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Quasi-compactness of dominated positive operators and C_0-semigroups

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1 Introduction

Quasi-compact operators were introduced by Kryloff and Bogoliouboff [15] in the context of Markov's process in order to prove a uniform ergodic theorem for Markov operators induced by transition probabilities. This result was generalized by Yosida and Kakutani [28] showing that any quasi-compact operator with bounded powers on a Banach space is uniformly ergodic. Since this result, many contributions have been made in probability and ergodic theory using the spectral properties of quasi-compact operators, see for instance [6, 14, 16–19] and [20]. Quasi-compactness for strongly continuous semigroups and its relationship to uniform ergodicity and to the asymptotic behavior of the solutions of abstract Cauchy problems is studied in [21]. In connection with this problem see also [8, 11, 12] and [26]. Also O'Brien [22] gave an application of quasi-compact semigroups to control theory.

An interesting problem in the theory of positive operators in Banach lattices is to know what properties of $T \in \mathscr{L}(E)$, where E is a Banach lattice, are inherited by $S \in \mathscr{L}(E)$ if we know that $0 \leqq S \leqq T$. Compactness properties have been considered for example in [1] and [9] (see, [2] and [29] for a survey) and spectral properties in [7]. In this paper we consider quasi-compactness of dominated positive operators and C_0-semigroups. The quasi-compactness of an operator means that the essential spectral radius is less than 1 (see, Prop. 2.2 below). Therefore S is quasi-compact whenever $0 \leqq S \leqq T$ and T is quasi-compact, if and only if $r_{ess}(S) \leqq r_{ess}(T)$. But the monotonicity of the essential spectral radius is known only in few cases, e.g., when S in an AM-compact operator (see [23], Theor. 3.2), or when $E = L^1(\Omega, \Sigma, \mu)$ where (Ω, Σ, μ) is a totally finite, complete measure space without atoms (see [27], 4.2, Coroll. 1).

Throughout this paper, E will denote a complex Banach lattice, i.e., $E = E_{\mathbf{R}} \oplus \imath E_{\mathbf{R}}$, the complexification of a real Banach lattice $E_{\mathbf{R}}$. The absolute value in $E_{\mathbf{R}}$ is extended to E by means of the formula $|f| = \sup\{(\cos\theta)g + (\sin\theta)h : 0 \leqq \theta \leqq 2\pi\}$, where $f = g + \imath h$ with $g, h \in E_{\mathbf{R}}$ (see [24], Chap. II, § 11 or [29],

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Sect. 91) and the norm in E satisfies $\|f\| = \||f|\|$ for all $f \in E$. By $\mathscr{L}(E)$ we denote the space of all bounded linear operators from E into E. For $T \in \mathscr{L}(E)$ there exist canonical operators Re T, Im $T \in \mathscr{L}(E_{\mathbf{R}})$ such that $Tf = (\text{Re } T)f + i(\text{Im } Tf)$ for all $f \in E_{\mathbf{R}}$. An operator $T \in \mathscr{L}(E)$ is said to be *positive* ($T \geqq 0$), if T is equal to Re T and $Tf \geqq 0$ for $f \in E_+$, T is called *regular* if T can be written as a linear combination of positive operators. The space of all regular operators from E into E is denoted by $\mathscr{L}^r(E)$. When E is Dedekind complete, $\mathscr{L}^r(E)$ is a Riesz space and moreover a Banach lattice with respect to the r-norm defined by $\|T\|_r := \||T|\|$ (see [24], II, § 11).

If E is a complex Banach space and $T \in \mathscr{L}(E)$ we denote, as usual, the spectrum by $\sigma(T)$ and the spectral radius by $r(T)$. The Fredholm domain of T is defined by

$$\Phi(T) := \{\lambda \in \mathbf{C}: \lambda I - T \textit{ is a Fredholm operator}\}$$
$$= \{\lambda \in \mathbf{C}: \operatorname{Ker}(\lambda I - T) \textit{ and } E/\operatorname{Ran}(\lambda I - T) \textit{ are finite dimensional}\}.$$

The set $\mathbf{C} \setminus \Phi(T)$ will be called the (*Wolf*) *essential spectrum* of T, and is denoted by $\sigma_{\text{ess}}(T)$. The essential spectrum of T is equal to the spectrum of the canonical image of T in the Calkin algebra $\mathscr{L}(E)/\mathscr{K}(E)$, where $\mathscr{K}(E)$ denotes the ideal of compact operators in E. A classical result of Atkinson (see e.g. [13], Prop. 25.2) states that $\lambda I - T$ is invertible modulo $\mathscr{K}(E)$ if and only if it is invertible modulo the ideal $\mathscr{F}(E)$ of finite rank operators. The essential spectral radius of T is defined by $r_{\text{ess}}(T) := \sup\{|\lambda|: \lambda \in \sigma_{\text{ess}}(T)\}$. We have that

$$r_{\text{ess}}(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|_{\text{ess}}},$$

where

$$\|T\|_{\text{ess}} = \operatorname{dist}(T, \mathscr{K}(E)) = \inf\{\|T - K\|: K \in \mathscr{K}(E)\}.$$

