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On the extension of C.R. functions

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In this work we study the following question:
Let M be a $\mathscr{C}^\infty$ manifold in $\mathbb{C}^n$ and f a Cauchy-Riemann (C.R. for short) function on M, is it possible to write f as a sum on M of holomorphic functions in wedges of edge M?

Of course, stated in this way, this question asks for an other one: what is a C.R. function, or a C.R. form, on a general manifold M? Hence we must start with the following natural definition:

Definition. Let M be a closed set in $\mathbb{C}^n$ and f a (p, q) form on M; we say that f is C.R. on M if there is an extension F of f in a neighborhood of M in $\mathbb{C}^n$, such that: $\bar{\partial}F$ is flat on M.

In the case M is an analytic set, then a function, i.e. a $(0, 0)$ form, C.R. in this sense is just a holomorphic function [3]; in the case where M is a C.R. manifold i.e. the dimension of the complex tangent space to M is constant, then this definition is equivalent to the classical one as it was proved by Andreotti and Hill [4].

Hence this definition makes a simple link between somewhat different notions of holomorphy. The aim of this paper is also to give simple proofs of the extension property which work in this general frame.

The literature on the extension property for C.R. functions on C.R. manifold is vaste since H. Lewy [18], J.J. Kohn and H. Rossi [17], A. Andreotti and C.D. Hill [4] [5], G. Henkin [14] and more recent papers by J.P. Rosay [20]; M.S. Baouendi, C.H. Chang and F. Trèves [6]; M.S. Baouendi, L.P. Rothschild and F. Trèves [7]; M.S. Baouendi and L.P. Rothschild [8]; J.M. Trépreau [21]. (There is an extensive bibliography in the paper by G. Henkin [14] for the first period and in the nice survey by F. Forstneric [13] for now.)

The ideas here are coming from two sources: the work I have done on extension and division problems for holomorphic functions [1] [2] [3] on one side and the talks I had with Salah Baouendi, Linda Rothschild and François Trèves on classical C.R. functions on the other side and I am happy to thank them for these interesting talks.

As in [2] and [3], the aim is to solve some $\bar{\partial}$-equations in open sets of $\mathbb{C}^n$ with $\mathscr{C}^\infty$ bounds, in order to solve in fact $\bar{\partial}$-equations for currents. In [5],

the authors studied also the case of manifolds of codimension higher than one but their point of view is very different, the results are different and they worked only in the case of C.R. manifolds. In § 1 we prove:

Theorem 1.1: *Let M be a closed set in $\mathbb{C}^n$ such that $0 \in M$ and M cuts the space near 0 in two parts M^- and M^+; if f is a C.R. (p, q) form on M near 0, then there are two (p, q) forms, f^-, $\mathscr{C}^\infty$ on $\bar{M}^-$ near 0 and f^+, $\mathscr{C}^\infty$ on $\bar{M}^+$ near 0, such that: $\bar{\partial} f^- = 0$ on M^-, $\bar{\partial} f^+ = 0$ on M^+ and $f = f^- + f^+$ on M near 0.*

The proof is very simple and almost contained in [4], with a change of language: the current $f \wedge \bar{\partial}\chi$ is $\bar{\partial}$-closed in a ball near 0, and there is a distribution u such that, near 0, $\bar{\partial} u = f \wedge \bar{\partial}\chi$ where χ is the indicatrix of M^-; then the restriction u^- of u to M^- is $\mathscr{C}^\infty$ in $\bar{M}^-$ and $\bar{\partial}$-closed there; the restriction u^+ of u to M^+ is $\mathscr{C}^\infty$ in $\bar{M}^+$ and $\bar{\partial}$-closed there and we have $f = u^- - u^+$. In the case $q = 0$ then *any* solution u verifies $f = u^- - u^+$.

This proof relies only on the solution of the $\bar{\partial}$-equation in an open ball of $\mathbb{C}^n$. This case is a good illustration for the idea behind the more complicated codimension two case.

Much more precise results in the case of C.R. manifolds with extension in only *one* wedge were done by Trépreau [21] and Tumanov [22], using the method of discs.

On the other hand Trépreau proved [unpublished] that, even in the case of C.R. manifolds, there is not always extension, no matter the number of wedges you choose. So we must put conditions in order to get the extension property.

In § 2, we prove a general local theorem of extension for the codimension two case.

In § 3, we give some applications; let S_1 and S_2 be two real $\mathscr{C}^\infty$ hypersurfaces in $\mathbb{C}^n$, then we get first a local result

Theorem 3.2. *Let $M = S_1 \cap S_2$, where S_1 is strictly pseudo-convex near 0, and S_1 and S_2 are transversal; then if $n \geqq 3$, any C.R. function on M near 0 is a sum of boundary values of holomorphic functions in wedges of edge M.*

In the case $n = 2$, we have to deal with the H. Lewy's obstruction, hence we get the same result provided that the C.R. function f verifies an extra regularity condition (Theor. 3.3).

In special cases, when $M \subset \mathscr{C}^2$ has only one complex tangency of elliptic type, Kenig and Webster [15], and Moser and Webster [19] obtained a better result: they characterized the local enveloppe of holomorphy of M.

In § 4 and § 5, we study the compact case and we get

Theorem 4.1. *Let $M = \partial\Omega_1 \cap \partial\Omega_2$, with Ω_i bounded pseudo convex domains with smooth $\mathscr{C}^\infty$ boundaries and defined by global p.s.h. functions; suppose that $\partial\Omega_1$ and $\partial\Omega_2$ are transversal, then if $n \geqq 3$, any C.R. function f on M is a sum of holomorphic functions defined on wedges of edge M.*

Theorem 5.2. *If $n = 2$, then there is a measure $d\pi$ on M such that if: $\int f \varphi \, d\pi = 0$ for all φ holomorphic near $\bar{\Omega}_i$, $i = 1, 2$ then f is a sum of holomorphic functions defined on wedges of edge M.*

The condition in the case $n = 2$ is naturally coming from the necessary condition to solve the $\bar{\partial}_b$-equation on $\partial\Omega_i$.

