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# Chamber structures of algebraic surfaces with Kodaira dimension zero and moduli spaces of stable rank two bundles

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## Introduction

Let $X$ be a smooth algebraic surface over the complex number field. Fix a line bundle $c_1$ over $X$, an integer $c_2$ and an ample divisor $L$ on $X$ [1]. Let $\mathscr{M}_L(c_1, c_2)$ be the moduli space of rank two vector bundles stable with respect to $L$ in the sense of Mumford-Takemoto. Consider the question: what is the difference between $\mathscr{M}_{L_1}(c_1, c_2)$ and $\mathscr{M}_{L_2}(c_1, c_2)$ for two different ample divisors $L_1$ and $L_2$. This arise naturally in the definition and the computation of the $\Gamma$-type diffeomorphic invariants ([D] and [FM]). In [Q], we have defined walls, chambers and equivalence classes for ample divisors, and discussed the case when $X$ is a ruled surface.

In this paper, we study the case when $X$ is a surface of Kodaira dimension equal to zero. For chamber structures, we have (see Sect. 1 for notations).

**Theorem 1** *Assume* $(4c_2 - c_1^2) > 0$ *and* $\neq 2\chi(\mathcal{O}_X)$. *Then,*

(i) *For any ample divisor $L$ lying in some chamber (denoted by $\mathscr{C}_L$)*

$$\Delta_L = \mathscr{E}_L = \mathscr{C}_L;$$

(ii) *For any ample divisor $L$ lying in some face of a chamber $\mathscr{C}$*

$$\mathscr{C} \subseteq \Delta_L - \mathscr{E}_L.$$

We then analyse the numbers of modulis of stable rank two bundles coming from walls, and obtain

**Theorem 2** *Let* $c_1 \in \mathrm{Pic}(X)$ *and* $c_2 \in \mathbf{Z}$ *with* $(4c_2 - c_1^2) > 0$. *Then for any two ample divisors* $L_1$ *and* $L_2$, $\mathscr{M}_{L_1}(c_1, c_2)$ *and* $\mathscr{M}_{L_2}(c_1, c_2)$ *are either birational or empty except the following two cases:*

(i) *$X$ is Abelian and* $(4c_2 - c_1^2) = 2$;
(ii) *$X$ is $K3$ and* $(4c_2 - c_1^2) = 6$ *or* 8.

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[1] *Conventions*: Throughout this paper, $X$ stands for an algebraic surface of Kodaira dimension zero over the complex number field. The stability of a rank two bundle over $X$ is in the sense of Mumford-Takemoto

In Sect. 1, we recall the definitions of walls, chambers and equivalence classes of ample divisors on algebraic surfaces, and some useful notations. For convenience, we also state two well-known results about the construction of rank two bundles on algebraic surfaces, which will be used in later sections.

In Section two, we study the chamber structures and prove the theorems above. First of all, stable rank two bundles will be constructed by walls, and the numbers of modulis of these bundles will be calculated by standard methods. Then, we discuss the chamber structures. Finally, we compare the moduli spaces of stable rank two bundles under different ample divisors.

We see that Theorem 2 has two exceptional cases. In the final section, we discuss one of them. Let $X$ be an elliptic $K3$-surface such that all fibers are irreducible. We begin with the description of divisors on $X$, and find a set of ample divisors $L_r$ where $r>2$. Then for values of $c_1$ and $c_2$ on $X$ which give $(4c_2-c_1^2)=6$ or 8, we will investigate moduli spaces with respect to $L_r$. In particular, we will obtain results which are parallel to those of Mukai [M]. Also, the equivalence classes of $L_r$ will be classified.

## 1 Preliminaries

### *1.1 Basic definitions* ([Q])

Let $X$ be a smooth algebraic surface over the complex number field $\mathbf{C}$, and let $\mathrm{Num}(X)$ to be the group of divisors modulo numerical equivalence relation. Then, $\mathrm{Num}(X)$ is a finitely generated free abelian group. There is an ample cone $\mathbf{C}_X$ in $\mathrm{Num}(X)\otimes\mathbf{R}$ which is spanned by ample divisors. The intersection theory on $\mathrm{Div}(X)$ induces a quadratic form on $\mathrm{Num}(X)\otimes\mathbf{R}$. By the Hodge Index Theorem on algebraic surfaces, this quadratic form has positive index 1. We begin with the definition of equivalence classes of ample divisors.

**Definition 1** Let $L_1$, $L_2$ be two ample divisors on $X$. Fix $c_1\in\mathrm{Pic}(X)$ and $c_2\in\mathbf{Z}$. We define $L_1\overset{s}{\geqq}L_2$ if every rank two vector bundle with $c_1$ and $c_2$ as its first and second chern classes is $L_1$-stable whenever it is $L_2$-stable. We define $L_1\overset{s}{=}L_2$ if both $L_1\overset{s}{\geqq}L_2$ and $L_2\overset{s}{\geqq}L_1$.

**Notations.** When $L$ is an ample divisor on $X$, we put

$$\Delta_L=\{L'\,|\,L' \text{ ample and } L'\overset{s}{\geqq}L\};$$

$$\mathscr{E}_L=\{L'\,|\,L' \text{ ample and } L'\overset{s}{=}L\}.$$

**Definition 2** Fix $c_1\in\mathrm{Pic}(X)$ and $c_2\in\mathbf{Z}$ with $(4c_2-c_1^2)>0$.

(i) Let $\zeta\in\mathrm{Num}(X)\otimes\mathbf{R}$. We define $W^\zeta$ as

$$\mathbf{C}\cap\{x\in\mathrm{Num}(X)\otimes\mathbf{R}\,|\,x\cdot\zeta=0\};$$

(ii) We define $\mathscr{W}(c_1,c_2)$ to be the set of $W^\zeta$ where $\zeta$ runs over all numerical equivalence classes such that $(\zeta-c_1)$ is numerically equivalent to $2F$ for some divisor $F$, and that

$$-(4c_2-c_1^2)\leqq\zeta^2<0;$$

(iii) A chamber of type $(c_1,c_2)$ is a connected component of $\mathbf{C}_X-\mathscr{W}(c_1,c_2)$.

*Remark 1* If $c_1=0$, then in (ii) we change $\zeta$ to $\zeta/2$, and change the numerical condition to

$$-c_2 \leqq \zeta^2 < 0.$$

*Remark 2* If $\zeta_1$ and $\zeta_2$ define the same wall of type $(c_1, c_2)$ then the linear equations $x\cdot\zeta_1=0$ and $x\cdot\zeta_2=0$ have the same set of solutions in $\mathrm{Num}(X)\otimes\mathbf{R}$, so $\zeta_1=\lambda\zeta_2$ for some $\lambda\in\mathbf{R}$. Thus, if moreover $\zeta_1^2=\zeta_2^2$, then $\zeta_1=\pm\zeta_2$.

*Remark 3* From § 3 in [8], we know each chamber is contained in some equivalence class, so we can say a rank two bundle is stable with respect to a chamber. Moreover, we conclude that an equivalence class is a union of chambers, and possibly some ample divisors lying on walls.

### *1.2 Rank two bundles constructed by extensions* ([F1] *or* [Q])

From §2 of [Q], we know that if $L_1$ and $L_2$ are two ample divisors which aren't equivalent, and if a rank two bundle $V$ is $L_1$-stable but not $L_2$-stable, then there exists an exact sequence

$$0\to\mathcal{O}_X(F)\to V\to\mathcal{O}_X(c_1-F)\otimes I_Z\to 0$$

where $Z$ is a locally complete intersection 0-cycle on $X$ and $(2F-c_1)$ defines a wall of type $(c_1, c_2)$ separating $L_1$ and $L_2$ (more precisely, $(2F-c_1)\cdot L_1<0<(2F-c_1)\cdot L_2$). In view of this, as in [Q], we make the following

**Definition 3** (i) Let $\zeta$ be a numerical equivalence class defining a wall of type $(c_1, c_2)$. We define $E_\zeta(c_1, c_2)$ to be the set of rank two bundles $V$ such that

$$0\to\mathcal{O}_X(F)\to V\to\mathcal{O}_X(c_1-F)\otimes I_Z\to 0$$

where $F$ is divisor with $2F-c_1\equiv\zeta$ and $Z$ is a locally complete intersection 0-cycle with length $l(Z)=c_2+(\zeta^2-c_1^2)/4$. Moreover, we require that $V$ isn't given by the trivial extension when $\zeta^2=-(4c_2-c_1^2)$;

(ii) We define $D(\zeta)$ to be the number of modulis of those $V$ in $E_\zeta(c_1, c_2)$, and put $d(\zeta)=D(\zeta)-(4c_2-c_1^2-3\chi(\mathcal{O}_X))$.

*Remark 4* By the standard construction in [HS], $E_\zeta(c_1, c_2)$ is quasi-projective and there exists a universal rank two bundle over $X\times E_\zeta(c_1, c_2)$. We will omit this process when we use $E_\zeta(c_1, c_2)$ later.