Recall that $\lambda \in \sigma(T)$ is a *Riesz point* of $\sigma(T)$ if λ is a pole of the resolvent $R(\mu, T) = (\mu - T)^{-1}$, whose residuum is a projection of finite rank. It is known that $r_{\text{ess}}(T)$ is a smallest $r \in \mathbf{R}_+$ such that every $\lambda \in \sigma(T)$ with $|\lambda| > r$ is a Riesz point of $\sigma(T)$ (see e.g. [13], § 52).

Following Schaefer [25], we define the *order spectrum* $\sigma_o(T)$ of a regular operator $T \in \mathscr{L}^r(E)$ to be the spectrum of T in the Banach algebra $\mathscr{L}^r(E)$. It is obvious that $\sigma(T) \subset \sigma_o(T)$ and $r(T) \leqq r_o(T)$ hold for the corresponding spectral radius; moreover, $r(T) = r_0(T)$ whenever $T \geqq 0$. Arendt and Sourour [5] introduced the order essential spectrum in the following way: Let $T \in \mathscr{L}^r(E)$ and let

$$\pi: \mathscr{L}^r(E) \to \mathscr{L}^r(E)/\mathscr{K}^r(E),$$

where $\mathscr{K}^r(E)$ is the ideal defined as the closure (in the r-norm $\|\ \|_r$) of the ideal $\mathscr{F}(E)$. The *order essential spectrum* $\sigma_{\text{oe}}(T)$ of T is the spectrum of $\pi(T)$ in the Banach algebra $\mathscr{L}^r(E)/\mathscr{K}^r(E)$.

Finally, note that our general references concerning the theory of positive operators in Banach lattice are the monographs [2, 24] and [29], and concerning the theory of positive C_0-semigroups are the monographs [8] and [21].

2 Quasi-compactness of dominated positive operators

We start this section recalling some characterizations of quasi-compact operators.

2.1 Definition. Let E be a Banach space. An operator $T\in\mathscr{L}(E)$ is called *quasi-compact* if there exists a compact operator $K\in\mathscr{K}(E)$ and $n\in\mathbf{N}$ such that

$$\|T^n-K\|<1.$$

The following characterization is well known (see e.g. [6]). We include here a short proof for the sake of completeness.

2.2 Proposition. *Let E be a Banach space. For $T\in\mathscr{L}(E)$ the following assertions are equivalent.*

(a) *T is quasi-compact.*
(b) $\lim_{n\to\infty}\|T^n\|_{ess}=0$.
(c) $r_{ess}(T)<1$.
(d) *$T=U+K$, where $K\in\mathscr{K}(E)$ is of finite rank and $U\in\mathscr{L}(E)$ has spectral radius $r(U)<1$.*

Proof. (a)⇒(b)⇒(c) are obvious.

(c)⇒(d): Since $r_{ess}(T)<1$, there exists $0<\alpha<1$ such that $\sigma_{ess}(T)\subset B(0,a)$, $B(0,a)$ being the open ball centered at 0 with radius a. Then, we have

$$\sigma(T)=(\sigma(T)\cap B(0,a))\cup\{\lambda_1,\ldots\lambda_n\}$$

each λ_j being a Riesz point of $\sigma(T)$. If P is the spectral projection associated with the spectral set $\{\lambda_1,\ldots\lambda_n\}$, we have that

$$T=TP+T(I-P),$$

the dimension of $P(E)$ is finite and

$$\sigma(T(I-P))=\sigma(T)\cap B(0,a)$$

Thus, if $U:=T(I-P)$ and $K:=TP$ we obtain (d).

(d)⇒(a): If $T=K+U$ with $r(U)<1$ and $K\in\mathscr{F}(E)$, then $T^n=K_n+U^n$ with $K_n\in\mathscr{F}(E)$. Now, since $r(U)=\lim_{n\to\infty}\sqrt[n]{\|U^n\|}<1$, there exists $n\in\mathbf{N}$ such that $\|U^n\|<1$. Thus, $\|T^n-K_n\|<1$, and so T is quasi-compact. Q.E.D.