Again in some special cases, when $M \subset \mathbb{C}^2$, Bedford and Gaveau [9] characterized the enveloppe of holomorphy of M.

§ 1 Codimension one

Suppose that the closed set M cuts the space in two parts M^+ and M^- and that $0 \in M$ then we have the following local result

Theorem 1.1. *If f is a C.R. (p, q) form on M near 0, then there are two (p, q) forms, f^+, $\mathscr{C}^\infty$ on $\bar{M}^+$ and f^-, $\mathscr{C}^\infty$ on $\bar{M}^-$, such that $\bar{\partial} f^+ = 0$ on M^+ and $\bar{\partial} f^- = 0$ on M^- and with $f = f^+ + f^-$ on M.*

Proof. The form

$$\omega := \chi \cdot \bar{\partial} F \quad \text{is a } \mathscr{C}^\infty(p, q+1) \quad \text{form near } 0 \text{ in } \mathscr{C}^n, \tag{1.1}$$

where χ is the characteristic function of the set M^-, because $\bar{\partial} F$ is flat on M. Moreover ω is $\bar{\partial}$-closed and, choosing a ball centered at 0 and contained in the set where ω is smooth, we can solve

$$\bar{\partial} v = \omega \quad \text{with } v \text{ in } \mathscr{C}^\infty \text{ in this ball.} \tag{1.2}$$

Let now

$$f^+ := v \quad \text{and} \quad f^- := F - v \tag{1.3}$$

then we have

$$\bar{\partial} f^+ = \bar{\partial} v = \omega = 0 \quad \text{in } \bar{M}^{-c}, \quad \text{hence in } M^+ \tag{1.4}$$

$$\bar{\partial} f^- = \bar{\partial} F - \bar{\partial} v = 0 \quad \text{in } M^- \tag{1.5}$$

and

$$F = f^- + f^+, \quad \text{hence,} \quad f = f^+ + f^- \quad \text{on } M. \tag{1.6}$$

Now the current $\omega' = F \wedge \bar{\partial} \chi$ verifies $\bar{\partial} \omega' = \bar{\partial} F \wedge \bar{\partial} \chi = 0$ because $\bar{\partial} F$ is flat on M and the support of $\bar{\partial} \chi$ is contained in M; moreover we have that

$$u = F\chi - v \quad \text{verifies } \bar{\partial} u = \omega' \quad \text{by (1.2)} \tag{1.7}$$

and

$$u^+ = -v = -f^+, \quad u^- = F - v = f^- \quad \text{by (1.3),} \tag{1.8}$$

where u^- is the restriction of u to M^- and u^+ is the restriction of u to $\bar{M}^{-c}$; hence we get

$$f = u^- - u^+ \quad \text{on } M. \tag{1.9}$$

Moreover, if $q = 0$ the difference between two solutions of

$$\bar{\partial} u = \omega' \tag{1.10}$$

is a $\bar{\partial}$-closed distribution, hence a holomorphic function near 0, hence $\mathscr{C}^\infty$ through M. So we get in that case that *any* solution u of (1.10) verifies (1.9).

§ 2-Codimension two

Let Ω be an open set in $\mathbb{C}^n$; we say that Ω has the E_q *property at* 0 if $0 \in \partial\Omega$ and it is possible to solve locally the $\bar{\partial}$-equation with $\mathscr{C}^\infty$ bounds up to $\partial\Omega$ for (p, q) forms; precisely:

let $\mathscr{U}$ be a neighbourhood of 0 in $\mathscr{C}^n$ and ω a (p, q) form in $\mathscr{C}^\infty(\mathscr{U} \cap \bar{\Omega})$, $\bar{\partial}$-closed, then there is a neighbourhood $\mathscr{V}$ of 0 in $\mathbb{C}^n$ and a $(p, q-1)$ form η in $\mathscr{C}^\infty(\mathscr{V} \cap \bar{\Omega})$ such that $\bar{\partial}\eta = \omega$ in $\mathscr{V} \cap \Omega$.

Examples:

Ω a $\mathscr{C}^\infty$ smoothly bounded pseudo-convex open set in $\mathbb{C}^n$, the existence of admissible neighbourhoods [1] and the results of J.J. Kohn [16] prove the property E_q for $1 \leqq q \leqq n$;

Ω with the property $Z(q)$, the Levi form of $\partial\Omega$ at 0 has at least $q+1$ negative eigenvalues or at least $n-q$ positive ones [4];

Ω a convex open set, proved in [12].

Now we are in position to prove the following local result

Theorem 2.1. *Let M be a closed set in $\mathbb{C}^n$ such that $0 \in M$ and there are open sets Ω_i, Ω_i', $i = 1, 2$ with:*

Ω_i and Ω_i' verify the property E_{q+1}, $i = 1, 2$ at 0;

$M = \partial\Omega_1 \cap \partial\Omega_2 = \partial\Omega_1' \cap \partial\Omega_2'$ near 0;

$M = \bar{\Omega}_1 \cap \bar{\Omega}_2'$ near 0;

$\partial\Omega_1$ and $\partial\Omega_2$ are transversal, $\partial\Omega_i$ and $\partial\Omega_i'$ are regularly situated. Then any C.R. form $\mathscr{C}^\infty$ on M near 0 is a sum of $\bar{\partial}$-closed forms defined in wedges of edge M.