In order to study chamber structures, we need to construct stable rank two bundles. We next introduce an exact sequence coming from the local to global spectral sequence ([F1]) and two standard results ([F1] or [Q]) for the sake of reference. Let $L$ and $L'$ be two divisors on an algebraic surface $X$; let $Z$ be a locally complete intersection 0-cycle on $X$. We have

$$\begin{aligned}0&\to H^1(\mathcal{O}_X(L-L'))\to\mathrm{Ext}^1(I_Z, \mathcal{O}_X(L-L'))\to H^0(\mathcal{O}_Z)\\&\to H^2(\mathcal{O}_X(L-L'))\to\mathrm{Ext}^2(I_Z, \mathcal{O}_X(L-L'))\to 0.\end{aligned}$$

**Proposition.** *Suppose $Z$ consists of $n$ distinct points $\{p_1, \ldots, p_n\}$. Then, a locally free extension of $\mathcal{O}_X(L')I_Z$ by $\mathcal{O}_X(L)$ exists if and only if every section of $\mathcal{O}_X(L'-L+K_X)$ which vanishes at all but one of the $p_i$ vanishes at the remaining point as well where $K_X$ is the canonical divisor of $X$.*

**Corollary.** *Suppose that $H^0(\mathcal{O}_X(L'-L+K_X))=0$. Then there exists a locally free extension of $\mathcal{O}_X(L')I_Z$ by $\mathcal{O}_X(L)$ for any $Z$.*

## 2 Chamber structures and comparison of moduli spaces

### *2.1 Construction of stable rank two bundles*

Before we study the chamber structures and equivalence classes of ample divisors on an algebraic surface $X$ with Kodaira dimension zero, we now construct stable rank two bundles by using walls. From the classification theory ([B]), we see that the Kodaira dimension of an algebraic surface $X$ is zero if and only if the canonical divisor $K_X$ is numerically equivalent to zero (i.e., $K_X \equiv 0$). Thus if $K_X \equiv 0$, then $X$ is one of the following:

(a) Enriques' surfaces: $p_g=0$, $q=0$, $\chi(\mathcal{O}_X)=1$;
(b) Bielliptic surfaces: $p_g=0$, $q=1$, $\chi(\mathcal{O}_X)=0$;
(c) $K3$-surfaces: $p_g=1$, $q=0$, $\chi(\mathcal{O}_X)=2$;
(d) Abelian surfaces: $p_g=1$, $q=2$, $\chi(\mathcal{O}_X)=0$.

Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbf{Z}$ such that $(4c_2-c_1^2)>0$. Let $\zeta$ be a numerical equivalence class defining a nonempty wall of type $(c_1, c_2)$ on $X$ and $F$ be any divisor such that $2F \equiv \zeta + c_1$. The following Lemma shows the vanishing of some cohomology groups.

**Lemma 1** $H^0(\mathcal{O}_X(c_1-2F+K_X))=0$ *and* $H^0(\mathcal{O}_X(2F-c_1))=0$.

*Proof.* By our assumption, $\zeta$ defines a nonempty wall, thus $\zeta \cdot L_1 > 0 > \zeta \cdot L_2$ for some ample divisors $L_1$ and $L_2$. Since $c_1-2F+K_X \equiv -\zeta+K_X \equiv -\zeta$, $(c_1-2F+K_X)\cdot L_1 = -\zeta \cdot L_1 < 0$, so $(c_1-2F+K_X)$ can never be effective, therefore $H^0(\mathcal{O}_X(c_1-2F+K_X))=0$. Also, $(2F-c_1)\cdot L_2 = \zeta \cdot L_2 < 0$, so $H^0(\mathcal{O}_X(2F-c_1))=0$. $\square$

Next, we give a necessary and sufficient condition for $E_\zeta(c_1, c_2)$ to be empty.

**Lemma 2** *$E_\zeta(c_1, c_2)$ is empty if and only if $\zeta^2 = (c_1^2-4c_2) \doteq -2\chi(\mathcal{O}_X)$.*

*Proof.* By the Lemma 1 above and the Corollary in Sect. 1, for any locally complete intersection 0-cycle $Z$ with length $l(Z)=c_2+(\zeta^2-c_1^2)/4$ on $X$, we can construct a rank two vector bundle $V$ such that

$$0 \to \mathcal{O}_X(F) \to V \to \mathcal{O}_X(c_1-F) \otimes I_Z \to 0.$$

Clearly, $c_1(V)=c_1$ and $c_2(V)=c_2$.

When $Z=\varnothing$, i.e., $l(Z)=0$, we have $\zeta^2=(c_1^2-4c_2)$. In this case, $\dim \mathrm{Ext}^1(\mathcal{O}_X(c_1-F), \mathcal{O}_X(F)) = h^1(\mathcal{O}_X(2F-c_1)) = -\chi(\mathcal{O}_X)-\zeta^2/2$, the second equity is obtained by the Riemann-Roch formula and Lemma 1. Thus, $\dim \mathrm{Ext}^1(\mathcal{O}_X(c_1-F), \mathcal{O}_X(F))=0$ if and only if $\zeta^2=-2\chi(\mathcal{O}_X)$. Therefore, when $\zeta^2 = (c_1^2-4c_2) = -2\chi(\mathcal{O}_X)$, $V$ can only be given by the trivial extension. $\square$

Let $V\in E_\zeta(c_1, c_2)$. Assume that $L_1$ is an ample divisor such that $L_1\cdot\zeta>0$, and that $\mathscr{C}$ is a chamber such that one of its faces is contained in $W^\zeta$ and $\mathscr{C}\cdot\zeta<0$ (that is, $L_2\cdot\zeta<0$ for any ample divisor $L_2$ in $\mathscr{C}$). Fix $L_2\in\mathscr{C}$, and $L\in W^\zeta$. Then, $\zeta\cdot L_2<0$ and $\zeta\cdot L=0$.

**Lemma 3** (i) *$V$ is $L_1$-unstable;*
(ii) *$V$ is strictly $L$-semistable;*
(iii) *$V$ is $L_2$-stable.*

*Proof.* (i) is clear. For (ii) and (iii), let $\mathcal{O}_X(F')$ be any sub-line-bundle of $V$ with torsion free quotient. Then, either $0\to\mathcal{O}_X(F')\to\mathcal{O}_X(F)$ or $0\to\mathcal{O}_X(F')\to\mathcal{O}_X(c_1-F)\otimes I_Z$. In the former case, $F-F'$ is effective, so $F'\cdot L_2\leqq F\cdot L_2<c_1\cdot L_2/2$ and $F'\cdot L\leqq F\cdot L=c_1\cdot L/2$ since $\zeta\cdot L_2<0$ and $\zeta\cdot L=0$. Suppose $0\to\mathcal{O}_X(F')\to\mathcal{O}_X(c_1-F)\otimes I_Z$. Then, $(c_1-F-F')$ is strictly effective since $V$ is indecomposable when $Z$ is an empty set. Thus, $0<(c_1-F-F')\cdot L$, so $F'\cdot L<(c_1-F)\cdot L=c_1\cdot L/2$, therefore $V$ is strictly $L$-semistable. If $F'\cdot L_2\geqq c_1\cdot L_2/2$, then $(2F'-c_1)$ defines a wall of type $(c_1, c_2)$ separating $L$ and $L_2$ with $(2F'-c_1)\cdot L<0$ for any $L\in W^\zeta$, but this is impossible since one of the faces of $\mathscr{C}$ is contained in $W^\zeta$. Thus, $F'\cdot L_2<c_1\cdot L_2/2$. So $V$ must be $L_2$-stable. □

From (iii) above, we conclude the following

**Corollary 1** *Let $\zeta$ be a numerical equivalence class defining a nonempty wall $W^\zeta$, and let $\mathscr{C}$ be a chamber such that one of its faces is contained in $W^\zeta$ and $\mathscr{C}\cdot\zeta<0$. Then, $V$ is $L$-stable for any $V\in E_\zeta(c_1, c_2)$ and $L\in\mathscr{C}$.*

We end with a simple (and well-known) result which will be used later and treats the moduli spaces in the case when the surface is either Abelian or $K3$.