2.3 Corollary. *Let E be a Banach lattice and $0\leqq T\in\mathscr{L}(E)$ with $r(T)=1$. Then the following statements are equivalent.*

(a) *T is quasi-compact.*
(b) $1\notin\sigma_{ess}(T)$.
(c) *1 is a Riesz point of $\sigma(T)$.*

Proof. (a)⇒(b)⇒(c) are obvious.

(c)⇒(a): Since $T\geqq 0$, it follows from ([24], V.5.5) that every point in $\sigma(T)\cap\{\lambda\in\mathbf{C}\colon|\lambda|=1\}$ is a pole of $R(\lambda,T)$. Moreover, a close inspection of the proof of that theorem shows that all these points are Riesz points of $\sigma(T)$, and hence $r_{ess}(T)<1$. Therefore, T is quasi-compact. Q.E.D.

In this section we are interested in the following problem.

(P) *Let E be a Banach lattice. Let $S, T\in\mathscr{L}(E)$ be such that $0\leqq S\leqq T$. Is S quasi-compact whenever T is quasi-compact?*

In general problem (P) has a negative answer as the following example shows.

2.4 Example. B. de Pagter and A. Schep in ([23], Example 3.7) build two convolution operators T_μ and T_ν in $L^2(\Delta, \lambda)$, Δ being the Cantor group and λ its Haar measure, such that $r_{ess}(T_\mu)=\frac{1}{3}$ and $r_{ess}(T_\mu+T_\nu)=\frac{2}{3}$. Then, if $S:=3T_\mu$ and $T:=3(T_\mu+T_\nu)$, we have that, $0\leqq S\leqq T$, $r_{ess}(S)=1$ and $r_{ess}(T)=\frac{2}{3}<1$. It follows from Prop. 2.2 that T is quasi-compact and S is not quasi-compact.

In the following result we will give a partial positive answer to problem (P).

2.5 Proposition. *Let E be a Banach lattice. Let $S, T\in\mathscr{L}(E)$ be such that $0\leqq S\leqq T$. If T is quasi-compact and $r(T)\leqq 1$, then S is quasi-compact.*

Proof. Since $r(S)\leqq r(T)\leqq 1$, we can assume that $r(S)=r(T)=1$. Now, by Corollary 2.3, $r(T)$ is a Riesz point of $\sigma(T)$. Then, it follows from Theorem 4.1 in [7] that $r(T)$ is a Riesz point of $\sigma(S)$, so S is quasi-compact by Corollary 2.3. Q.E.D.

As a consequence of Props. 2.2 and 2.4 we have the following corollary.

2.6 Corollary. *Let E be a Banach lattice. Let $S, T\in\mathscr{L}(E)$ be such that $0\leqq S\leqq T$. Then*

$$r_{ess}(S)<r(T) \quad \text{if } r_{ess}(T)<r(T).$$

In the following theorem we obtain that the order essential spectral radius (that we denote by $r_{oe}(T)$) has the monotonicity property. As a consequence we will see that in AL-spaces or AM-spaces problem (P) has a positive answer. We first need the following lemma, whose proof we omit because it is a slight modification of the proof of Theor. 18.25 in [2].

2.7 Lemma. *Let E be a Dedekind complete Banach lattice and let $S, T\in\mathscr{L}(E)$ be such that $0\leqq S\leqq T$. If $T\in\mathscr{K}^r(E)$, then $S^3\in\mathscr{K}^r(E)$.*

2.8 Theorem. *Let E be a Dedekind complete Banach lattice and let $S, T\in\mathscr{L}(E)$ be such that $0\leqq S\leqq T$. Then*

$$r_{oe}(S)\leqq r_{oe}(T).$$

Proof. Evidently, it is sufficient to prove that $r_{oe}(T)<1$ implies $r_{oe}(S)<1$. Thus, suppose $r_{oe}(T)<1$. Since $r_{ess}(T)\leqq r_{oe}(T)<1$, T is quasi-compact. Then, by Prop. 2.2, $T=K+D$, where $K\in\mathscr{F}(E)$ and $r(D)<1$. Since $K\in\mathscr{F}(E)$, D is a regular operator. Now, it follows from $K\in\mathscr{K}^r(E)$ that $r_{oe}(D)=r_{oe}(T)<1$. Moreover, by Corollary 3.6 in [5], we have

$$\sigma_o(D)\subset\sigma(D)\cup\sigma_{oe}(D).$$

Hence, $r_o(D)=\max\{r(D), r_{oe}(D)\}<1$. Thus, there exists $n\in\mathbf{N}$ such that $\|D^n\|_r<1$.