In fact the wedges are precisely the sets: $\Omega_1 \cap \Omega_2$, $\Omega_1 \cap \Omega_2'$, $\Omega_1' \cap \Omega_2$ and $\Omega_1' \cap \Omega_2'$.

Let γ_1 a function such that

$$\gamma_1 = 1 \quad \text{in } \Omega_1; \quad \gamma_1 = 0 \quad \text{in } \Omega_1' \tag{2.1}$$

moreover γ_1 can be chosen such that $\gamma_1 \in \mathscr{C}^\infty(M^c)$ because $\partial\Omega_1$ and $\partial\Omega_1'$ are regularly situated.

Similarly, let γ_2 a function such that

$$\gamma_2 = 1 \quad \text{in } \Omega_2; \quad \gamma_2 = 0 \quad \text{in } \Omega_2' \quad \text{and} \quad \gamma_2 \in \mathscr{C}^\infty(M^c). \tag{2.2}$$

Now let f a C.R. (p, q) form on M and F a $\mathscr{C}^\infty$-extension of f in $\mathscr{C}^n$ which is $\bar{\partial}$-flat on M.

Let

$$\omega := F \wedge \bar{\partial}\gamma_1 \wedge \bar{\partial}\gamma_2, \tag{2.3}$$

this $(p, q+2)$ current has M as support and we have

$$\bar{\partial}\omega = \bar{\partial}F \wedge \bar{\partial}\gamma_1 \wedge \bar{\partial}\gamma_2 = 0 \tag{2.4}$$

because $\bar{\partial}F$ is flat on M and $\bar{\Omega}_i$ and $\bar{\Omega}_i'$ are regularly situated.

Now $\bar{\partial}F\cdot\gamma_1\wedge\bar{\partial}\gamma_2$ is $\bar{\partial}$-closed for the same reason as above and this form is $\mathscr{C}^\infty$ in a full neighbourhood of 0. Hence we can find in a ball $\mathbb{B}$ centered at 0 a $\mathscr{C}^\infty(p, q+1)$ form X such that

$$\bar{\partial}X=-\bar{\partial}F\cdot\gamma_1\wedge\bar{\partial}\gamma_2. \tag{2.5}$$

Let us put

$$\mu_0:=F\gamma_1\wedge\bar{\partial}\gamma_2+X \tag{2.6}$$

then we have $\bar{\partial}\mu_0=F\wedge\bar{\partial}\gamma_1\wedge\bar{\partial}\gamma_2$.

We can solve the equation $\bar{\partial}v=X$ where $\bar{\partial}X=0$, hence in Ω_2 and in Ω_2', so we get

$$v_2\in\mathscr{C}^\infty(\bar{\Omega}_2),\quad \text{with } \bar{\partial}v_2=X \quad \text{in } \Omega_2 \tag{2.7}$$

and

$$v_2'\in\mathscr{C}^\infty(\bar{\Omega}_2'),\quad \text{with } \bar{\partial}v_2'=X \quad \text{in } \Omega_2'. \tag{2.8}$$

Now let τ_2 a smooth form such that $\tau_2=v_2$ in Ω_2, $\tau_2=v_2'$ in Ω_2'; we can choose τ_2 to be in $\mathscr{C}^\infty(M^c)\cap L^\infty(\mathbb{B})$. Then clearly

$$\mu=\mu_0-\bar{\partial}\tau_2=F\wedge\gamma_1\cdot\bar{\partial}\gamma_2+X-\bar{\partial}\tau_2 \tag{2.9}$$

still verifies $\bar{\partial}\mu=\omega$, and the support of μ is contained in $(\Omega_1^c\cap\Omega_1'^c)\cup(\Omega_2^c\cap\Omega_2'^c)$.

Let us see what happen in Ω_1; we have that $\gamma_1=1$ there and

$$\mu=F\wedge\bar{\partial}\gamma_2+X-\bar{\partial}\tau_2; \tag{2.10}$$

this can be rewritten in Ω_1

$$\mu=\bar{\partial}(F\cdot\gamma_2)+X-\gamma_2\cdot\bar{\partial}F-\bar{\partial}\tau_2 \tag{2.11}$$

but the form $\gamma_2\cdot\bar{\partial}F$ is smooth on $\bar{\Omega}_1$ because $\bar{\partial}F$ is flat on M, so we get that $X-\gamma_2\cdot\bar{\partial}F$ is a $\mathscr{C}^\infty(\bar{\Omega}_1)$ $\bar{\partial}$-closed smooth $(p, q+1)$ form and thus there is a (p, q) form α_1 such that

$$\bar{\partial}\alpha_1=X-\gamma_2\cdot\bar{\partial}F \quad \text{and} \quad \alpha_1\in\mathscr{C}^\infty(\bar{\Omega}_1), \tag{2.12}$$

hence the form $h_1:=F\cdot\gamma_2-\tau_2+\alpha_1$ verifies

$$\bar{\partial}h_1=\mu \quad \text{on } \Omega_1. \tag{2.13}$$

Similarly in Ω_1' we have $\gamma_1=0$ hence $\mu=X-\bar{\partial}\tau_2$; still X is in $\mathscr{C}^\infty(\bar{\Omega}_1')$ hence there is a (p, q) form α_1' in $\mathscr{C}^\infty(\bar{\Omega}_1')$ such that

$$\bar{\partial}\alpha_1'=X, \quad \text{in } \Omega_1', \tag{2.14}$$

hence the form $h_2:=-\tau_2+\alpha_1'$ verifies

$$\bar{\partial}h_2=\mu \quad \text{on } \Omega_1'. \tag{2.15}$$

But recalling that μ is supported on $(\Omega_1^c \cap \Omega_1'^c) \cup (\Omega_2^c \cap \Omega_2'^c)$ we get that

(2.16) $h_1 = F \cdot \gamma_2 - \tau_2 + \alpha_1$ is $\bar{\partial}$-closed in $\Omega_1 \cap \Omega_2$ and in $\Omega_1 \cap \Omega_2'$.

