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Index and stability of harmonic Gauss maps

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By a well-known result of Ruh [R] an immersed surface $M \in \mathbb{E}^3$ has constant mean curvature h if and only if its Gauss map

$$\nu: M \to S^2$$

is a harmonic map. Harmonic maps being critica of the energy functional, it is natural to attempt to relate the second variation of energy to the geometry of M when $h =$ constant. We show

Theorem I *Let $M \subset \mathbb{E}^3$ be a compact surface of constant mean curvature and genus g. Then Index $(\nu) \geqq$ (largest integer $< 1 + g/2$).*

1 Preliminaries

The second variation of energy for harmonic maps was given by R.T. Smith [Sm], Here we recompute the second variation ina manner more suited to the present situation. The emphasis is on utilizing harmonic vector fields on M which may be considered as sections of the pull back bundle $\nu^* TS^2$ using the parallelism of $\mathbb{E}^3$. Let

$$X: M \to \mathbb{E}^3$$

be an immersion of a surface with constant mean curvature $h \neq 0$. Let

$$\nu: M \to S^2$$

be the Gauss map of M. The notation $\Omega \Subset M$ will indicate that Ω is a relatively compact subdomain of M with smooth boundary $\partial\Omega$. Let $\Gamma_c(\nu^* TS^2, \Omega)$ denote the space of smooth sections of the pull-back bundle $\nu^* TS^2$ with compact support in Ω.

Consider a variation of ν of the form

$$\nu_\varepsilon = \nu + \varepsilon\xi + O(\varepsilon^2), \qquad \xi \in \Gamma_c(\nu^* TS^2, \Omega).$$

The energy of $v|_\Omega$ is given by

$$E = {}^1/_2 \int_\Omega dv \wedge *dv$$

where $*$ is the Hodge operator of M. Letting "$\cdot$" denote differentiation w.r.t. ε, we have the first variation

$$D_\xi E = \int d\dot{v} \wedge *dv$$

and the second variation

$$\begin{aligned} D_\xi^2 E &= \int (d\dot{v} \wedge *d\dot{v} + d\ddot{v} \wedge *dv \\ &= \int (d\dot{v} \wedge *d\dot{v} - \langle \ddot{v}, d*dv \rangle) \\ &= \int \|d\xi\|^2 - \langle \ddot{v}, \Delta v \rangle * 1. \end{aligned}$$

Using the fact that v is harmonic and hence satisfies

$$\Delta v + \|dv\|^2 v = 0$$

we obtain

$$D_\xi^2 E = \int \|d\xi\|^2 + \langle \ddot{v}, v \rangle \|dv\|^2 * 1$$

Since $\langle \ddot{v}, v \rangle = -\|\dot{v}\|^2 = -\|\xi\|^2$ one has

$$D_\xi^2 = \int_\Omega \|d\xi\|^2 - \|dv\|^2 \|\xi\|^2 * 1.$$

Def. $v|_\Omega$ is stable iff

$$D_\xi^2 E > 0$$

for all non-trivial $\xi \in \Gamma_c(v^* TS^2, \Omega)$. We say v is unstable if $v|_\Omega$ is not stable for some $\Omega \Subset M$.

Definition. The index of $v|_\Omega$, $\mathrm{Index}_\Omega(v)$, is the dimension of the largest subspace of $\Gamma_c(v^* TS^2, \Omega)$ on which

$$\xi \mapsto D_\xi^2 E$$

is negative definite. When M in compact we set $\mathrm{Index}(v) = \mathrm{Index}_M(v)$.

Observe that there is a natural inclusion of $\Gamma_c(TM, \Omega)$, the space of section of TM with support in Ω in $\Gamma_c(v^* TS^2, \Omega)$. For $\xi \in \Gamma_c(TM, \Omega)$, the Weingarten equation gives

$$d\xi = \nabla \xi - \langle \xi, dv \rangle v \tag{1.1}$$

where ∇ is the connection of TM. The two terms in the right hand side of (1.1) are orthogonal so that

$$\|d\xi\|^2 = \|\nabla \xi\|^2 + \|dv(\xi)\|^2.$$

By a well-known formula ([S], pg. 88, Propos. 6)

$$\|dv(\xi)\|^2 = -2h\langle dv(\xi), \xi \rangle - \kappa \|\xi\|^2$$

where κ is the curvature of M. Therefore

$$D_\xi^2 E = \int_\Omega \|\nabla \xi\|^2 - 2h\langle dv(\xi), \xi\rangle + \kappa\|\xi\|^2 - 4h^2\|\xi\|^2 * 1 \tag{1.2}$$

since $\|dv\|^2 = 4h^2 - 2\kappa$. Finally, using integration by parts we express this as

$$D_\xi^2 E = -\int_\Omega \langle \xi, J(\xi)\rangle * 1 \tag{1.3}$$

where J is the Jacobi operator

$$J(\xi) \equiv \text{tr. } \nabla\nabla\xi - \kappa\xi + 2h\,dv(\xi) + 4h^2\xi. \tag{1.4}$$