On the other hand, $T^n=K_n+D^n$ with $K_n\in\mathscr{F}(E)$. Then, applying the decomposition property in the Riesz space $\mathscr{L}^r(E)$ (see e.g. [2], Theor. 19), there exists $F, U\in\mathscr{L}^r(E)$ such that

$$S^n=F+U$$

with

$$0\leqq F\leqq|K_n| \quad \text{and} \quad 0\leqq U\leqq|D^n| \ (\Rightarrow\|U\|_r<1).$$

Given $\lambda \in \mathbf{C}$ with $|\lambda| \geqq 1$ we have that

$$\lambda I - S^n = (\lambda I - U)(I - R(\lambda, U)F). \tag{1}$$

Now, since $K_n \in \mathscr{F}(E)$, it follows from Theor. 1.3 in [4] that $|K_n| \in \mathscr{K}^r(E)$. Then, we conclude from

$$0 \leqq |R(\lambda, U)F| \leqq R(|\lambda|, U)|K_n|$$

and Lemma 2.7 that $|R(\lambda, U)F|^3 \in \mathscr{K}^r(E)$. So, keeping in mind the spectral mapping theorem (in the Banach algebra $\mathscr{L}^r(E)/\mathscr{K}^r(E)$), we have that $r_{oe}(|R(\lambda, U)F|) = 0$. Now, it is easy to see that

$$r_{oe}(R(\lambda, U)F) \leqq r_{oe}(|R(\lambda, U)F|).$$

Therefore $r_{oe}(R(\lambda, U)F) = 0$ and $I - R(\lambda, U)F$ is invertible in the algebra $\mathscr{L}^r(E)/\mathscr{K}^r(E)$. Thus, since $\lambda \notin \sigma_o(U)$, it follows from (1) that $\lambda - S^n$ is invertible in the algebra $\mathscr{L}^r(E)/\mathscr{K}^r(E)$, so $\lambda \in \sigma_{oe}(S^n)$. Hence $r_{oe}(S) < 1$ since $r_{oe}(S^n) < 1$. Q.E.D.

2.9 Corollary. *Let E be a Dedekind complete Banach lattice and let $S, T \in \mathscr{L}(E)$ be such that $0 \leqq S \leqq T$. If $\sigma(T) = \sigma_o(T)$, then $r_{ess}(S) \leqq r_{ess}(T)$.*

Proof. Since $\sigma(T) = \sigma_o(T)$, $\mathbf{C} \setminus \sigma_{ess}(T)$ and $\mathbf{C} \setminus \sigma_{oe}(T)$ have the same unbounded component by Theor. 7.1 in [5], hence $r_{ess}(T) = r_{oe}(T)$. Thus the corollary follows from the above theorem. Q.E.D.

There are some cases in which $\sigma(T) = \sigma_o(T)$ holds for instance, in [25], Theor. 3.3, it is shown that if E has property (P) (i.e., if there exists a positive, contractive projection $E^{**} \to E$) and $0 \leqq T \in \mathscr{L}(E)$ is majorizing or cone absolutely summing (see [24], IV.3.1, for the definitions), then $\sigma(T) = \sigma_o(T)$. On the other hand it is obvious that if E in an AL-space, then $\sigma(T) = \sigma_o(T)$ for every $T \in \mathscr{L}^r(E) = \mathscr{L}(E)$. Therefore, having in mind that every dual Banach lattice has property (P) and the corollary of Theor. IV, 3.8 in [24], we obtain the following result.

2.10 Corollary. *Let $S, T \in \mathscr{L}(E)$ be such that $0 \leqq S \leqq T$. In each one of the following cases we have that*

$$r_{ess}(S) \leqq r_{ess}(T).$$

(a) *E is an* AL*-space or an* AM*-space.*

(b) *E is a Dedekind complete Banach lattice and T is majorizing or cone absolutely summing.*

We know that for $0 \leqq S \leqq T \in \mathscr{L}^r(L^p(\mu))$ with $1 < p < +\infty$ the quasi-compactness of T does not imply the quasi-compactness of S, in general. Now, as consequence of some results of Lotz [20] it is easy to establish some positive results under certain restrictions on S. To that purpose we recall the following definition. A positive linear operator T on a Banach lattice E is said to satisfy a *Doeblin condition* if there exist $m \in \mathbf{N}$, $0 \leqq \phi \in E^*$, and a real number $\eta < 1$ such that

$$\|T^m f\| \leqq \langle f, \phi \rangle + \eta \|f\| \quad \text{for all } 0 \leqq f \in E.$$

In [20] it is proved that every positive quasi-compact operator satisfies a Doeblin condition, and it is also proved (Corollary 6) that if $E=L^p(\mu)$ $(1<p\leqq\infty)$, $0\leqq T\in\mathscr{L}(E)$ satisfies a Doeblin condition and the means $T_n:=\frac{1}{n}\sum_{i=0}^{n-1}T^i$ are uniformly bounded, then T is quasi-compact. Since a Doeblin condition is inherited by a dominated operator, we can establish the following result.