So we get that

(2.17) $h_1^+ := F - v_2 + \alpha_1$ is $\bar{\partial}$-closed in Ω_1 in $\Omega_1 \cap \Omega_2$,

(2.18) $h_1^- := -v_2' + \alpha_1$ is $\bar{\partial}$-closed in $\Omega_1 \cap \Omega_2'$.

Similarly

(2.19) $h_2^+ := -v_2 + \alpha_1'$ is $\bar{\partial}$-closed in $\Omega_2 \cap \Omega_1'$,

(2.20) $h_2^- := -v_2' + \alpha_2'$ is $\bar{\partial}$-closed in $\Omega_2' \cap \Omega_1'$.

Because all the forms involved are $\mathscr{C}^\infty$ up to the boundary of their domain, we get that

(2.21) $F = f = -h_2^+ + h_2^- + h_1^+ - h_1^-$, on M.

Hence the theorem is proved.

§ 3 Applications

In all the applications we shall have to make the hypothesis that our domains will be defined by a p.s.h. function in a neighbourhood of their closure.

In order to use the § 2, we need the following lemma

Lemma 3.1. *Let $M = S_1 \cap S_2$, where S_2 is strictly pseudo-convex and S_1 and S_2 are transversal, then there are two pairs of domains Ω_i and Ω_i', $i = 1, 2$, such that:*
a) *Ω_i and Ω_i' are smoothly bounded and either pseudo-convex or strictly pseudo-concave,*
b) *$\bar{\Omega}_i \cap \bar{\Omega}_i' = M$, $i = 1, 2$,*
c) *$\bar{\Omega}_i$ and $\bar{\Omega}_i'$ are regularly situated, $i = 1, 2$.*

Proof of the lemma. a) S_1 and S_2 are strictly pseudo-convex; then we take $\Omega_1 = \{\rho_1 > 0\}$ then Ω_1 is strictly pseudo-concave. Now let $\rho_1' = \rho_1 + \varepsilon \rho_2^2$, where ρ_i is the defining function of S_i; a simple computation then shows that if ε is small enough then $\Omega_1' := \{\rho_1' < 0\}$ is still s.p.c. We do the same to get Ω_2 and Ω_2', just reversing the role of ρ_1 and ρ_2.

The other properties of the lemma are easy to see.

b) S_1 s.p.c. then let $r_2 := t\rho_1 + (1-t)\rho_2$, again if $(1-t)$ is small enough, S_2' $:= \{r_2 = 0\}$ is s.p.c. and because $M = S_1 \cap S_2'$ we now are in the case a).

a) Local case and $n \geqq 3$

Let S_i be smooth real hypersurfaces, then we have the following

Theorem 3.2. *Let $M = S_1 \cap S_2$, where S_1 is strictly pseudo-convex and S_1 and S_2 are transversal. Then any C.R. function on M is a sum of boundary values of holomorphic functions defined in wedges of edge M.*

Proof. We apply the lemma above to get the sets Ω_i and Ω_i'; if Ω_i (or Ω_i') is pseudo-convex then they verify the E_1 property; if Ω_i is strictly pseudo-concave, because $n \geqq 3$, it still verifies the E_1 property. Hence in any case we are in position to apply the results of § 2.

b) Local case and $n=2$

In this case we have to distinguish between the two possibilities:

S_1 and S_2 are complex transversal,

then $M = S_1 \cap S_2$ is totally real, hence the result is well known.

S_1 and S_2 are not complex transversal,

then suppose that S_i verify the same hypothesis as in the theorem above; using the lemma again we introduce the sets Ω_i and Ω_i'. So suppose that Ω_2 is strictly pseudo-concave, i.e. S_2 is strictly pseudo-convex; to give a theorem in that case we must return to the proof.

Let f be a C.R. function on $M = S_2 \cap S_2$. Using the lemma 3.1 we can introduce the sets Ω_i' and γ_1 and γ_2 as before and let F be the extension of f in a ball $\mathbb{B}$ centered at 0, $\bar{\partial}F$ flat on M. Again we can solve the problem

$$\bar{\partial}X = \bar{\partial}F \cdot \gamma_1 \wedge \bar{\partial}\gamma_2$$

with $X \in \mathscr{C}^{\infty}_{(0,1)}(\mathbb{B})$ and the question now is to solve

$$\bar{\partial}v = X \quad \text{in } \Omega_2. \tag{3.7}$$

This time we are in a strictly pseudo-concave set in $\mathbb{C}^2$ and, in order to solve this equation we must overcome the Lewy's obstruction

it is necessary and sufficient that X verifies a condition given by G. Henkin [14, § 5]: there is a kernel K' acting on X such that the condition is

$$K'(X) \quad \text{is real analytic near } 0; \tag{3.8}$$

but X is a solution of a $\bar{\partial}$-equation in a ball, hence using $\bar{\partial}$-solving kernels, K'', [11], we have

$$X = K''(-\gamma_1 \bar{\partial}F \wedge \bar{\partial}\gamma_2) \tag{3.9}$$

so, composing K' and K'', we get the existence of a kernel K acting on $\bar{\partial}F$ such that the necessary and sufficient condition for solving (3.7) locally can be written

$$K(\bar{\partial}F) \quad \text{is real-analytic near } 0; \tag{3.10}$$

Once this is done, the rest of the proof is exactly the same.
Hence in that case we have

Theorem 3.3. *Let $M = S_1 \cap S_2$, $0 \in M$, where S_1 is strictly pseudo-convex and S_1 and S_2 are transversal, in $\mathbb{C}^2$. Let f be a C.R. function on M near 0 and F an extension of f in $\mathbb{C}^2$ near 0 such that $\bar{\partial}F$ is flat on M. Suppose moreover that* (3.10) *holds then f is a sum of boundary values of holomorphic functions defined in wedges of edge M.*

Remark 3.4. If the C.R. function f on M is holomorphic in a thin tubular neighbourhood of M, then the condition (3.10) is fullfilled, meaning that independantly of the thinness of this neighbourhood, f is the sum on M of boundary values of holomorphic functions defined on fixed wedges.