**Lemma 4** *If $X$ is either Abelian or $K3$, then any moduli space is smooth and of dimension $[4c_2-c_1^2-3\chi(\mathcal{O}_X)]$ whenever it is nonempty.*

*Proof.* Note that in both cases $K_X$ is trivial. Thus, for any stable rank two bundle $V$, $h^2(\operatorname{ad} V)=h^2(\mathscr{E}nd\, V)-p_g=h^0(\mathscr{E}nd\, V)-1=0$ where $\operatorname{ad} V$ is the trace-free sub-bundle of $\mathscr{E}nd\, V=V\otimes V^*$. So the moduli space is smooth everywhere, and has dimension $[4c_2-c_1^2-3\chi(\mathcal{O}_X)]$ whenever it is nonempty. □

### *2.2 Chamber structures*

Now, we begin to study the chamber structures. Fix $c_1\in\operatorname{Pic}(X)$ and $c_2\in\mathbf{Z}$ with $(4c_2-c_1^2)>0$. We prove the following

**Theorem 1** *Assume $(4c_2-c_1^2)\neq 2\chi(\mathcal{O}_X)$. Then,*
(i) *For any ample divisor $L$ lying in some chamber (denoted by $\mathscr{C}_L$)*

$$\Delta_L=\mathscr{E}_L=\mathscr{C}_L;$$

(ii) *For any ample divisor $L$ lying in some face of a chamber $\mathscr{C}$*

$$\mathscr{C}\subseteq\Delta_L-\mathscr{E}_L.$$

*Proof.* (i) Clearly, we have $\mathscr{C}_L\subseteq\mathscr{E}_L\subseteq\Delta_L$ by the Remark 3 in Sect. 1. Let $L_1\in\Delta_L-\mathscr{C}_L$. Then by definition, every rank two bundle which is $L$-stable must be

$L_1$-stable. On the other hand, since $L_1$ isn't in $\mathscr{C}_L$, there exits a wall $W$ of type $(c_1, c_2)$ separating $L_1$ and $\mathscr{C}_L$. We may assume that $W$ contains a face of $\mathscr{C}_L$ and is represented by $\zeta$ with $\zeta \cdot \mathscr{C}_L < 0$. Then, $L_1 \cdot \zeta \geqq 0$. Since $(4c_2 - c_1^2) \neq 2\chi(\mathcal{O}_X)$, $E_\zeta(c_1, c_2)$ isn't empty by Lemma 2. By Lemma 3, for any $V$ in $E_\zeta(c_1, c_2)$, $V$ is $L$-stable, so it is $L_1$-stable. On the other hand, $V$ has a sub-line-bundle $\mathcal{O}_X(F)$ with $2F - c_1 \equiv \zeta$. Since $F \cdot L_1 = [(c_1 + \zeta) \cdot L_1]/2 = (c_1 \cdot L_1)/2 + (\zeta \cdot L_1)/2 \geqq (c_1 \cdot L_1)/2$, $V$ isn't $L_1$-stable. Thus, we come to a contradiction. Therefore, $\Delta_L - \mathscr{C}_L$ is empty, so we conclude (i).
(ii) From a general result in §3 of [Q], we know that $\mathscr{C} \subseteq \Delta_L$. By (i), $L$ isn't equivalent to ample divisors in $\mathscr{C}$, so $\mathscr{C}$ isn't contained in $\mathscr{E}_L$. Therefore, $\mathscr{C} \subseteq \Delta_L - \mathscr{E}_L$. □

*Remark 1* If $X$ is either bielliptic or Abelian, then $(4c_2 - c_1^2) > 0 = 2\chi(\mathcal{O}_X)$, so the condition in the theorem is automatically satisfied. If $X$ is $K3$, then by Lemma 4, no stable rank two bundle exists when $(4c_2 - c_1^2) = 2\chi(\mathcal{O}_X)$, thus the entire ample cone $\mathbf{C}_X$ is an equivalence class in this case. Let $X$ be an Enriques' surface and $(4c_2 - c_1^2) = 2\chi(\mathcal{O}_X)$. Let $\zeta$ is a numerical equivalence class defining a nonempty wall of type $(c_1, c_2)$. Then $E_\zeta(c_1, c_2)$ is nonempty if and only if $\zeta^2 \neq -(4c_2 - c_1^2)$. Thus, if $\zeta^2 = -(4c_2 - c_1^2)$, $\zeta$ makes no contribution. Let $\mathscr{C}'$ is any connected component of $\mathbf{C}_X - \{W^\zeta\}$ where $W^\zeta$ are walls of type $(c_1, c_2)$ such that $0 > \zeta^2 > -(4c_2 - c_1^2)$ and $\mathbf{C}_X$ is the ample cone in $\mathrm{Num}(X) \otimes \mathbf{R}$. Then, we conclude as before the following

$$\Delta_L = \mathscr{E}_L = \mathscr{C}'_L$$

for any $L \in \mathscr{C}'$ and

$$\mathscr{C}' \subseteq \Delta_L - \mathscr{E}_L$$

for any $L$ contained in some face of $\mathscr{C}'$.

### 2.3 *Calculation of* $d(\zeta)$

Let $\zeta$ be a numerical equivalence class defining a nonempty wall $W^\zeta$. By definition, for any $V$ in $E_\zeta(c_1, c_2)$, we have

$$0 \to \mathcal{O}_X(F) \to V \to \mathcal{O}_X(c_1 - F) \otimes I_Z \to 0$$

where $F$ is a divisor with $2F - c_1 \equiv \zeta$ and $Z$ is a locally complete intersection 0-cycle with $l(Z) = c_2 + (\zeta^2 - c_1^2)/4$.

**Lemma 5** (i) *Such* $\mathcal{O}_X(F)$ *is unique;*
(ii) $h^0(V \otimes \mathcal{O}_X(-F)) = 1$.

*Proof.* (i) Suppose $\mathcal{O}_X(F_1)$ is another sub-line-bundle of $V$ with $2F_1 - c_1 \equiv \zeta$. Then, either $0 \to \mathcal{O}_X(F_1) \to \mathcal{O}_X(F_1)$ or $0 \to \mathcal{O}_X(F_1) \to \mathcal{O}_X(c_1 - F) \otimes I_Z$. In the second case, $c_1 - F - F_1$ must be effective, so is $2(c_1 - F - F_1)$; on the other hand, $2(c_1 - F - F_1) \equiv -2\zeta$, we conclude that $2(c_1 - F - F_1)$ can't be effective since $\zeta$ defines a nonempty wall. Thus, we have $0 \to \mathcal{O}_X(F_1) \to \mathcal{O}_X(F)$, so $F - F_1$ is effective. By symmetry, $F_1 - F$ is also effective, thus $\mathcal{O}_X(F_1) = \mathcal{O}_X(F)$.

(ii) Tensoring $\mathcal{O}_X(-F)$, we get

$$0 \to \mathcal{O}_X \to V \otimes \mathcal{O}_X(-F) \to \mathcal{O}_X(c_1 - 2F) \otimes I_Z \to 0.$$

Since $c_1 - 2F \equiv \zeta$, $c_1 - 2F$ can't be effective, so $H^0(\mathcal{O}_X(c_1 - 2F) \otimes I_Z) = 0$. Thus, $h^0(V \otimes \mathcal{O}_X(-F)) = h^0(\mathcal{O}_X) = 1$. $\square$

The above Lemma shows that for any $V$ in $E_\zeta(c_1, c_2)$, the extension

$$0 \to \mathcal{O}_X(F) \to V \to \mathcal{O}_X(c_1 - F) \otimes I_Z \to 0$$

is canonical. Next, we calculate the dimension of the extension group

$$\mathrm{Ext}^1(\mathcal{O}_X(c_1 - F) \otimes I_Z, \mathcal{O}_X(F)) = \mathrm{Ext}^1(I_Z, \mathcal{O}_X(2F - c_1)).$$

**Lemma 6** $\dim \mathrm{Ext}^1(I_Z, \mathcal{O}_X(2F - c_1)) = c_2 - (\zeta^2 + c_1^2)/4 - \chi(\mathcal{O}_X)$.

*Proof.* Using Lemma 1 and the exact sequence in Sect. 1, we obtain the following

$$0 \to H^1(\mathcal{O}_X(2F - c_1)) \to \mathrm{Ext}^1(I_Z, \mathcal{O}_X(2F - c_1)) \to \mathcal{O}_Z \to 0$$

since $h^2(\mathcal{O}_X(2F - c_1)) = h^0(\mathcal{O}_X(c_1 - 2F + K_X)) = 0$. Note that $h^0(\mathcal{O}_X(2F - c_1)) = 0$ and $l(Z) = c_2 + (\zeta^2 - c_1^2)/4$. Thus,

$$\begin{aligned}\dim \mathrm{Ext}^1(I_Z, \mathcal{O}_X(2F - c_1)) &= l(Z) + h^1(\mathcal{O}_X(2F - c_1)) = l(Z) - \chi(\mathcal{O}_X(2F - c_1)) \\ &= l(Z) - \chi(\mathcal{O}_X) - \zeta^2/2 = c_2 - (\zeta^2 + c_1^2)/4 - \chi(\mathcal{O}_X). \quad \square\end{aligned}$$

By the two Lemmas above,

$$\begin{aligned}D(\zeta) &= \#\mathrm{moduli}(\mathcal{O}_X(F)) + \#\mathrm{moduli}(Z) + \mathrm{Ext}^1(\mathcal{O}_X(c_1 - F) \otimes I_Z, \mathcal{O}_X(F)) - 1 \\ &= q + 2[c_2 + (\zeta^2 - c_1^2)/4] + [c_2 - (\zeta^2 + c_1^2)/4 - \chi(\mathcal{O}_X)] - 1 \\ &= [4c_2 - c_1^2 - 3\chi(\mathcal{O}_X)] + [(c_1^2 - 4c_2)/4 + \zeta^2/4 + \chi(\mathcal{O}_X) + p_g].\end{aligned}$$

Thus, we have

**Corollary 2** $d(\zeta) = [(c_1^2 - 4c_2)/4 + \zeta^2/4 + \chi(\mathcal{O}_X) + p_g]$.

### *2.4 Comparison of moduli spaces*

Suppose that $L_1$ and $L_2$ are two different ample divisors on $X$. Let $V$ be a rank two bundle which is $L_1$-stable but $L_2$-unstable. Then by 1.2, $V \in E_\zeta(c_1, c_2)$ where $\zeta$ defines a nonempty wall of type $(c_1, c_2)$ such that $\zeta \cdot L_1 < 0 < \zeta \cdot L_2$. So if $d(\zeta) < 0$ (i.e., $D(\zeta) < [4c_2 - c_1^2 - 3\chi(\mathcal{O}_X)]$) for any $\zeta$ which defines a nonempty wall of type $(c_1, c_2)$, then any two moduli spaces are birational whenever nonempty.