2.11 Proposition. *Let $E=L^p(\mu)$ with $1<p\leqq\infty$, and let $S, T\in\mathscr{L}(E)$ such that $0\leqq S\leqq T$. Assume that T is a quasi-compact operator. Then, if the means S_n are uniformly bounded, S is quasi-compact.*

2.12 Remark. Note that if S satisfies the assumption of the above proposition we have $\sup_{n\in\mathbb{N}}\|S_n\|\leqq M<\infty$. Hence $\|S^n\|=\|(n+1)S_{n+1}-nS_n\|\leqq(2n+1)M$ and $r(S)=\lim_{n\to\infty}\sqrt[n]{\|S^n\|}\leqq 1$. Thus the assumption $r(S)\leqq 1$ is weaker than the assumption of the proposition. Moreover, it is easy to see that the operator T_μ of Example 2.4 has spectrum $\sigma(T_\mu)=\left\{1, 0, (-1)^n\frac{1}{3^n}, n\in\mathbb{N}\right\}$. For $S=2T_\mu$ we obtain $r(S)=3$. This yields to the following open problem.

Is Proposition 2.11 true changing the assumption $\sup_{n\in\mathbb{N}}\|S_n\|\leqq M<\infty$ into $r(S)\leqq 1$?

As it is well known, $r(T)\in\sigma(T)$ for any positive operator T on a Banach lattice (see e.g. [8, 24] or [29]). It is a natural question to ask whether an analogous result is valid for the essential spectrum of positive operator. As shown in [21] the operator T_μ of Example 2.4 satisfies $r_{ess}(T_\mu)\notin\sigma(T_\mu)$. For certain classes of positive operators, however, the question has an affirmative answer. For instance, B. de Pagter and A. Schep ([23], Theor. 3.4) proved that $r_{ess}(T)\in\sigma_{ess}(T)$ for every positive AM-compact operator T defined on a Banach lattice, and L. Weis and M. Wolf in [27] showed $r_{ess}(T)\in\sigma_{ess}(T)$ for any positive operator in $L^1(\Omega, \Sigma, \mu)$ with (Ω, Σ, μ) a totally finite, complete measure space without atoms. In the next theorem we show, using the idea of the proof of Prop. 3.3 in [23], that the above question has an affirmative answer for the order essential spectral radius. We also obtain, as a consequence, some positive answer for the essential spectral radius.

2.13 Theorem. *Let E be a Dedekind complete Banach lattice and let T be a positive operator in E. Then $r_{oe}(T)\in\sigma_{oe}(T)$.*

Proof. Without loss of generality we assume $r_{oe}(T)=1$. Assume by contradiction that $1\notin\sigma_{oe}(T)$. Then, there exists a real number $0<a<1$ such that

$$\sigma_{oe}(T)\subset\{\lambda\in\mathbf{C}: \operatorname{Re}\lambda\leqq a\}.$$

Now, by the spectral mapping theorem (in the algebra $\mathscr{L}^r(E)/\mathscr{K}^r(E)$), it follows that,

$$\sigma_{oe}(e^{tT})=e^{t\sigma_{oe}(T)}\subset\{\lambda\in\mathbf{C}: |\lambda|\leqq e^{ta}\}.$$

Hence $r_{oe}(e^{tT})\leqq e^{ta}$ for all $t\geqq 0$. On the other hand

$$0\leqq\frac{t^n}{n!}T^n\leqq e^{tT}$$

for all $n\in\mathbf{N}$ and $t\geqq 0$. By Theorem 2.8 we obtain

$$r_{oe}\left(\frac{t^n}{n!}T^n\right)\leqq r_{oe}(e^{tT})\leqq e^{ta}.$$

Since

$$r_{oe}\left(\frac{t^n}{n!}T^n\right)=\frac{t^n}{n!}r_{oe}(T)^n=\frac{t^n}{n!},$$

we conclude $0\leqq\frac{t^n}{n!}\leqq e^{ta}$ for all $n\in\mathbf{N}$ and $t\geqq 0$. Substituting $t=\frac{n}{a}$, it follows from Stirling's formula that $a\geqq 1$. This contradiction completes the proof of the theorem. Q.E.D.

2.14 Corollary. *Let E be a Dedekind complete Banach lattice and let T be a positive operator in E such that $\sigma(T)=\sigma_o(T)$. Then $r_{ess}(T)\in\sigma_{ess}(T)$.*

Proof. Since $\sigma(T)=\sigma_o(T)$, the sets $\mathbf{C}\setminus\sigma_{ess}(T)$ and $\mathbf{C}\setminus\sigma_{oe}(T)$ have the same unbounded component by Theor. 7.1 in [5]. Thus the corollary follows from the above theorem. Q.E.D.