Remark 3.5. The case of a surface $M \subset \mathbb{C}^2$ of real dimension two with an isolated complex tangency at the origin studied by Kenig and Webster [15] and by Moser and Webster [19] is in the field of application of theorem 3.1, but in the case when the origin is an elliptic point, they have a better result: they characterize the local hull of holomorphy of M near 0 as the union of a one parameter family of discs.

§ 4 Compact case and $n \geqq 3$

We shall need the following

Lemma 4.1. *Let $M = S_1 \cap S_2$, with S_i defined by global p.s.h. functions, and S_1 and S_2 transversal. Then there are 2 pairs of pseudo-convex domains Ω_i and Ω_i' such that $M = \partial\Omega_i \cap \partial\Omega_i'$, $i = 1, 2$, $\partial\Omega_i$ and $\partial\Omega_i'$ are regularly situated, and $\partial\Omega_1$ and $\partial\Omega_2$ are still transversal.*

Let us suppose, without loss of generality, that $\rho_i \geqq -1$, where ρ_i is the defining function for S_i; let $\varepsilon > 0$ be such that ρ_i is p.s.h. in $\{\rho_i < \varepsilon\}$ and let $t \in]0, 1[$ fixed small and finally let $A = 2(1-t)/t$.

Now let us take a function χ, convex and increasing on $\mathbb{R}$ such that

$$\chi \equiv 0 \quad \text{on } [-\infty, \varepsilon/3] \quad \text{and} \quad \chi(2\varepsilon/3) = 1/2.$$

Now let $\rho_i' := \rho_i + A\chi(\rho_i)$, $i = 1/2$ and $\rho_t := (1-t)\rho_1' + t\rho_2'$ then we have

Claim. ρ_t is a p.s.h. function near the closure of $\Omega_t := \{\rho_t < 0\}$.

The point is to see that ρ_i' are p.s.h. functions near Ω_t; we have $\rho_t < 0 \Rightarrow (1-t)\rho_1' + t\rho_2' < 0$, hence

$$\rho_1' < t/(1-t) \Rightarrow \rho_1 + A\chi(\rho_1) < t/(1-t) \Rightarrow \chi(\rho_1) < t^2/2(1-t)^2 \Rightarrow \rho_1 < 2\varepsilon/3;$$

ρ_1 is a p.s.h. function, hence ρ_1' too because χ is convex increasing.

For ρ_2 we have $\rho_2' < (1-t)/t \Rightarrow \chi(\rho_2) < 1/2 \Rightarrow \rho_2 < 2\varepsilon/3$, hence the same conclusion. Let

$$(*) \qquad \Omega_t := \{\rho_t < 0\}$$

Then by the claim Ω_t is pseudo-convex.

Now let (with a new ε!)

$$\Omega_t' := \{\rho_t' := \rho_t + \varepsilon\rho_2^2 < 0\}$$

We can calculate the Levi form for ρ_t'

$$\partial\bar{\partial}\rho_t' = (1-t)\,\partial\bar{\partial}\rho_t + (t + 2\varepsilon\rho_2')\,\partial\bar{\partial}\rho_2' + 2\varepsilon\,\partial\rho_2' \wedge \bar{\partial}\rho_2'.$$

Now we can choose ε so small that $(t + 2\varepsilon\rho_2') \geqq 0$, hence ρ_t' is a p.s.h. function and hence Ω_t' is still p.c. with smooth boundary.

We have that

$$\rho_t=\rho'_t=0\Rightarrow\rho'_2=0 \quad \text{and hence } \rho'_1=0,$$
$$\text{but } \rho'_i=0\Rightarrow\rho_i=0 \quad \text{hence } \Omega_t\cap\Omega'_t=M.$$

Moreover, the order of contact is clearly two between Ω_t and Ω'_t, hence they are regularly situated.

This pair Ω_t and Ω_t' may play the role of the pair Ω_1 and Ω_1' of the lemma, and in order to get the pair Ω_2 and Ω_2', we have just to play the same game, exchanging the role of 1 and 2. Hence the lemma.

Theorem 4.1. *Let $M=\partial\Omega_1\cap\partial\Omega_2$, with Ω_i bounded pseudo-convex domains with smooth $\mathscr{C}^\infty$ boundaries and defined by global p.s.h. functions; suppose that $\partial\Omega_1$ and $\partial\Omega_2$ are transversal then any C.R. function on M is a sum of holomorphic functions defined in wedges of edge M.*

Proof. The proof is very analogous to the one of § 2 but we have to use an extra result by J.P. Rosay [20] and there is a change of notations so we prefer to do it completely.