Here tr. denotes the trace taken with respect to the induced metric. Note that the operator

$$\Box\xi \equiv \text{tr. } \nabla\nabla\xi - \kappa\xi \tag{1.5}$$

is the Laplacian acting on vector fields to that

$$J = \Box + 2h\,dv + 4h^2. \tag{1.6}$$

(pf. of Theor. I). Let $H \in T(M)$ denote the space of harmonic vector fields on M. Then H is a $2g$ dimensional vector space equipped with an anti-involution

$$\xi \to \xi^\perp := \tau(\xi)$$

where τ denotes the almost complex structure of M. Note that $\|\xi\| = \|\xi^\perp\|$ holds.

Note that J is a strongly elliptic, self-adjoint operator acting on $\Gamma T(M)$. Denote its eigenvalues by $\lambda_1 \leqq \lambda_2 \leqq \ldots$.

Let $Y_1, Y_2, \ldots, Y_{m-1}$ be arbitrary elements in $\Gamma T(M)$. For $g > 2m-2$, there exists a non-trivial $\xi \in H$ satisfying $2m-2$ linear conditions

$$\int\langle \xi, Y_j\rangle = 0 = \int\langle \xi^\perp, Y_j\rangle; \quad j = 1, \ldots, m-1.$$

Since

$$J(\xi) = 2h\,dv(\xi) + 4h^2\xi$$
$$J(\xi^\perp) = 2h\,dv(\xi^\perp) + 4h^2\xi^\perp,$$

one has

$$D_\xi^2 E = -\int 2h\langle dv(\xi), \xi\rangle + 4h^2\|\xi\|^2 * 1$$
$$D_{\xi^\perp}^2 E = -\int 2h\langle dv(\xi^\perp), \xi^\perp\rangle + 4h^2\|\xi^\perp\|^2 * 1$$

and hence one finds by adding the two previous lines and using the definition of h,

$$\begin{aligned} D_\xi^2 E + D_{\xi^\perp}^2 E &= -\int(-4h^2\|\xi\| + 8h^2\|\xi\|^2) * 1 \\ &= -\int 4h^2\|\xi\|^2 * 1 < 0. \end{aligned}$$

It follows that at least one of $D_\xi^2 E$ or $D_{\xi^\perp}^2 E$ is negative.

Since λ_m is characterized by

$$\lambda_m = \sup_{\{Y_1,\ldots,Y_{m-1}\}\subset\Gamma(TM)} \inf_{\eta\perp\{Y_1,\ldots,Y_{m-1}\}} -\frac{\int\langle\eta, J(\eta)\rangle}{\int\|\eta\|^2}$$

one finds that λ_m is negative and the theorem follows.

Remark. Let $M\subset\mathbb{E}^3$ be a complete surface of constant non-zero mean curvature. By multiplying ξ by a suitable cut-off function and using the method of Theorem I, we can show that the index of v is bounded below by the largest integer less than $1+\gamma/2$ where 2γ denotes the dimension of the space of L^2 harmonic vector fields on M. Details along with results for surfaces in the three sphere will be given elsewhere.

Remark. Recall that surfaces of constant mean curvature $\hbar$ in $\mathbb{E}^3$ are derived from a variational principle

$$\delta J=0$$

where

$$J=\text{area}+2h \quad \text{(enclosed 3-volume)}.$$

For a variation of the form

$$X+\varepsilon\psi\nu+O(\varepsilon^2), \qquad \psi\in C_c^\infty(M)$$

the second variation of J is given by

$$\delta^2 J=-\int\psi L\psi$$

where

$$L\psi=\Delta\psi+(4h^2-2\mathscr{K})\psi.$$

Following Barbosa and DoCarmo [B, DC] we say a domain $\Omega\Subset M$ is stable iff $\delta^2 J\geqq 0$ for all nontrivial $\psi\in C_c^\infty(M)$ with

$$\int\psi=0.$$

A natural question is whether there is any connection between the stability of Ω and the stability of $v|_\Omega$. In this direction note that if $\xi\in\Gamma_c(v^*TS^2)$ then Kato's inequality gives

$$\|\nabla\|\xi\|\|^2\leqq\|d\xi\|^2$$

and therefore

$$\int(\|\nabla\|\xi\|\|^2-(4h^2-2\mathscr{K})\|\xi\|^2)*1\leqq\int\|d\xi\|^2-\|dv\|^2\|\xi\|^2*1.$$

It follows that if $\delta^2 J\geqq 0$ for all $\psi\in C_c^\infty$ then $v|_\Omega$ is stable.

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