Using the same reasoning used to prove Corollary 2.10 we obtain the following result as a consequence of the above corollary.

2.15 Corollary. *Let E a Banach lattice and $0\leqq T\in\mathscr{L}(E)$. In each of the following cases we have*

$$r_{ess}(T)\in\sigma_{ess}(T).$$

(a) *E is an* AL*-space or an* AM*-space.*

(b) *E is a Dedekind complete Banach lattice and T is majorizing or cone absolutely summing.*

3 Quasi-compactness of dominated positive C_0-semigroups

In this section we are going to apply the results of the former section to positive C_0-semigroups. Let us recall first some standard definition and notations (see [21]).

Let $(T(t))_{t\geqq 0}$ be a C_0-semigroup of bounded linear operators with generator A in the Banach space E. By the *growth bound* of the C_0-semigroup $(T(t))_{t\geqq 0}$ we understand the number

$$\begin{aligned}\omega(A)&:=\inf\{w\in\mathbf{R}:\exists M\in\mathbf{R}_+ \text{ such that } \|T(t)\|\leqq Me^{wt} \text{ for } t\geqq 0\}\\&=\lim_{t\to\infty}\frac{1}{t}\log\|T(t)\|,\end{aligned}$$

and by the *essential growth bound* the number

$$\omega_{ess}(A):=\lim_{t\to\infty}\frac{1}{t}\log\|T(t)\|_{ess}.$$

It is known that

$$r(T(t))=e^{t\omega(A)} \quad \text{and} \quad r_{\text{ess}}(T(t))=e^{t\omega_{\text{ess}}(A)}$$

for all $t\geqq 0$. One important relation between this two bounds is

$$\omega(A)=\max\{\omega_{\text{ess}}(A), s(A)\},$$

being $s(A):=\sup\{\operatorname{Re}\lambda: \lambda\in\sigma(A)\}$ (i.e., the spectral bound of the generator A). Finally, recall that the *boundary spectrum* $\sigma_b(A)$ of A is defined by

$$\sigma_b(A):=\{\lambda\in\sigma(A): \operatorname{Re}\lambda=s(A)\}.$$

3.1 Definition. A C_0-semigroup $(T(t))_{t\geqq 0}$ on a Banach space E is called *quasi-compact* if

$$\lim_{t\to\infty} \operatorname{dist}(T(t), \mathscr{K}(E))=0.$$

In [21], B-IV, Prop. 2.8 the following characterization of quasi-compactness is given.

3.2 Proposition. *For a C_0-semigroup $(T(t))_{t\geqq 0}$ with generator A in a Banach space E the following conditions are equivalent.*

(i) *$(T(t))_{t\geqq 0}$ is quasi-compact.*

(ii) *$\omega_{\text{ess}}(A)<0$.*

(iii) *There exists $t_0>0$ such that $T(t_0)$ is a quasi-compact operator.*

A typical situation where quasi-compact semigroups occur is the following. Consider a C_0-semigroup $(T(t))_{t\geqq 0}$ with $\omega_{\text{ess}}(A)<\omega(A)$. Then the rescaled semigroup $(e^{-t\omega(A)}(T(t))$ is quasi-compact. For this reason, and also because the inequality $\omega_{\text{ess}}(A)<\omega(A)$ has important spectral consequences (for example, in this situation $s(A)$ is a strictly dominant spectral value if $(T(t))_{t\geqq 0}$ is assumed to be positive. See [21], C-III, 3.6 or [8], Theor. 9.10) it is interesting to know if $\omega_{\text{ess}}(A)<\omega(A)$.

As a consequence of Prop. 2.5, Corollary 2.6 and the above characterization of quasi-compact C_0-semigroup we have the following result.

3.3 Proposition. *Let $(T(t))_{t\geqq 0}$ and $(S(t))_{t\geqq 0}$ be positive C_0-semigroups on the Banach lattice E with generators A and B, respectively, such that*

$$o\leqq S(t)\leqq T(t) \quad \text{for all } t\geqq 0.$$

Then the following statements hold.

(i) *If $(T(t))_{t\geqq 0}$ is quasi-compact and $\omega(A)\leqq 0$, then $(S(t))_{t\geqq 0}$ is quasi-compact.*

(ii) *$\omega_{\text{ess}}(B)<\omega(A)$ if $\omega_{\text{ess}}(A)<\omega(A)$.*

One of the most important phenomenon in population dynamics is the property of asynchronous or balanced exponential growth. Following G.F. Webb [26], a C_0-semigroup $(T(t))_{t\geqq 0}$ in a Banach space E is said to have *synchronous*

exponential growth with intrinsic growth constant $\lambda \in \mathbf{R}$ provided there exists a nonzero finite rank operator P in E such that

$$\lim_{t \to \infty} e^{-\lambda t} T(t) = P$$

where the limit is in the operator norm topology. Implicitly, this property is also studied in [8, 11, 12] and [21]. As a consequence of the above proposition we have.