Introduce the pairs Ω_i, Ω_i' given by lemma 4.1: then Ω_i and Ω_i' are pseudo-convex; moreover we have

$$\bar{\Omega}'_i\setminus M\subset\Omega_i, \qquad i=1,2. \tag{4.1}$$

Let γ_1 a function such that

$$\gamma_1=0 \text{ in } \Omega_1^c; \quad \gamma_1=1 \text{ in } \Omega'_1; \quad \gamma_1\in\mathscr{C}^\infty(\mathbb{C}^n\setminus M). \tag{4.2}$$

Let γ_2 the analogous function for Ω_2

$$\gamma_2=0 \text{ in } \Omega_2^c; \quad \gamma_2=1 \text{ in } \Omega'_2; \quad \gamma_2\in\mathscr{C}^\infty(\mathbb{C}^n\setminus M).$$

Let F the extension of f such that $\bar\partial F$ is flat on M and let

$$\omega:=F\bar\partial\gamma_1\wedge\bar\partial\gamma_2 \tag{4.3}$$

it is a $\bar\partial$-closed current as before.

Now we have that $\bar\partial F\gamma_1\wedge\bar\partial\gamma_2$ is a $\mathscr{C}^\infty(\mathbb{C}^n)$ $\bar\partial$-closed form; moreover we can manage to have F with compact support, so it is possible to solve

$$\bar\partial G=\bar\partial F\gamma_1\wedge\bar\partial\gamma_2 \quad \text{with } G \text{ in } \mathscr{C}^\infty(\mathbb{C}^n). \tag{4.4}$$

So we get that $\mu_0:=F\gamma_1\cdot\bar\partial\gamma_2-G$ verifies $\bar\partial\mu_0=\omega$.

Now we want to diminish the support of G. Clearly we have that

$$G\in\mathscr{C}^\infty_{(0,1)}(\bar{\Omega}'_2), \quad \bar\partial G\equiv 0 \text{ in } \Omega'_2. \tag{4.5}$$

Hence, using J. Kohn's $\mathscr{C}^\infty$ estimates for $\bar\partial$, we can find

$$v'_2\in\mathscr{C}^\infty(\bar{\Omega}'_2), \quad \text{such that } \bar\partial v'_2=G. \tag{4.6}$$

G is still in $\mathscr{C}^\infty(\Omega_2^c)$ and $\bar\partial$-closed, so on the boundary $\partial\Omega_2$, G induces a $\mathscr{C}^\infty$, $\bar\partial_b$-closed form; hence using J.P. Rosay's result [20] with $n\geqq 3$, we can extend

$$(*) \qquad G \text{ in } \bar\Omega_2 \quad \text{as a } \mathscr{C}^\infty\bar\partial\text{-closed } (0,1) \text{ form } G_2.$$

Now let us take a big ball $\mathbb{B}$ which contains $\bar{\Omega}_2$ and $\bar{\Omega}_1$, and call again G_2 the (0, 1) form G extended by G_2; then G_2 is in $\mathscr{C}^\infty(\mathbb{B})$ and $\bar{\partial}$-closed there.

We can solve the $\bar{\partial}$-equation $\bar{\partial}v_2 = G_2$ with v_2 in $\mathscr{C}^\infty(\mathbb{B})$. In particular we have

$$\bar{\partial}v_2 = G \quad \text{in } \Omega_2^c \cap \mathbb{B}. \tag{4.7}$$

Let us still denote by v_2' any $\mathscr{C}^\infty$ continuation of v_2' to $\mathbb{B}$ and define

$$v := \gamma_2 v_2' + (1-\gamma_2) v_2. \tag{4.8}$$

We are now in a position to consider the following current

$$\mu := F\gamma_1 \cdot \bar{\partial}\gamma_2 - G + \bar{\partial}v. \tag{4.9}$$

Clearly this current still verifies $\bar{\partial}\mu = \omega$ but now μ is supported by $(\bar{\Omega}_1 \cap \Omega_1'^c) \cup (\bar{\Omega}_2 \cap \Omega_2'^c)$.

Let us see what happens in Ω_1'; there $\gamma_1 = 1$ and $\mu = F \cdot \bar{\partial}v$; this can be rewritten in Ω_1'

$$\mu = \bar{\partial}(F \cdot \gamma_2) - \gamma_2\, \bar{\partial}F - G + \bar{\partial}v, \tag{4.10}$$

but the form $\gamma_2 \cdot \bar{\partial}F$ is smooth in $\bar{\Omega}_1'$ because $\bar{\partial}F$ is flat on M, so we get that $G + \gamma_2 \cdot \bar{\partial}F$ is a $\mathscr{C}^\infty(\bar{\Omega}_1')$ $\bar{\partial}$-closed smooth (0, 1) form and thus there is a function α_1' such that

$$\bar{\partial}\alpha_1' = G + \gamma_2 \cdot \bar{\partial}F, \quad \text{and} \quad \alpha_1' \in \mathscr{C}^\infty(\bar{\Omega}_1'); \tag{4.11}$$

hence the function $h_1' := F \cdot \gamma_2 + v - \alpha_1'$ verifies

$$\bar{\partial}h_1' = \mu \quad \text{on } \Omega_1'. \tag{4.12}$$

Similarly in Ω_1^c we have $\gamma = 0$ hence $\mu = -G + \bar{\partial}v$; still G is in $\mathscr{C}^\infty(\bar{\Omega}_1^c)$ hence using again Rosay's result, there is a function α_1 in $\mathscr{C}^\infty(\bar{\Omega}_1^c)$ such that

$$\bar{\partial}\alpha_1 = G, \quad \text{in } \Omega_1^c; \tag{**}$$

hence the function $h_1 := v - \alpha_1$ verifies

$$\bar{\partial}h_1 = \mu \quad \text{in } \Omega_1^c. \tag{4.13}$$

But recalling that μ is supported on $(\Omega_1'^c \cap \bar{\Omega}_1) \cup (\bar{\Omega}_2 \cap \Omega_2'^c)$ we get that

$$h_1 = F \cdot \gamma_2 + v - \alpha_1 \quad \text{is holomorphic in } \Omega_2' \cap \bar{\Omega}_1^c \quad \text{and in } \bar{\Omega}_1^c \cap \bar{\Omega}_2^c. \tag{4.14}$$