Now, we discuss case by case. By the Corollary 2, $d(\zeta) = [(c_1^2 - 4c_2)/4 + \zeta^2/4 + \chi(\mathcal{O}_X) + p_g]$ for any numerical equivalence class $\zeta$ defining a nonempty wall of type $(c_1, c_2)$.

If $X$ is bielliptic, then $p_g + \chi(\mathcal{O}_X) = 0$, so we always have $d(\zeta) < 0$. Therefore, all moduli spaces are either empty or birational.

Let $X$ be an Enriques' surface. Assume $d(\zeta) = [(c_1^2 - 4c_2)/4 + \zeta^2/4 + 1] \geqq 0$. Since $c_1^2 - 4c_2 \leqq \zeta^2 < 0$, we must have $[(c_1^2 - 4c_2)/4 + \zeta^2/4 + 1] = 0$. Thus, $\zeta^2 = -2$

(notice that $\zeta^2$ is even), so $c_1^2-4c_2=\zeta^2=-2=-2\chi\mathcal{O}_X$. But then, $E_\zeta(c_1, c_2)$ is empty by Lemma 2. Therefore, either $d(\zeta)<0$ or $E_\zeta(c_1, c_2)$ is empty, so all moduli spaces are either empty or birational.

Let $X$ be Abelian. Then as in the case when $X$ is an Enriques' surface, $d(\zeta)<0$ unless $c_1^2-4c_2=\zeta^2=-2$. So from the calculations in 2.3, $\dim \operatorname{Ext}^1(c_1-F, F)=1$ if $2F-c_1\equiv\zeta$. Thus, $E_\zeta(c_1, c_2)$ is isomorphic to $\operatorname{Pic}^0(X)$ which is dual to $X$ itself. Therefore, if $(4c_2-c_1^2)\neq 2$, then all moduli spaces are either empty or birational. When $(4c_2-c_1^2)=2$ and $W$ is a wall of type $(c_1, c_2)$ let $k(W)$ be the number of $\zeta$'s such that $\zeta$ represents $W$ and $\zeta^2=-2$. Then, $k(W)=0$ or 2 by the Remark 2 in Sect. 1. Using $W$, we can construct $k(W)$ irreducible components $E_\zeta(c_1, c_2)$ each of which is an Abelian surface dual to $X$. Assume $\mathscr{C}_1$ and $\mathscr{C}_2$ be two chambers sharing a common face contained in $W$. If $k(W)=0$, then $\mathscr{M}_{\mathscr{C}_1}(c_1, c_2)$ and $\mathscr{M}_{\mathscr{C}_2}(c_1, c_2)$ are either empty or birational. If $k(W)=2$, then $\mathscr{M}_{\mathscr{C}_i}(c_1, c_2)$ $(i=1, 2)$ contains exactly one irreducible component which is isomorphic to $\operatorname{Pic}^0(X)$ and will be lost after crossing the wall $W$, that is,

$$\mathscr{M}_{\mathscr{C}_i}(c_1, c_2)=\mathscr{M}_L(c_1, c_2)\coprod P_i$$

where $P_i$ is isomorphic to $\operatorname{Pic}^0(X)$ and $L$ is any ample divisor in the common face $\mathscr{C}_1\cap\mathscr{C}_2$ of $\mathscr{C}_1$ and $\mathscr{C}_2$.

Finally, let $X$ be a $K3$-surface. By the Lemma 4, if $d(\zeta)\geqq 0$, then $d(\zeta)=0$, so $[(c_1^2-4c_2)/4+\zeta^2/4+3]=0$. Since $c_1^2-4c_2\leqq\zeta^2<0$, $\zeta^2\geqq -6$. Thus, $\zeta^2=-4$ or $-6$ for $h^1(2F-c_1)=-\chi(\mathcal{O}_X)-\zeta^2/2\geqq 0$, so accordingly $(4c_2-c_1^2)=8$ or 6. Assume $(4c_2-c_1^2)=8$ and $\zeta^2=-4$. Then, $c_2+((\zeta^2-c_1^2)/4)=1$ and $\dim \operatorname{Ext}^1(I_Z, 2F-c_1)=1$ for any point $Z$ on $X$. Thus, $E_\zeta(c_1, c_2)$ is isomorphic to $\operatorname{Hilb}^1(X)=X$. Let $k(W)$ be the number of $\eta$'s such that $W^\eta=W^\zeta$ and $\eta^2=-4$. If $(4c_2-c_1^2)=6$ and $\zeta^2=-6$, then $[4c_2-c_1^2-3\chi(\mathcal{O}_X)]=0$, $c_2+((\zeta^2-c_1^2)/4)=0$ and $\dim \operatorname{Ext}^1(c_1-F, F)=1$. Thus $E_\zeta(c_1, c_2)$ consists of only one element which is given by the nontrivial extensions in $\dim \operatorname{Ext}^1(c_1-F, F)$. Let $k(W)$ be the number of $\eta$'s such that $W^\eta=W^\zeta$ and $\eta^2=-6$. In both cases, $k(W)=0$ or 2.

Therefore, if $(4c_2-c_1^2)\neq 6$ or 8, then all moduli spaces are either empty or birational. When $(4c_2-c_1^2)=6$ or 8, let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two chambers sharing a common face contained in $W$. If $k(W)=0$, then $\mathscr{M}_{\mathscr{C}_1}(c_1, c_2)$ and $\mathscr{M}_{\mathscr{C}_2}(c_1, c_2)$ are either empty or birational. If $k(W)=2$, and $(4c_2-c_1^2)=6$, then they contain the same number of reduced points and each will lose one point after crossing $W$:

$$\mathscr{M}_{\mathscr{C}_i}(c_1, c_2)=\mathscr{M}_L(c_1, c_2)\coprod P_i$$

where $P_i$ is a point and $L$ is any ample divisor in the common face $\mathscr{C}_1\cap\mathscr{C}_2$ of $\mathscr{C}_1$ and $\mathscr{C}_2$. When $k(W)=2$ and $(4c_2-c_1^2)=8$, then each of $\mathscr{M}_{\mathscr{C}_1}(c_1, c_2)$ and $\mathscr{M}_{\mathscr{C}_2}(c_1, c_2)$ contains one irreducible component which is isomorphic to $X$ and will be lost after crossing $W$:

$$\mathscr{M}_{\mathscr{C}_i}(c_1, c_2)=\mathscr{M}_L(c_1, c_2)\coprod P_i$$

where $P_i$ is isomorphic to $X$ and $L$ is any ample divisor in the common face $\mathscr{C}_1\cap\mathscr{C}_2$ of $\mathscr{C}_1$ and $\mathscr{C}_2$.

In summary, we have proved

**Theorem 2** *Let $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbf{Z}$ with $(4c_2 - c_1^2) > 0$. Then for any two ample divisors $L_1$ and $L_2$, $\mathcal{M}_{L_1}(c_1, c_2)$ and $\mathcal{M}_{L_2}(c_1, c_2)$ are either birational or empty except the following two cases:*
(i) *$X$ is Abelian and $(4c_2 - c_1^2) = 2$;*
(ii) *$X$ is $K3$ and $(4c_2 - c_1^2) = 6$ or $8$.*

*Remark 2* Let $X$ be a $K3$-surface. If $c_1 = 0$ and $c_2 \leqq 3$, then all moduli spaces are empty using an argument involved the Riemann-Roch formula. By definition, the entire ample cone $\mathbf{C}_X$ in $\mathrm{Num}(X) \otimes \mathbf{R}$ is an equivalence class of type $(c_1, c_2)$ when $c_2 \leqq 3$. If $c_2 > 3$, then $4c_2 > 12$, so all moduli spaces are either empty or birational by the Theorem above. It is known that the moduli spaces are non-empty and irreducible when $c_2 > 3$ (see [F2], for instance). Thus, all moduli spaces are birational when $c_2 > 3$.

## 3 Moduli spaces of stable rank two bundles on elliptic $K3$-surfaces

### *3.1 Divisors on elliptic $K3$-surfaces*

Let $j: X \to \mathbf{P}^1$ be the elliptic fibration of $X$ such that any fiber of $j$ in irreducible with at worst ordinary double points as singularities. Let $l$ be a smooth fiber, and let $\Sigma$ be a section. Then, $l^2 = 0$, $l \cdot \Sigma = 1$ and $\Sigma^2 = -2$. Put $L_r = \Sigma + r\,l$ where $r$ is any real number. The following Proposition gives a necessary and sufficient condition for a divisor $L_r$ (we require that $r$ is an integer) to be ample, and shows that $r$ is bounded above if $L_r$ with $r > 2$ lies on some wall.

