication). Partial homotopy type may be able to distinguish homotopy types.

Example 3 Let G be any finite group satisfying Eichler's condition. Let ρ: $G\to Gl(n, Z)$ be a representation. The representations are in one-to-one correspondence with the semi-direct products $G\rtimes Z^n$. Let θ: $G\rtimes Z^n\to G$, then $\ker(\theta)$ will have free second homology. Therefore we may use the theory of partial homotopy types to try to distinguish the homotopy types of 2-complexes with fundamental group $G\rtimes Z^n$.

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