So

$$h_1^+ := F + v_2' - \alpha_1 \quad \text{is holomorphic in } \Omega_2' \cap \Omega_1^c; \tag{4.15}$$

$$h_1^- := v_2 - \alpha_1 \quad \text{is holomorphic in } \bar{\Omega}_1^c \cap \bar{\Omega}_2^c. \tag{4.16}$$

Similarly

$$h_2^+ := v_2 - \alpha_1' \quad \text{is holomorphic in } \Omega_2^c \cap \Omega_1' \tag{4.17}$$

$$h_2^- := v_2' - \alpha_1' \quad \text{is holomorphic in } \Omega_2' \cap \Omega_1'. \tag{4.18}$$

Because all the functions involved are $\mathscr{C}^\infty$ up to the boundaries of their domain we get

$$F=f=h_2^+-h_2^-+h_1^+-h_1^-, \quad \text{on } M, \tag{4.19}$$

and the theorem.

§ 5 Compact case and $n=2$

The only problem is how to obtain (*) and (**) in the previous proof, because we are now in the critical case with $n=2$. In order to study what happens here we have to introduce some notations

$$\Omega_i \quad \text{is defined by } \rho_i<0, \quad i=1,2, \tag{5.1}$$

$$\Omega_i' \quad \text{is defined by } \rho_i+\varepsilon_i^2<0, \quad i=1,2, \tag{5.2}$$

using Lemma 4.1; ε_i being a smooth function. Ω_i are bounded pseudo-convex domains with smooth $\mathscr{C}^\infty$ boundaries and $\partial\Omega_1$ and $\partial\Omega_2$ are transversal.

Let f be a C.R. function on M and F be its $\mathscr{C}^\infty$ $\bar\partial$-flat extension. We have to be precise in defining the functions γ_i

$$\gamma_i:=\chi(-\rho_i/\varepsilon_i^2), \quad i=1,2 \tag{5.3}$$

where χ is a function such that

$$\chi(t)=0 \quad \text{if } t\leqq 0; \quad \chi(t)=1 \quad \text{if } t\geqq 1; \quad \chi\in\mathscr{C}^\infty(\mathbb{R}). \tag{5.4}$$

Let us start in the same way as above up to the point where we have the (0, 1) form G such that

$$\bar\partial G=\bar\partial F\gamma_1\wedge\bar\partial\gamma_2 \quad \text{with } G \text{ in } \mathscr{C}^\infty(\mathbb{C}^n). \tag{5.5}$$

Of course, because Ω_2' is pseudo-convex, we can solve

$$\bar\partial v_2'=G \quad \text{in } \Omega_2' \quad \text{with } v_2'\in\mathscr{C}^\infty(\bar\Omega_2'). \tag{5.6}$$

We want to solve the same equation in Ω_2^c and now we have the global condition [20]

$$\int_{\partial\Omega_2}\varphi\cdot G\wedge\beta=0, \quad \text{for all } \varphi \text{ holomorphic near } \bar\Omega_2, \tag{*'}$$

where $\beta:=dz_1\wedge dz_2$.

The idea is to show that this condition is in fact an integral condition on f over M.

Proposition 5.1. There is a measure $d\pi$ on M such that the condition (*') is equivalent to the condition (A):

$$\int_M f\cdot\varphi\, d\pi=0, \quad \text{for all } \varphi \text{ holomorphic near } \bar\Omega_2. \tag{A}$$

Proof. By Stokes' theorem we have

$$\int_{\partial\Omega_2} \varphi \cdot G \wedge \beta = \int_{\Omega_2} \varphi \cdot \bar{\partial} G \wedge \beta. \tag{5.7}$$

We can replace $\bar{\partial}G$ by its value: $\gamma_1 \cdot \bar{\partial}F \wedge \bar{\partial}\gamma_2$ hence we have to estimate

$$I := \int_{\Omega_2} \varphi \gamma_1 \, \bar{\partial}F \wedge \bar{\partial}\gamma_2 \wedge \beta. \tag{5.8}$$

Let us introduce smooth approximations for the γ_i's

$$\gamma_1^\varepsilon := \chi\left[\frac{-\rho_1}{\varepsilon_1^2+\varepsilon^2}\right]; \quad \gamma_2^\delta := \chi\left[\frac{-\rho_2}{\varepsilon_2^2+\delta^2}\right]. \tag{5.9}$$

We can choose F with compact support in $\mathbb{C}^2$, hence by Stokes' formula applied in a domain Ω containing the support of $\bar{\partial}\gamma_2^\delta$ we get

$$\int_{\mathbb{C}^2} \varphi \gamma_1^\varepsilon \, \bar{\partial}F \wedge \bar{\partial}\gamma_2^\delta \wedge \beta + \int_{\mathbb{C}^2} \varphi F \bar{\partial}\gamma_1^\varepsilon \wedge \bar{\partial}\gamma_2^\delta \wedge \beta = \int_{\partial\Omega} \varphi F \gamma_1^\varepsilon \, \bar{\partial}\gamma_2^\delta \wedge \beta = 0, \tag{5.10}$$

provided that φ is holomorphic near $\bar{\Omega}$.