**Proposition 1** (i) *A divisor $L_r$ is ample if and only if $r > 2$;*
(ii) *If for some $r > 2$, $L_r$ lies on a wall $W$ of type $(c_1, c_2)$, then either $r \leqq (1 + (4c_2 - c_1^2)/2)$ or $W$ contains all $L_r$ with $r > 2$;*
(iii) *All ample divisors $L_r$ with $r > (1 + (4c_2 - c_1^2)/2)$ are in one equivalence class of type $(c_1, c_2)$.*

*Proof.* (i) We use the Nakai-Moishezon Criterion for ample divisors several times. If $L_r$ is ample, then $0 < L_r \cdot \Sigma = (r-2)$, so $r > 2$. Suppose $r > 2$. Then, $(L_r)^2 = 2r - 2 > 0$, and $L_r \cdot \Sigma = (r-2) > 0$. Let $E$ be any irreducible curve different from $\Sigma$. If $E \cdot l > 0$, then $L_r \cdot E \geqq r(l \cdot E) > 0$. If $E \cdot l = 0$, then *$E$ is a fiber, then* $L_r \cdot E = 1$. Therefore, $L_r$ is ample.

(ii) Let $\zeta$ define the wall $W$ of type $(c_1, c_2)$ containing $L_r$. By Proposition 2(i) below, we put $\zeta = a\Sigma + b\Sigma_1 + c\,l$ where $\Sigma_1$ is a section different from $\Sigma$ and $b = 0$ or $1$. Also, we may assume $a \geqq 0$. If $b = 0$, then $0 = \zeta \cdot L_r$ gives $c = a(2-r)$, so $\zeta^2 = 2a^2(1-r)$ and $a \neq 0$ since $\zeta^2 < 0$, thus $-(4c_2 - c_1^2) \leqq \zeta^2 = 2a^2(1-r)$, and $r \leqq 1 + (4c_2 - c_1^2)/(2a^2) \leqq (1 + (4c_2 - c_1^2)/2)$. If $b = 1$, then $0 = \zeta \cdot L_r$ gives $c = 2a - (\Sigma \cdot \Sigma_1) - (a+1)r$, so $\zeta^2 = 2a^2 + 4a - 2 - 2(\Sigma \cdot \Sigma_1) - 2(a+1)^2 r = -2[(a+1)^2(r-1) + (\Sigma \cdot \Sigma_1) + 2]$, thus $-(4c_2 - c_1^2) \leqq -2[(a+1)^2(r-1) + (\Sigma \cdot \Sigma_1) + 2]$. If $(a+1) \neq 0$, then $r < 1 + (4c_2 - c_1^2)/[2(a+1)^2] \leqq (1 + (4c_2 - c_1^2)/2)$; if $(a+1) = 0$, then clearly $L_r \cdot \zeta = 0$ for any $r > 2$.

(iii) Note that if $\zeta$ defines a wall of type $(c_1, c_2)$ which separates two ample divisors $L_{r_1}$ and $L_{r_2}$, then $\zeta \cdot L_r = 0$ for some $L_r$ with $r > \min(r_1, r_2)$ (here, $r$ may not be an integer). Thus, (iii) follows immediately from (ii). □

Next, we prove a result about divisors on $X$.

**Proposition 2** (i) *Let $F$ be a divisor on $X$. Then, $F=n\Sigma+\Sigma_1+ml$ where $n$, $m\in\mathbf{Z}$, and $\Sigma_1$ is a section to $j$;*
(ii) *If $F^2>-4$ and $F\cdot l=0$, then $F=dl$ for some integer $d$.*

*Proof.* (i) Let $F\cdot l=(n+1)$, and let $G=F-n\Sigma$. Then, $G\cdot l=1$. By the Semicontinuity Theorem, $j_*(\mathcal{O}_X(G))$ is an invertible sheaf on $\mathbf{P}^1$. Put $j_*(\mathcal{O}_X(G))=\mathcal{O}_{\mathbf{P}^1}(m_1)$. Then from the canonical injection $0\to j^*j_*(\mathcal{O}_X(G))\to\mathcal{O}_X(G)$, we obtain that $G-m_1 l$ is effective. Since $(G-m_1 l)\cdot l=1$, $G-m_1 l$ consists of a section $\Sigma_1$ and possibly some fibers. Therefore, $F=n\Sigma+\Sigma_1+ml$ for some $n$, $m\in\mathbf{Z}$.
(ii) Put $F_1=F+(1-F\cdot\Sigma)l$. Since $L_r\cdot(-F_1)=-1$, $(-F_1)$ can't be effective, so $0=h^0(\mathcal{O}_X(-F_1))=h^2(\mathcal{O}_X(F_1))$. By the Riemann-Roch formula, $\chi(\mathcal{O}_X(F_1))=2+F^2/2>0$. Thus, $h^0(\mathcal{O}_X(F_1))>0$, so $F_1$ is effective. Since $F_1\cdot l=0$, $F_1=d_1 l$ for some $d_1$, so $F=dl$ for some $d$. □

Fix $c_1\in\mathrm{Pic}(X)$ and $c_2\in\mathbf{Z}$ with $(4c_2-c_1^2)>0$. Note that a rank two bundle $V$ is stable with respect to some ample divisor if and only if $V\otimes\mathcal{O}_X(F)$ is stable with respect to the same ample divisor where $F$ is any divisor. Thus, in view of Proposition 2(i), we may assume that $c_1$ is one of the following:

$$0,\ \mathcal{O}_X(l),\ \mathcal{O}_X(\Sigma),\ \mathcal{O}_X(\Sigma-l),\ \mathcal{O}_X(\Sigma-\Sigma_1),\ \mathcal{O}_X(\Sigma-\Sigma_1+l)$$

where $\Sigma_1$ is a section different from $\Sigma$. For simplicity, we assume that $(\Sigma\cdot\Sigma_1)=0$ in the cases of $c_1=\mathcal{O}_X(\Sigma-\Sigma_1)$ and $\mathcal{O}_X(\Sigma-\Sigma_1+l)$. Therefore, for $(4c_2-c_1^2)=6$ or 8, all possible values of $c_1$ and $c_2$ are the following

(i) $c_1=0$, $c_2=2$, $(4c_2-c_1^2)=8$;
(ii) $c_1=\mathcal{O}_X(l)$, $c_2=2$, $(4c_2-c_1^2)=8$;
(iii) $c_1=\mathcal{O}_X(\Sigma)$, $c_2=1$, $(4c_2-c_1^2)=6$;
(iv) $c_1=\mathcal{O}_X(\Sigma-l)$, $c_2=1$, $(4c_2-c_1^2)=8$;
(v) $c_1=\mathcal{O}_X(\Sigma-\Sigma_1)$ where $\Sigma_1$ is a section different from $\Sigma$ with $(\Sigma\cdot\Sigma_1)=0$, $c_2=1$, $(4c_2-c_1^2)=8$;
(vi) $c_1=\mathcal{O}_X(\Sigma-\Sigma_1+l)$ where $\Sigma_1$ is a section different from $\Sigma$ with $(\Sigma\cdot\Sigma_1)=0$, $c_2=1$, $(4c_2-c_1^2)=8$.

By the Remark 2 in Sect. 2, we see that no stable bundle exists in case (i). In this section, we will study the rest five cases.

### 3.2 $c_1=\mathcal{O}_X(l)$ *and* $c_2=2$

We begin with the analysis of any rank two bundle $V$ with chern classes $c_1=\mathcal{O}_X(l)$ and $c_2=2$. By the Riemann-Roch formula, $\chi(V)=2$, so either $h^0(V)>0$ or $h^2(V)>0$, the second case gives $h^0(\mathcal{O}_X(-l)\otimes V)>0$. Thus, either $0\to\mathcal{O}_X\to V$ or $0\to\mathcal{O}_X(l)\to V$.

If $V$ is stable for some ample divisor, then $0\to\mathcal{O}_X(l)\to V$ is impossible, so we must have $0\to\mathcal{O}_X\to V$. Thus,

$$0\to\mathcal{O}_X(F)\to V\to\mathcal{O}_X(l-F)\otimes I_Z\to 0$$

for some effective divisor $F$ and some locally complete intersection 0-cycle $Z$.

By Proposition 1(iii), all ample divisors $L_r$ with $r>5$ are equivalent. We now study those bundles which are $L_r$-stable for $r>5$.

**Theorem 1** (i) *A rank two bundle $V$ is $L_r$-stable for $r>5$ if and only if it is given by an extension*

$$0 \to \mathcal{O}_X \to V \to \mathcal{O}_X(l) \otimes I_Z \to 0$$

*where $Z$ is a 0-cycle of length 2 supported in a fiber of the elliptic fibration;*
(ii) *For any 0-cycle $Z$ consisting two distinct points lying in a fiber, there exists a rank two bundle as in* (i).

*Proof.* (i) Assume $V$ is $L_r$-stable where $r>5$. From the analysis above, $V$ has a sub-line-bundle $\mathcal{O}_X(F)$ where $F$ is effective. Then $F \cdot L_r < (c_1 \cdot L_r)/2 = 1/2$ for $r>5$, so $F \cdot L_r = 0$, thus $F=0$. Therefore, we have

$$0 \to \mathcal{O}_X \to V \to \mathcal{O}_X(l) \otimes I_Z \to 0$$

where $Z$ has length equal to 2. Suppose $Z$ is reduced. Since $V$ is locally free, any section of $\mathcal{O}_X(l)$ which vanishes at one of the two points of $Z$ must also vanishes at the other point, thus these two points must lie in the same fiber of the elliptic fibration.