3.4 Corollary. *Let $(T(t))_{t \geqq 0}$ and $(S(t))_{t \geqq 0}$ be positive C_0-semigroups on the Banach lattice E with generators A and B, respectively, such that*

$$0 \leqq S(t) \leqq T(t) \quad \text{for all } t \geqq 0, \quad \text{and} \quad s(A) = s(B).$$

Then, $(S(t))_{t \geqq 0}$ has asynchronous exponential growth whenever $(T(t))_{t \geqq 0}$ has it.

Proof. By ([26] Prop. 2.3) $(T(t))_{t \geqq 0}$ has asynchronous exponential growth if and only if $\omega_{\text{ess}}(A) < s(A)$, $\sigma_b(A) = \{s(A)\}$, and $s(A)$ is a simple pole of the resolvent $R(\lambda, A)$. We show that the same holds for B. Since $\omega_{\text{ess}}(A) < s(A)$, $\omega(A) = s(A)$, we obtain $\omega_{\text{ess}}(A) < \omega(A)$. Then by Prop. 3.3, $\omega_{\text{ess}}(B) < \omega(A) = s(A) = s(B)$ and $\omega_{\text{ess}}(B) < \omega(B)$. Now, applying ([21], C-III, 3.16 or [8], Theor. 9.10) we have $\sigma_b(B) = \{s(B)\}$. On the other hand,

$$s(B) \in \sigma(B) \setminus \sigma_{\text{ess}}(B)$$

and consequently, $s(B)$ is a pole of the resolvent $R(\lambda, B)$. Moreover,

$$\|(\lambda - s(B))^k R(\lambda, B)\| \leqq \|(\lambda - s(A))^k R(\lambda, A)\|.$$

Therefore, $s(B)$ is a simple pole of $R(\lambda, B)$. Q.E.D.

The above result was obtained in ([3], Coroll. 2.4) with the additional assumption that one of the operators $S(t)$ is AM-compact.

The assumption $s(A) = s(B)$ in the above corollary is necessary as the following example shows.

3.5 *Example.* Consider $\mathbf{R}_+$ with the Lebesgue measure m and define $E_p := L^p(\mathbf{R}_+, dm) \cap L^1(\mathbf{R}_+, e^x\, dm)$, $1 < p < \infty$, which is a Banach lattice with respect to the norm

$$N_p(f) := \left(\int_0^\infty |f(x)|^p \, dm(x) \right)^{\frac{1}{p}} + \int_0^\infty |f(x)|\, e^x \, dm(x).$$

For $t \geqq 0$ define the operators $S(t)$ in E_p by $S(t) f(x) := f(x+t)$. Then, it is not difficult to show that $(S(t))_{t \geqq 0}$ is a positive C_0-semigroup on E_p with generator B satisfying $s(B) = -1$ and $\omega(B) = 0$. Thus, $\omega_{\text{ess}}(B) = \omega(B)$.

We define the operator $K: E_p \to E_p$ by

$$Kf := \left(6 \int_0^\infty f(x)\, dm(x) \right) f_0$$

with $f_0 := e^{-2x}$. Since K is a bounded operator $A := B + K$ generates a C_0-semigroup $(T(t))_{t \geqq 0}$ on E_p, and $0 \leqq S(t) \leqq T(t)$ for all $t \geqq 0$, since K is positive.

Moreover, since K is a compact operator, $\omega_{ess}(A)=\omega(B)=0$. On the other hand, since

$$Af_0=Bf_0+Kf_0=-2f_0+3f_0=f_0,$$

it follows $s(A)\geqq 1$. Hence, $\omega_{ess}(A)<\omega(A)$. Finally, the irreducibility of K implies that $(T(t))_{t\geqq 0}$ is a irreducible semigroup (see [21], C-III, 3.3 or [8], Theor. 9.22). Therefore, $(T(t))_{t\geqq 0}$ has asynchronous exponential growth (see [26], Remark 2.2). However, $(S(t))_{t\geqq 0}$ does not have this property since $\omega_{ess}(B)=\omega(B)$.

The semigroup $(S(t))_{t\geqq 0}$ in this example is a modification given in [8] of an example due to Greiner-Voigt-Wolff [10].

As a consequence of Corollaries 2.9 and 2.10 we obtain the following result about the monotonicity of the essential growth bounds.