Because of the transversality of $\partial\Omega_1$ and $\partial\Omega_2$, when $\varepsilon \to 0$, $\delta \to 0$, we have that

$$\bar{\partial}\gamma_1^\varepsilon \wedge \bar{\partial}\gamma_2^\delta \wedge \beta \to \bar{\partial}\gamma_1 \wedge \bar{\partial}\gamma_2 \wedge \beta, \tag{5.11}$$

which is a measure supported by M. Moreover, if φ is holomorphic near $\bar{\Omega}_2$ for ε and δ small enough, φ will be holomorphic near the support of $\bar{\partial}\gamma_2$ hence we have

(5.12)

$$I = \int_{\Omega_2} \varphi \gamma_1 \, \bar{\partial}F \wedge \bar{\partial}\gamma_2 \wedge \beta = -\int_M \varphi f \bar{\partial}\gamma_1 \wedge \bar{\partial}\gamma_2 \wedge \beta,$$

$$\forall \varphi \quad \text{holomorphic near } \bar{\Omega}_2$$

Hence the Proposition 5.1 with $d\pi := \bar{\partial}\gamma_1 \wedge \bar{\partial}\gamma_2 \wedge \beta$.

Of course, doing exactly the same thing for (**'), if

$$\int_M \varphi f \, d\pi = 0, \tag{5.18}$$

for all φ holomorphic near $\bar{\Omega}_1$, then we can solve

$$\bar{\partial}\alpha_1 = G, \quad \text{in } \Omega_1^c; \tag{**'}$$

the end of the proof is the same as in § 4, hence we proved the following

Theorem 5.2. *Let $M=\partial\Omega_1\cap\partial\Omega_2$, with Ω_i bounded pseudo-convex domains with $\mathscr{C}^\infty$ smooth boundaries; suppose that $\partial\Omega_1$ and $\partial\Omega_2$ are transversal then any C.R. function f on M such that:*

$$\int_M \varphi f d\pi = 0 \quad \forall \varphi \quad \text{holomorphic near } \bar{\Omega}_i, \quad i=1,2$$

is a sum of holomorphic functions defined in wedges of edge M.

Remark 5.3. In special cases, Bedford and Gaveau [9] obtained a better result: they characterized the enveloppe of holomorphy of M.

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References

1. Amar, E.: Cohomologie Complexe at Applications. J. London Math. Soc. 29, 127–140 (1984)
2. Amar, E.: Extension de fonctions holomorphes et courants. Bull. Soc. Math., 2ième série **107**, 25–48 (1983)
3. Amar, E.: Extension of holomorphic functions and residue current. Université de Bordeaux L. Preprint (1988)
4. Andreotti, A., Hill, C.D.: Levi convexity and the Hans Lewy problem. III. Ann. Sc. Norm. Super. Pisa **26**, 325–363; ibid. 747–806 (1972)
5. Andreotti, A., Hill, C.D., Lojasiewicz, S., Mackichan, B.: Complexes of Differential Operators. Invent Math. **35**, 43–86 (1976)
6. Baouendi, S.M., Chang, C.H., Treves, F.: Microlocal hypo-ellipticity and extension of C.R. functions. J. Differ. Geom. **18**, 331–391 (1983)
7. Baouendi, S.M., Rothschild, L.P., Treves, F.: C.R. structures with group action and extendability of C.R. functions. Invent Math. **82**, 359–396 (1985)
8. Bouendi, S.M., Rothschild, L.P.: Transversal Lie group actions on abstract C.R. manifolds. Preprint
9. Bedford, E., Gaveau, B.: Envelopes of holomorphy of certain 2-spheres in $\mathbb{C}_2$. Am. J. Math. **105**, 975–1009 (1983)
10. Chaumat, J., Chollet, A.M.: Ensembles pics pour $A^\infty(\mathbf{D})$. Ann. Inst. Fourier **29**, 171–200 (1979)
11. Charpentier, P.: Formules explicites pour les solutions minimales de l'équation $\bar{\partial}u=f$... Ann. Inst. Fourier **30**, 121–154 (1980)
12. Dufresnoy, A.: Sur l'opérateur d" et et les fonctions différentiables au sens de Whitney. Ann. Inst. Fourier **29**, 229–238 (1979)
13. Forstneric, F.: Proper holomorphic mappings: a survey. Preprint
14. Henkin, G.M.: The Lewy equation and analysis on pseudoconvex manifolds. Russ. Math. Surv. **32**, 57–118 (1977)
15. Kenig, C., Webster, S.: The local hull of holomorphy of a surface in the space of two variables. Invent. Math. **67**, 1–21 (1982)
16. Kohn, J.J.: Global regularity of $\bar{\partial}$ on weakly pseudo-convex manifolds. T.A.M.S. **181**, 273–292 (1973)
17. Kohn, J.J., Rossi, H.: On the extension of holomorphic functions from the boundary of a complex manifold. Ann. Math. **81**, 451–472 (1965)
18. Lewy, H.: An example of a smooth linear partial differential equation without solution. Ann. Math. **64**, 155–158 (1957)
19. Moser, J.K., Webster, S.: Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations. Acta. Math. **150**, 255–296 (1983)

20. Rosay, J.P.: Equation de Lewy-résolubilité globale de l'équation $\bar{\partial}_b u = f$ sur la frontière de domaines faiblement pseudo-convexes de $\mathbb{C}^2$ (ou de $\mathbb{C}^n$). Duke Math. J. **49**, 121–128 (1982)
21. Trepreau, J.M.: Sur le prolongement des fonctions C.R. définies sur une hypersurface réelle de classe $\mathscr{C}^2$ dans $\mathbb{C}^n$. Invent. Math. **83**, 583–592 (1986)
22. Tumanov, A.E.: Extending C.R. functions to a wedge from manifolds of finite type. Mat. Sb. **136**, 128–139 (1988)