Conversely, let $V$ be given by the above extension, and let $\mathcal{O}_X(F)$ be any sub-line-bundle of $V$ with torsion free quotient. If $0 \to \mathcal{O}_X(F) \to \mathcal{O}_X$, then $F \cdot L_r \leqq 0 < (c_1 \cdot L_r)/2$. If $0 \to \mathcal{O}_X(F) \to \mathcal{O}_X(l) \otimes I_Z$, then $(l-F)$ is strictly effective. Put $(l-F)=E$. Then, $F \cdot L_r = 1 - E \cdot L_r \leqq 0 < (c_1 \cdot L_r)/2$. Therefore, $V$ is actually $L_r$-stable for all $r>2$.
(ii) Since any section of $\mathcal{O}_X(l)$ which vanishes at one of the two points of $Z$ must also vanishes at the other point, $V$ exists by the Corollary in Sect. 1. □

In the proof above, we see that $V$ is stable with respect to all $L_r$ with $r>2$. In fact, we have

**Proposition 3** *All ample divisors $L_r$ are in one equivalence class.*

*Proof.* We need only to show that if $\zeta$ defines a wall containing some $L_r$, then $W^\zeta$ must contains all $L_r$. Put $\zeta = a\Sigma + b\Sigma_1 + c\,l$ where $\Sigma_1$ is a section different from $\Sigma$ and $b=0$ or 1. Note that $\zeta^2$ is divided by 4 by the formula for $d(\zeta)$ in 2.3.

Assume $b=0$. Then, $0=\zeta \cdot L_r$ gives $c=a(2-r)$, so $\zeta^2 = -2a^2(r-1)$ and $a \neq 0$. If $\zeta^2=-4$, then $\zeta = \pm(\Sigma - l)$, so $\zeta \cdot l$ is odd, but this is impossible since by definition $\zeta = 2F - c_1$ for some divisor $F$. If $\zeta^2 = -8$, then $r = 1 + 4/a^2$, so $a = \pm 1$ and $r=5$, thus $\zeta = \pm(\Sigma - 3l)$, but this is impossible again.

Assume $b=1$. Then, $0=\zeta \cdot L_r$ gives $c = 2a - \Sigma \cdot \Sigma_1 - (a+1)r$, so $\zeta^2 = -2[(a+1)^2(r-1) + \Sigma \cdot \Sigma_1 + 2]$. If $\zeta^2=-4$, then $(a+1)^2(r-1) + \Sigma \cdot \Sigma_1 = 0$, so $a=-1$ and $\Sigma \cdot \Sigma_1 = 0$, thus $\zeta \cdot L_r = 0$ for all $r>2$. Therefore, $W^\zeta$ contains all $L_r$ with $r>2$. If $\zeta^2 = -8$, then $0 \leqq \Sigma \cdot \Sigma_1 \leqq 2$; when $\Sigma \cdot \Sigma_1 = 0$, $r = 1 + 2/(a+1)^2$, so $r=3$ and $a=0$ or $-2$, in both cases $\zeta \cdot l$ are odd, impossible; when $\Sigma \cdot \Sigma_1 = 1$, $r = 1 + 1/(a+1)^2 \leqq 2$, but we know that $r>2$; *finally, when* $\Sigma \cdot \Sigma_1 = 2$, $a=-1$, again $\zeta \cdot L_r = 0$ for all $r>2$, so $W^\zeta$ contains all $L_r$ with $r>2$. □

### *3.3 $c_1 = \mathcal{O}_X(\Sigma)$ and $c_2 = 1$*

We use the process in 3.2. Let $V$ be any rank two bundle with chern classes $c_1$ and $c_2$. Then, $\chi(V \otimes \mathcal{O}_X(l-\Sigma)) = 1$. Thus, either $h^0(V \otimes \mathcal{O}_X(l-\Sigma)) > 0$, then we have $0 \to \mathcal{O}_X(\Sigma - l) \to V$; or $h^2(V \otimes \mathcal{O}_X(l-\Sigma)) > 0$, then $h^0(V \otimes \mathcal{O}_X(-l)) > 0$, so we have $0 \to \mathcal{O}_X(l) \to V$.

**Lemma 1.** *$\mathcal{M}_{L_4}(c_1, c_2)$ is empty.*

*Proof.* Note that $(\Sigma-2l)\cdot L_4=0$. If $0\to\mathcal{O}_X(\Sigma-l)\to V$, then $(\Sigma-l)\cdot L_4=(c_1\cdot L_4)/2$; if $0\to\mathcal{O}_X(l)\to V$, then $l\cdot L_4=(c_1\cdot L_4)/2$. Thus, $V$ can't be $L_4$-stable in both cases. □

*Remark 1* Let $\mathcal{M}_L(c_1,c_2)$ be any nonempty moduli space. Then $\mathcal{M}_L(c_1,c_2)$ consists of finite many reduced points. By Lemma 1, any bundle corresponding to a point of $\mathcal{M}_L(c_1,c_2)$ is the only element in some $E_\zeta(c_1,c_2)$ where $\zeta$ defines a nonempty wall of type $(c_1,c_2)$ separating $L$ and $L_4$. In this case, $\zeta^2=-6$ by the discussion in 2.4.

Now, we determine the equivalence classes of the ample divisors $L_r$ with $r>2$, and the moduli spaces corresponding to these ample divisors. Put $\zeta=(\Sigma-2l)$. Then, $\zeta$ defines a nonempty wall of type $(c_1,c_2)$ which contains $L_4$. Since $\zeta^2=-6$, both $E_\zeta(c_1,c_2)$ and $E_{(-\zeta)}(c_1,c_2)$ consist of exactly one element.

**Lemma 2** *If $\zeta_1$ with $\zeta_1^2=-6$ defines a wall of type $(c_1,c_2)$ containing some $L_r$ with $r>2$, then $\zeta_1=\pm\zeta$ and $r=4$.*

*Proof.* We use the same method as in the proof of Proposition 3. Put $\zeta_1=a\Sigma+b\Sigma_1+cl$ where $\Sigma_1$ is a section different from $\Sigma$, and $b=0$ or 1. If $b=0$, then $\zeta_1=\pm\zeta$ and $r=4$. If $b=1$, we obtain that $c=2a-\Sigma\cdot\Sigma_1-(a+1)r$ and that $(a+1)^2(r-1)=1-\Sigma\cdot\Sigma_1$. When $\Sigma\cdot\Sigma_1=0$, $r\leqq 2$, but this is impossible. When $\Sigma\cdot\Sigma_1=1$, then $a=-1$ and $\zeta_1=-\Sigma+\Sigma_1-3l$, so $\zeta_1\cdot l=0$; on the other hand, $2F-c_1=\zeta_1$ for some divisor $F$, so $0=2(F\cdot l)-1$, impossible. □

**Theorem 2** *The ample divisors $L_r$ with $r>2$ are divided into the following three equivalence classes:*

(i) *$L_3$. $\mathcal{M}_{L_3}(c_1,c_2)$ is identified with $E_\zeta(c_1,c_2)$;*
(ii) *$L_4$. $\mathcal{M}_{L_4}(c_1,c_2)$ is empty;*
(iii) *$L_r$ with $r>4$. $\mathcal{M}_{L_r}(c_1,c_2)$ is identified with $E_{(-\zeta)}(c_1,c_2)$.*

*Proof.* Note that if $\zeta_1$ defines a nonempty wall of type $(c_1,c_2)$ and $\zeta_1^2\neq-6$, then both $E_{(-\zeta_1)}(c_1,c_2)$ and $E_{\zeta_1}(c_1,c_2)$ are empty. If $L_{r_1}$ and $L_{r_2}$ with $r_1,r_2>4$ are not equivalent, then there exists a wall $W^\eta$ containing some $L_r$ with $r\geqq\max(r_1,r_2)>4$ where $\eta^2=-6$, but this is impossible by Lemma 2. Thus, all $L_r$ with $r>4$ are equivalent. By Lemma 1, $\mathcal{M}_{L_4}(c_1,c_2)$ is empty. If we can show $\mathcal{M}_{L_3}(c_1,c_2)=E_\zeta(c_1,c_2)$ and $\mathcal{M}_{L_r}(c_1,c_2)=E_{(-\zeta)}(c_1,c_2)$ for $r>4$, then $L_4$ isn't equivalent to $L_r$ with $r\neq 4$, and $L_3$ isn't equivalent to $L_r$ with $r\geqq 4$.