3.6 Proposition. *Let $(T(t))_{t\geqq 0}$ and $(S(t))_{t\geqq 0}$ be positive C_0-semigroups on the Banach lattice E with generators A and B, respectively, such that*

$$0\leqq S(t)\leqq T(t) \quad \text{for all } t\geqq 0.$$

In each of the following cases we have

$$\omega_{ess}(B)\leqq\omega_{ess}(A).$$

(a) *E is Dedekind complete and there exists $t_0>0$ such that $\sigma_a(T(t_0))=\sigma(T(t_0))$.*

(b) *E is an* AL*-space or an* AM*-space.*

(c) *E is a Dedekind complete Banach lattice and there exists $t_0>0$ such that $T(t_0)$ is majorizing or cone absolutely summing.*

3.7 Corollary. *Let $(T(t))_{t\geqq 0}$ and $(S(t))_{t\geqq 0}$ be positive C_0-semigroups on the Banach-lattice E with generators A and B, respectively, such that*

$$0\leqq S(t)\leqq T(t) \quad \text{for all } t\geqq 0.$$

Assume that E is an AL*-space or an* AM*-space. Then, if $(S(t))_{t\geqq 0}$ is irreducible and $\omega_{ess}(A)<\omega(B)$, the semigroup $(S(t))_{t\geqq 0}$ has asynchronous exponential growth.*

Proof. Consider the rescaled C_0-semigroups $S_1(t):=e^{-t\omega(B)}S(t)$ with generator $B_1=B-\omega(B)I$, and $T_1(t):=e^{-t\omega(B)}T(t)$ with generator $A_1=A-\omega(B)I$. Then $0\leqq S_1(t)\leqq T_1(t)$ for all $t\geqq 0$ and $\omega_{ess}(A_1)=\omega_{ess}(A)-\omega(B)<0$. It follows from the above proposition that $\omega_{ess}(B_1)=\omega_{ess}(B)-\omega(B)<0$, so $\omega_{ess}(B)<\omega(B)$. Therefore, by the irreducibility of $(S(t))_{t\geqq 0}$, it follows that $(S(t))_{t\geqq 0}$ has asynchronous exponential growth (see e.g. [8], Theor. 9.10). Q.E.D.

For L^p spaces we have the following result.

3.8 Proposition. *Let $E=L^p(\mu)$ with $1<p\leqq\infty$ and let $(T(t))_{t\geqq 0}$ and $(S(t))_{t\geqq 0}$ be positive C_0-semigroups on E with generators A and B, respectively, such that*

$$0\leqq S(t)\leqq T(t) \quad \text{for all } t\geqq 0.$$

Assume that $(T(t))_{t\geqq 0}$ is quasi-compact. Then, if $(S(t))_{t\geqq 0}$ is uniformly bounded, it is quasi-compact.

Proof. By Prop. 3.2, there exists $t_0>0$ such that $T(t_0)$ is a quasi-compact operator. Now, since the semigroup $(S(t))_{t\geqq 0}$ is uniformly bounded, the means of

the operator $S(t_0)$ are uniformly bounded. Then, since $0 \leqq S(t_0) \leqq T(t_0)$, it follows from Prop. 2.11 that $S(t_0)$ is a quasi-compact operator, so the semigroup $(S(t))_{t \geqq 0}$ is quasi-compact. Q.E.D.

3.9 Corollary. *Let $E = L^p(\mu)$ with $1 < p \leqq \infty$ and let $(T(t))_{t \geqq 0}$ and $(S(t))_{t \geqq 0}$ be positive C_0-semigroups on E with generators A and B, respectively, such that*

$$0 \leqq S(t) \leqq T(t) \qquad \text{for all } t \geqq 0.$$

If $\omega_{ess}(A) < \omega(B)$ and $\|S(t)\| \leqq M e^{t\omega(B)}$ for some $M > 0$ and all $t \geqq 0$ then the semigroup $(S(t))_{t \geqq 0}$ has asynchronous exponential growth.

Proof. Considering the same rescaled semigroups as in the proof of Coroll. 3.7 and using the above proposition we obtain $\omega_{ess}(B) < \omega(B)$. On the other hand, $s(B_1) = 0$ and the semigroup $S_1(t) := e^{-t\omega(B)} S(t)$ is uniformly bounded. Then, since $(S_1(t))_{t \geqq 0}$ is quasi-compact, it follows from Remark C-IV, 2.2. (c) in [21], that $s(B_1)$ is a simple pole of the resolvent $R(\lambda, B_1)$. Hence $s(B)$ is a simple pole of $R(\lambda, B)$. Therefore, $(S(t))_{t \geqq 0}$ has asynchronous exponential growth. Q.E.D.

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