We will only prove that $\mathcal{M}_{L_r}(c_1,c_2)=E_{(-\zeta)}(c_1,c_2)$ for $r>4$. By Remark 1, $\mathcal{M}_{L_r}(c_1,c_2)$ is contained in the union of $E_{(-\eta)}(c_1,c_2)$ and $E_\eta(c_1,c_2)$ where $\eta$ runs over all numerical equivalence classes which define walls separating $L_r$ and $L_4$, and $\eta^2=-6$. By Lemma 2, $\eta=\pm\zeta$ where $\zeta=(\Sigma-2l)$. Since $L_r\cdot\zeta=(r-4)>0$, we see that $\mathcal{M}_{L_r}(c_1,c_2)$ is contained in $E_{(-\zeta)}(c_1,c_2)$. Let $V$ be the unique element in $E_{(-\zeta)}(c_1,c_2)$. Then, we have

$$0\to\mathcal{O}_X(l)\to V\to\mathcal{O}_X(\Sigma-l)\to 0.$$

Suppose that $V$ isn't $L_r$-stable for $r>4$, and that $\mathcal{O}_X(F)$ is a sub-line-bundle of $V$ with torsion free quotient and $F\cdot L_r\geqq(c_1\cdot L_r)/2$. If $0\to\mathcal{O}_X(F)\to\mathcal{O}_X(l)$, then $F\cdot L_r\leqq l\cdot L_r<(c_1\cdot L_r)/2$, a contradiction. Thus, we must have $0\to\mathcal{O}_X(F)\to\mathcal{O}_X(\Sigma-l)$, so $(\Sigma-l-F)$ is strictly effective. Put $(\Sigma-l-F)=E$ and $\zeta_1=(2F-c_1)$. Since $\zeta_1\cdot L_r\geqq 0$ for $r>4$ and $\zeta_1\cdot L_4=-2(E\cdot L_4)<0$, $\zeta_1$ defines a wall separating $L_4$

and $L_r$, so $W^{\zeta_1}$ contains some $L_s$ with $s>4$. Now $\zeta_1^2=-6$ and $W^{\zeta_1}$ doesn't contain $L_4$, this contradicts to Lemma 2. Therefore, $V$ must be $L_r$-stable for $r>4$. Hence, $\mathcal{M}_{L_r}(c_1, c_2)=E_{(-\zeta)}(c_1, c_2)$. □

### *3.4 $c_1=\mathcal{O}_X(\Sigma-l)$ and $c_2=1$*

As before, let $V$ be any rank two bundle with chern classes $c_1$ and $c_2$. By the Riemann-Roch formula, we have $\chi(V\otimes\mathcal{O}_X(-\Sigma+l))=1$. Thus, either $h^0(V\otimes\mathcal{O}_X(-\Sigma+l))>0$, then $0\to\mathcal{O}_X(\Sigma-l)\to V$; or $h^2(V\otimes\mathcal{O}_X(-\Sigma+l))>0$, then $h^0(V)>0$, so $0\to\mathcal{O}_X\to V$.

**Lemma 3** *$\mathcal{M}_{L_3}(c_1, c_2)$ is empty.*

*Proof.* Note that $(\Sigma-l)\cdot L_3=0$. Then, the conclusion follows from the analysis above. □

**Theorem 3** *Any irreducible component of a nonempty moduli space is birational to $X$.*

*Proof.* Let $L$ be an ample divisor such that $\mathcal{M}_L(c_1, c_2)$ is nonempty, let $\mathcal{M}$ be an irreducible component of $\mathcal{M}_L(c_1, c_2)$. Then, $\mathcal{M}$ is smooth by the Lemma 4 in Sect. 2. Since $\mathcal{M}_{L_3}(c_1, c_2)$ is empty, an open dense subset $\mathcal{U}$ of $\mathcal{M}$ is contained in some $E_\zeta(c_1, c_2)$, where $\zeta$ defines a nonempty wall of type $(c_1, c_2)$ separating $L$ and $L_3$. Then, we must have $d(\zeta)=0$ and $\zeta^2=-4$. Now, $D(\zeta)=2=\dim\mathcal{M}$. By 2.4, $E_\zeta(c_1, c_2)$ is isomorphic to the $K3$-surface $X$. Therefore, $\mathcal{M}$ is birational to $X$. □

*Remark 2.* Mukai [M] has proved that on a $K3$-surface $X$, any two dimensional compact irreducible component of a moduli space of rank two bundles Gieseker-stable with respect to an ample divisor is isogenus to $X$. Thus, any two dimensional irreducible component of a moduli space of rank two bundles stable in our sense with respect to an ample divisor is birational to some $K3$-surface isogenus to $X$. We see that Theorem 3 illustrates this result.

Next, we classify the ample divisors $L_r$ where $r>2$. Put $\zeta=(\Sigma-l)$. Then $\zeta^2=-4$, and $\zeta$ defines a wall of type $(c_1, c_2)$ containing $L_3$. First, we state a lemma similar to Lemma 2.

**Lemma 4** *Let $\zeta_1$ define a wall of type $(c_1, c_2)$ containing some $L_r$ with $r>2$. Then,*

(i) *either $r=3$ or $r=5$;*
(ii) *$\zeta_1=\pm\zeta$ if $\zeta_1^2=-4$ and $r=3$;*
(iii) *$\zeta_1=\pm(\Sigma-3l)$ if $r=5$.*

*Proof.* Put $\zeta_1=a\Sigma+b\Sigma_1+cl$ where $\Sigma_1$ is a section different from $\Sigma$, and $b=0$ or 1. Assume $b=0$. If $\zeta_1^2=-4$, then $\zeta_1=\pm\zeta$ and $r=3$. If $\zeta_1^2=-8$, then $a=\pm1$, $r=5$, and $\zeta_1=\pm(\Sigma-3l)$. Assume $b=1$. If $\zeta_1^2=-4$, then $a=-1$ and $\zeta_1=-\Sigma+\Sigma_1+cl$, but this is impossible. If $\zeta_1^2=-8$, we have $r=3$ and $(a+1)=\pm1$. □

Now, we show the following

**Theorem 4** *The ample divisors $L_r$ with $r>2$ are divided into the following five equivalence classes:*

(i) $L_r$ *with* $r>5$. $\mathcal{M}_{L_r}(c_1, c_2)$ *is birational to* $E_{(-\zeta)}(c_1, c_2)$ *which is isomorphic to* $X$;
(ii) $L_5$. $\mathcal{M}_{L_5}(c_1, c_2)$ *is birational to* $E_{(-\zeta)}(c_1, c_2)$;
(iii) $L_4$. $\mathcal{M}_{L_4}(c_1, c_2)$ *is isomorphic to* $E_{(-\zeta)}(c_1, c_2)$;
(iv) $L_3$. $\mathcal{M}_{L_3}(c_1, c_2)$ *is empty.*

*Proof.* Note that (iv) comes from Lemma 3. Since $D(\pm(\Sigma-3l))=1$, $E_{(\pm(\Sigma-3l))}(c_1, c_2)$ has dimension one. Using Lemma 4 and the same argument as in the proof of Theorem 2, we conclude that (1) any bundle in $E_{(-(\Sigma-3l))}(c_1, c_2)$ is $L_r$-stable if $r>5$ but not $L_r$-stable if $r\leqq 5$; (2) any bundle in $E_{(\Sigma-3l)}(c_1, c_2)$ is $L_r$-stable if $r<5$ but not $L_r$-stable if $r\geqq 5$; (3) any bundle in $E_{(-\zeta)}(c_1, c_2)$ is $L_r$-stable if $r>3$ but not $L_3$-stable. Thus, all $L_r$ with $r>2$ are divided into the above four equivalence classes, and the moduli spaces $\mathcal{M}_{L_r}(c_1, c_2)$ have the stated properties. □

### 3.5 $c_1=\mathcal{O}_X(\Sigma-\Sigma_1)$ *where* $\Sigma_1$ *with* $\Sigma\cdot\Sigma_1=0$ *is a section* $\neq\Sigma$ *and* $c_2=1$

Put $G_r=(\Sigma+\Sigma_1+rl)$ where $r$ is any real number. As in Proposition 1(i), we can show that $G_r$ is ample if and only if $r>2$. Clearly, $G_r\cdot c_1=0$. Let $V$ be any rank two bundle with chern classes $c_1$ and $c_2$. Since $\chi(V)=1$, either $h^0(V)>0$, then we get $0\to\mathcal{O}_X\to V$; or $h^2(V)>0$, so $0<h^0(V^*)=h^0(V\otimes\mathcal{O}_X(-\Sigma+\Sigma_1)$, thus we get $0\to\mathcal{O}_X(\Sigma-\Sigma_1)\to V$. In either case, $V$ has a sub-line-bundle $\mathcal{O}_X(F)$ with $F\cdot G_r=0=(c_1\cdot G_r)/2$, so $V$ can't be $G_r$-stable. Therefore, $\mathcal{M}_{G_r}(c_1, c_2)$ is empty for all ample divisors $G_r$ with $r>2$.

Now, let us consider $\mathcal{M}_{L_r}(c_1, c_2)$ for ample divisors $L_r$ with $r>2$. Since $\chi(V\otimes\mathcal{O}_X(l))=1$, either $h^0(V\otimes\mathcal{O}_X(l))>0$, then we get $0\to\mathcal{O}_X(-l)\to V$; or $h^2(V\otimes\mathcal{O}_X(l))>0$, so $0<h^0(V^*\otimes\mathcal{O}_X(-l))=h^0(V\otimes\mathcal{O}_X(-\Sigma+\Sigma_1-l)$, thus we get $0\to\mathcal{O}_X(\Sigma-\Sigma_1+l)\to V$. In either case, $V$ has a sub-line-bundle $\mathcal{O}_X(F)$ with $F\cdot L_r=-1=(c_1\cdot L_r)/2$, so $V$ can't be $L_r$-stable. Therefore, $\mathcal{M}_{L_r}(c_1, c_2)$ is also empty for all ample divisors $L_r$ with $r>2$.

Using the above discussion and the same method as in the proof of Theorem 3, we conclude the following

**Theorem 5** (i) *The moduli spaces* $\mathcal{M}_{L_r}(c_1, c_2)$ *and* $\mathcal{M}_{G_r}(c_1, c_2)$ *are empty for all* $r>2$;
(ii) *Any irreducible component of a nonempty moduli space is birational to* $X$.

By the definition of equivalence classes, we immediately have

**Corollary.** *All ample divisors* $L_r$ *and* $G_r$ *with* $r>2$ *are in one equivalence class.*

Next, we show that there exist ample divisors with respect to which the moduli spaces are nonempty. Put $\zeta=(\Sigma-\Sigma_1)$. Then, $\zeta^2=c_1^2=-4$. Since $\zeta\cdot G_r=0$ for any $r$, $\zeta$ defines a nonempty wall of type $(c_1, c_2)$ containing the ample divisors $G_r$ where $r>2$. Since $\zeta^2=-4$, $E_{(\pm\zeta)}(c_1, c_2)$ are two dimensional and are isomorphic to $X$. Let $\mathcal{C}$ be any chamber of type $(c_1, c_2)$ whose boundary contains part of the wall $W^\zeta$. We may assume $\mathcal{C}\cdot\zeta<0$. Then, $E_\zeta(c_1, c_2)$ is contained in $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$. In particular, the moduli space $\mathcal{M}_L(c_1, c_2)$ is nonempty for any ample divisor $L$ in $\mathcal{C}$.

### 3.6 $c_1=\mathcal{O}_X(\Sigma-\Sigma_1+l)$ *where* $\Sigma_1$ *with* $\Sigma\cdot\Sigma_1=0$ *is a section* $\neq\Sigma$ *and* $c_2=1$

We now study the last case. Put $\zeta=(\Sigma-\Sigma_1+l)$, and $H_r=2L_r+G_r$ where $L_r=(\Sigma+rl)$ and $G_r=(\Sigma+\Sigma_1+rl)$. Then, $\zeta^2=-4$ and $H_r$ is ample when $r>2$. Since

$\zeta \cdot L_r = -1$ and $\zeta \cdot G_r = 2$, $\zeta \cdot H_r = 0$. Thus, $\zeta$ defines a nonempty wall of type $(c_1, c_2)$ containing all $H_r$ where $r > 2$. Moreover, $E_{(\pm\zeta)}(c_1, c_2)$ are isomorphic to $X$ by 2.4.

Let $V$ be any rank two bundle with chern classes $c_1$ and $c_2$. Since $\chi(V) = 1$, either $h^0(V) > 0$, so $0 \to \mathcal{O}_X \to V$; or $h^2(V) > 0$, so $0 \to \mathcal{O}_X(\Sigma - \Sigma_1 + l) \to V$.

**Lemma 5** *For any ample divisor $L_r$, the moduli space $\mathcal{M}_{L_r}(c_1, c_2)$ is identified with $E_\zeta(c_1, c_2)$.*

*Proof.* Suppose $V$ is $L_r$-stable. Since $(c_1 \cdot L_r)/2 = -1$, we can't have $0 \to \mathcal{O}_X \to V$. Thus, $0 \to \mathcal{O}_X(\Sigma - \Sigma_1 + l) \to V$. So

$$0 \to \mathcal{O}_X(\Sigma - \Sigma_1 + l + E) \to V \to \mathcal{O}_X(-E) \otimes I_Z \to 0$$

for some effective divisor $E$ and some locally complete intersection 0-cycle $Z$. Then, $(\Sigma - \Sigma_1 + l + E) \cdot L_r < (c_1 \cdot L_r)/2 = -1$, so $-1 + E \cdot L_r < -1/2$. We conclude that $E \cdot L_r = 0$ and $E = 0$. Therefore, we get

$$0 \to \mathcal{O}_X(\Sigma - \Sigma_1 + l) \to V \to I_Z \to 0$$

for some point $Z$ in $X$. Hence, $V \in E_\zeta(c_1, c_2)$.

Conversely, suppose that $V \in E_\zeta(c_1, c_2)$ and that $\mathcal{O}_X(F)$ is any sub-line-bundle of $V$ with torsion free quotient. If $0 \to \mathcal{O}_X(F) \to \mathcal{O}_X(\Sigma - \Sigma_1 + l)$, then $F \cdot L_r < (c_1 \cdot L_r)/2$. If $0 \to \mathcal{O}_X(F) \to I_Z$, then $(-F)$ is strictly effective, thus $F \cdot L_r \leqq -1 < (c_1 \cdot L_r)/2$. Therefore, $V$ is $L_r$-stable. □

Next, we discuss the moduli spaces $\mathcal{M}_{G_r}(c_1, c_2)$. Put $\zeta' = (\Sigma - \Sigma_1 - l)$. Then, $(\zeta')^2 = -4$. Since $\zeta' \cdot L_r = -3$ and $\zeta' \cdot (\Sigma_1 + r\,l) = 1$ for any $r$, we conclude that $\zeta'$ defines a nonempty wall of type $(c_1, c_2)$ and that $E_{(\pm\zeta')}(c_1, c_2)$ are isomorphic to $X$.

**Lemma 6** *For any ample divisor $G_r$ with $r > 3$, the moduli spaces $\mathcal{M}_{G_r}(c_1, c_2)$ is identified with $E_{(\zeta')}(c_1, c_2)$.*

*Proof.* As in the proof of Lemma 5, we can show that every bundle in $\mathcal{M}_{G_r}(c_1, c_2)$ is contained in $E_{(\zeta')}(c_1, c_2)$ for $r \geqq 3$. Conversely, suppose that $V \in E_{(\zeta')}(c_1, c_2)$. Then, there is an exact sequence

$$0 \to \mathcal{O}_X(\Sigma - \Sigma_1) \to V \to \mathcal{O}_X(l) \otimes I_Z \to 0$$

where $Z$ is a point in $X$. Let $\mathcal{O}_X(F)$ be any sub-line-bundle of $V$ with torsion free quotient. If $0 \to \mathcal{O}_X(F) \to \mathcal{O}_X(\Sigma - \Sigma_1)$, then $F \cdot G_r \leqq 0 < (c_1 \cdot G_r)/2$. Assume $0 \to \mathcal{O}_X(F) \to \mathcal{O}_X(l) \otimes I_Z$. Then, $(l - F) = E$ for some strictly effective divisor $E$. Thus, $F \cdot G_r < (c_1 \cdot G_r)/2$ unless $E \cdot G_r = 1$, but then $E$ is irreducible and must be either $\Sigma$ or $\Sigma_1$ and $r = 3$ since $G_r = (\Sigma + \Sigma_1 + r\,l)$. Therefore, $V$ is $G_r$-stable if $r > 3$. □

As before, for the irreducible components of moduli spaces, we have

**Theorem 6** *Any irreducible component of a nonempty moduli space is birational to $X$.*

*Proof.* Let $\mathcal{M}_L(c_1, c_2)$ be nonempty. If $L = L_r$ for some $r > 2$, then $\mathcal{M}_L(c_1, c_2)$ is isomorphic to $X$ by Lemma 5. If $L \neq L_r$ for any $r > 2$, then any irreducible component $\mathcal{M}$ of $\mathcal{M}_L(c_1, c_2)$ is birational to $X$ by the same argument as in the proof of Theorem 3. □

Finally, we classify the ample divisors $L_r$ and $G_r$ where $r>2$. Put $\zeta_1=-(\Sigma+\Sigma_1-l)$. Then $\zeta_1^2=-8$, $\zeta_1\cdot G_3=0$ and $\zeta_1\cdot G_r<0$ if $r>3$. So $\zeta_1$ defines a nonempty wall of type $(c_1, c_2)$ containing $G_3$. Since $D(\zeta_1)=1$, $E_{(\zeta_1)}(c_1, c_2)$ has dimension one. It is easy to see that any bundle in $E_{(\zeta_1)}(c_1, c_2)$ is $G_r$-stable if $r>3$ and isn't $G_3$-stable.

**Proposition 4** *The ample divisors $L_r$ and $G_r$ with $r>2$ are divides into the following three equivalence classes:*

(i) $L_r$ *with* $r>2$; (ii) $G_r$ *with* $r>3$; (iii) $G_3$.

*Proof.* By Lemma 5, all $L_r$ with $r\geqq 3$ are in one equivalence class $\mathscr{E}_1$. Since $\zeta\cdot G_r=2>0$, any bundle in $E_\zeta(c_1, c_2)$ is $G_r$-unstable, so $G_r$ isn't contained in $\mathscr{E}_1$ for any $r>2$. By Lemma 6, all $G_r$ with $r>3$ are in one equivalence class $\mathscr{E}_2$. Since any bundle in $E_{(\zeta_1)}(c_1, c_2)$ is $G_r$-stable if $r>3$ and isn't $G_3$-stable, $G_3$ isn't contained in $\mathscr{E}_2$. □

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